Design and Implementation of Spatial Feedback Control on a Flexible Plate
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Abstract
This paper is aimed to design and experimentally evaluate the performance of an optimal feedback control strategy to suppress global broadband structural vibration of a flexible plate. The feedback controller was designed to minimize the spatial $H_2$ norm of the transfer function from the disturbance signal to the structural response at every point over the plate. This approach ensures the vibration contributed by all the in-bandwidth (0-500Hz) vibration modes is minimized, and hence is capable of minimizing vibration throughout the entire plate. The controller was implemented on a simply supported plate to show the effectiveness of the proposed controller in cancelling global structural vibration and noise radiation. The effectiveness of the spatial collocated feedback controller was then compared experimentally with that of the standard point-wise controller. Surface-bonded piezoceramic transducers were used as the disturbance generator, control source, and feedback sensor. Experiment results demonstrated that the spatial collocated feedback controller is able to reduce the energy transfer from the disturbance to the structural output across the structure.

Introduction
The trend of mechanical, aeronautical and civil design requires structures to become lighter, more flexible and stronger. These requirements cause the structure to be more easily influenced by unwanted vibrations which may lead to problems such as fatigue, instability, performance reduction, and may even cause damage to highly stressed structures. Furthermore, vibration problems may also cause acoustic disturbances which can be annoying and harmful to health, and in some cases illegal.

In practice, any structure that deforms under some loading can be regarded as flexible structure and is a distributed parameter systems. This implies that vibration at one point is related to vibration at the rest of the points over the structure. Thus, it is desirable to design a controller that takes into account vibration at every point over the structure by incorporating a global performance measure for vibration.

One of the global performance measures that can be used for distributed parameter systems is the spatial $H_2$ norm that was originally introduced in [1] and has been used to successfully design feedback controllers for structural vibration control [2]. The spatial feedback controllers minimized the spatial $H_2$ norm of transfer function from input disturbance to structural deflection at every point over the structure. Successful experimental studies have been conducted in [2]. However, previous research using spatial control approach [2] has only looked into one-dimensional flexible beam structures, and only aim to attenuate global structural vibration.

It was realized that more work is required to experimentally investigate the performance of the spatial controllers for vibration attenuation over a wider range of structures. It is natural to question whether the spatial control approach can be successfully implemented for such structures. Motivated by this, the present work aims to demonstrate the applicability of spatial controllers to attenuate global vibration (and the noise radiation indirectly) of a two-dimensional structure, in this case a flexible plate.

This paper focuses on systems in which the original excitation of the structure, due to the primary source, cannot be directly observed or in which there are too many primary sources to economically obtain reference signals from each one. In such systems, feedback vibration control strategy can be used for disturbance rejection. The work here uses the spatial $H_2$ norm of the spatially distributed output as the performance measure. The paper is organised as follows: The first section briefly describes the model of a flexible plate with surface-bonded piezoceramic actuators. In the second section, the design of a spatial $H_2$ collocated-feedback control for disturbance attenuation of flexible structures is presented. The third section discusses the design of spatial $H_2$ and point-wise $H_2$ feedback controllers for suppressing vibration of the first three modes of a simply supported plate. The fourth section presents experimental implementation of the controllers on a plate.

Model of Flexible Plate Structure
In this section, modelling of the piezoelectric laminate plate is briefly described.

There are many ways to model a flexible structure. In general cases of piezoelectric laminate structures, approximate methods such as Finite Element (FE) method may be required. There are other more comprehensive methods for modelling piezoelectric laminate structures. The modelling includes the contribution of piezoelectric patches to the properties of the structures and non-linear modelling of piezoelectric laminate structures. The purpose of the work here is to obtain a model sufficiently accurate to be used for control design.
Since the piezoelectric patches used in experiment are relatively thin with respect to the dimensions of the plate, it was assumed that the piezoelectric laminate plate is a uniform structure. The piezoelectric laminate plate used in the experiment can therefore be sufficiently modelled using the modal analysis method, which is confirmed experimentally in the following section.

Consider a homogeneous simply supported plate with length \( L_x \), width \( L_y \), and thickness \( h \) as shown in Figure 1. Each surface-bonded piezoelectric actuator has length \( L_{px} \), width \( L_{py} \) and thickness \( h_p \). Suppose there are \( j \) piezoelectric patches bonded over the plate. The applied voltages to the patches are \( V_a = [V_{a_1}...V_{a_j}]^T \).

The partial differential equation (PDE) that describes flexural vibration of thin plates is [2]:

\[
\rho h \frac{\partial^4 w}{\partial t^4} + D \frac{\partial^4 w}{\partial x^4} = \frac{\partial^2 M_{px}}{\partial x^2} + \frac{\partial^2 M_{py}}{\partial y^2},
\]

where \( D = \nabla^2 + \frac{\partial^2 W}{\partial x^4} + \frac{\partial^2 W}{\partial y^4} \), is the Green Function:

\[
G(s,x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{h_m(x,y) g_{mn}}{s^2 + 2 \zeta_m \omega_m s + \omega_m^2},
\]

where \( h_m \) and \( g_{mn} \) depend on the properties of the structure and the piezoelectric patches, and the damping ratio is denoted by \( \zeta_m \). The model in (2) describes the spatial and spectral properties of the plate system, which will be used to design the spatial H2 feedback controller.

**Spatial H2 Feedback Control of a Piezoelectric Laminate Plate**

This section is concerned with the problem of spatial H2 control for flexible structures. Consider a typical system of a flexible plate structure such as the one shown in Figure 2. The system consists of only one piezoelectric actuator-sensor pair for the sake of clarity. Here, the purpose of the controller \( K(s) \) is to reduce the effect of disturbance, \( d(t) \), on the entire structure, using piezoelectric actuators and sensors.

A spatially distributed Linear Time Invariant (LTI) dynamical system such as the plate in Figure 2 can be described in state space form:

\[
x(t) = Ax(t) + B_1 d(t) + B_2 u(t)
\]

\[
w(x,y,t) = C_1(x,y) x(t) + D_{11}(x,y) d(t) + D_{12}(x,y) u(t)
\]

where \( x(t) \) is the state vector and \( w(x,y,t) \) is the performance output (displacement and velocity) at time \( t \) over the plate. \( V(f) \) is the feedback signal from collocated sensor, \( u(t) \) is the control signal to the control source, and \( V(f) \) is the signal applied to the PZT. Here, the disturbance \( d(t) \) is assumed to enter through the same actuator channel, i.e. \( D_{22} = D_{12}, D_{11} = D_{12} \) and \( B_1 = B_2 \). However, a general case of disturbance can be dealt with in a similar manner.

The feedthrough term, \( D_{ij} \) can be estimated by matching with the dc gains of the measured experimental frequency responses. Experiments showed that it was sufficient to assume that the feedthrough terms from \( d(t) \) and \( u(t) \) to the displacement \( w(x,y,t) \) to be close zero, i.e. \( D_{11} = D_{12} = 0 \). The feedthrough terms from \( d(t) \) and \( u(t) \) to the PZT sensor output \( V(f) \) were found to be \( D_{21} = D_{22} = 0.02 \). They are not zero because there exists some direct structural connection path from the actuators to the sensor.

A spatial feedback controller is developed to reduce the effect of disturbance over the entire structure. The system with feedback control can be represented as a general Linear Fractional Transformation (LFT) as shown in Figure 3, where \( G_{ij} \) is the transfer function from \( j \) to \( i \). Thus, the aim for control design is to find a stable feedback controller \( K \) that minimizes a particular cost function \( J \). This cost function which is based on Spatial H2 norm concept represents the overall structural vibration of the plate.

According to the definition in [2], the weighted spatial H2 norm of a stable system \( G(j\omega,x,y) \) with \( x \in [0,L_x] \) and \( y \in [0,L_y] \) is defined as:

![Figure 1. A plate with surface-bonded piezoceramic actuator](image1)

![Figure 2. Collocated Feedback control of a flexible structure (section view)](image2)
where $Q(x, y)$ is a spatial weighting function and can be used for emphasizing the region where the vibration reduction is to be more concentrated.

The spatial $H_2$ norm of the system takes into account the spatial information embedded in the system. In this work, in order to minimize the disturbance of a spatially distributed system, the spatial 2-norm of the spatially distributed performance output over the plate is used as the cost function in the feedback control design.

In contrast to previous work in spatial controllers [2], the combination of displacement $w(x, y, t)$ and velocity term $\dot{w}(x, y, t)$ has been chosen here as the performance function in both spatial and point-wise feedback controller design. A cost function weighting, $\alpha$ is used to weight between the displacement and velocity terms. Displacement term emphasises on vibration reduction of the low frequency modes, while velocity term emphasises on the high frequency modes. This type of cost function can be shown to be related to the vibrational energy of the structure in physical sense. The output matrix of the system is defined as:

$$
\begin{bmatrix}
(1-\alpha) \times C_1 \\
\alpha \times C_{1v}
\end{bmatrix}
$$

where $C_1$ is the output displacement matrix and $C_{1v}$ is the output velocity matrix.

A control weighting, $\varepsilon$ is also added on the control signal to avoid an excessive controller gain. Excessive gain may lead to reduction of closed-loop robustness as well as other problems such as actuator saturation and noise sensitivity.

Here, the performance measure (cost function, $J$), the spatial 2-norm of $z(x, y, t)$ is defined as [1]:

$$
<< z >>^2_{2,0} = \iint_{D_x D_y} |z(x, y, t)|^2 Q(x, y) z(x, y, t) \, dx \, dy \, dt
$$

where $z(x, y, t) =$ {Displacement, $(1-\alpha) \times w(x, y, t)$
Velocity, $\alpha \times \dot{w}(x, y, t)$
Control Effort, $\varepsilon \times u(t)$

The spatial $H_2$ control problem is to design a stable feedback controller:

$$
\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + B_c V_S(t) \\
u(t) &= C_c x_c(t) + D_c V_S(t)
\end{align*}
$$

such that the weighted spatial $H_2$ norm of the closed loop system:

$$
<< T_{cl}(j\omega, x, y) >>^2_{2,0} = \iint_{D_x D_y} |T_{cl}(j\omega, x, y)|^2 Q(x, y) T_{cl}(j\omega, x, y) \, dx \, dy \, d\omega
$$

is minimized. Where $T_{cl}$ is the closed loop transfer function from $d$ to $z$. Following the approach in [1], and weight the entire plate equally with $Q(x, y) = 1$. The above control problem can be transformed into a standard $H_2$ control problem with the following system:

$$
\begin{align*}
\dot{x}(t) &= A x(t) + B d(t) + B u(t) \\
z(t) &= \Gamma x(t) + 0 u(t) \\
V_S(t) &= C z(t) + D_2 d(t) + D_2 u(t)
\end{align*}
$$

where $\Gamma$ is any matrix that satisfies:

$$
\int_{0}^{L_x} \int_{0}^{L_y} \Gamma \Gamma^T \, dx \, dy = \int_{0}^{L_x} \int_{0}^{L_y} C_1(x, y) \, dx \, dy
$$

In practice, a suitable $\varepsilon$ needs to be chosen to compromise between the vibration reduction level and controller gain. The optimal spatial feedback controller can then be obtained using a standard $H_2$ control technique [4].

**Feedback Controller Design**

This section explains the details of a spatial $H_2$ controller that is designed and implemented on a simply supported flexible plate in the Vibration Laboratory at the University of Adelaide, South Australia. A pair of G-1195 rectangular piezoceramic patches of 25mm wide, 25mm long, 0.25mm thick was attached symmetrically to either side of the plate by means of a thin epoxy film. Their locations on the patches are shown in Figure 4. The physical parameters are given in the Table 1.
Table 1. Properties of the Piezoelectric Laminate Plate

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Plate ( X )-length, ( L_x )</td>
<td>0.4 m</td>
</tr>
<tr>
<td>Plate ( Y )-length, ( L_y )</td>
<td>0.35 m</td>
</tr>
<tr>
<td>Plate thickness, ( h )</td>
<td>0.0028 m</td>
</tr>
<tr>
<td>Plate Young’s Modulus, ( E )</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Plate density, ( \rho )</td>
<td>7800 kg/m(^3)</td>
</tr>
<tr>
<td>Plate Poisson’s ratio, ( \nu )</td>
<td>0.292</td>
</tr>
<tr>
<td>Piezoceramic ( X )-length, ( L_{px} )</td>
<td>0.025 m</td>
</tr>
<tr>
<td>Piezoceramic ( Y )-length, ( L_{py} )</td>
<td>0.025 m</td>
</tr>
<tr>
<td>Piezoceramic thickness, ( h_p )</td>
<td>0.00025 m</td>
</tr>
<tr>
<td>Piezoceramic Young’s Modulus, ( E_p )</td>
<td>63 GPa</td>
</tr>
<tr>
<td>Charge constant, ( d_{31}, d_{32} )</td>
<td>( -1.66 \times 10^{-10} ) m/V</td>
</tr>
<tr>
<td>Voltage constant, ( g_{31} )</td>
<td>( -1.15 \times 10^{-2} ) Vm/N</td>
</tr>
<tr>
<td>Capacitance, ( C )</td>
<td>( 1.05 \times 10^{-7} ) F</td>
</tr>
<tr>
<td>Electromechanical coupling factor, ( k_{31} )</td>
<td>0.34</td>
</tr>
<tr>
<td>Piezoceramic Poisson’s ratio, ( \nu_p )</td>
<td>0.35</td>
</tr>
</tbody>
</table>

In this work, a SISO controller is designed for the purpose of controlling only the first five vibration modes (0-500Hz) of the plate. Hence, the model is truncated to include only the first 5 bending modes. The system in (8) can then be modelled in state space form (see [1, 2] for example). Figure 5 compares the frequency response from the actuator voltage to the sensor voltage between simulation and experiment for the open loop case.

![Figure 5](image1.png)

**Figure 5** Frequency response of Feedback Sensor signal (V/V) - Simulation VS Experiment

To compare the performance between the spatial and point-wise controllers, a point-wise controller was designed using the displacement and velocity information at a point close to the bottom centre of the plate (0.205m, 0.06m) as the cost function (see Figure 4). The cost function weighting \( \varepsilon \) was chosen as \( \varepsilon = 10^{-3} \). The penalty/weighting \( \varepsilon \) on the control signal of the point-wise controller was chosen such that the total control energy is the same as that of the spatial controller. This would ensure a fair comparison between the two controllers. MATLAB Robust Control Toolbox was used to design the spatial \( H_2 \) and point-wise \( H_2 \) controllers based on the system in (8) via state space approach. The frequency responses of the designed controllers are shown in Figure 6.

![Figure 6](image2.png)

**Figure 6** Frequency response of the controllers (input voltage to output voltage (V/V))

As expected, the controller has a resonant nature because of the highly resonant nature of the system. Spatial controller applies high gain at each resonant frequency in an attempt to reduce the resonance response, while the point-wise controller only applies high gain at 1\(^{st}\), 3\(^{rd}\) and 5\(^{th}\) modes.

In order to compare the performance of both controllers in achieving vibration reduction for spatially distributed systems, the \( H_2 \) norm at every point over the controlled and uncontrolled plate has been plotted based on the designed controllers. These simulation results are shown in Figure 7.

![Figure 7](image3.png)

**Figure 7** \( H_2 \) norm over the plate system

These \( H_2 \) norm plots demonstrate the energy transfer from the disturbance input to the structural response at
every point over the plate. From the plots, it can be seen that the spatial controller suppresses $H_2$ norm of the entire plate more uniformly, while the point-wise controller only tends to suppress $H_2$ norm at the point of interest, i.e. at point (0.205m, 0.06m).

When the overall structural velocity drops, it was expected that the noise radiation will drop as well. To get a better idea of the noise radiation before and after control; the sound power radiated from the plate is simulated using the elemental radiator approximation [3]. A set of 6x6 elemental radiators over the plate is defined. The acoustic power radiated by this array of elements can then be estimated by the following equation:

$$ W = v^H R v = v^H Q^T \Lambda Q v = y^H \Lambda y \approx \sum_{i=1}^{\infty} \lambda_i |y_i|^2 \quad (10) $$

where $R$ is the radiation resistance matrix for elemental radiators, $Q$ is an orthogonal matrix of eigenvectors, and $\Lambda$ is a diagonal matrix of eigenvalues $\lambda_i$, which are all positive real numbers, and $y = Qv$, is the vector of radiation modes in terms of the velocities of the individual elements. The simulated results are shown in Figure 8.

![Simulation Result - Noise Radiated (Sound Power Level, PWL) over the plate.](image)

**Figure 8. Simulation - Sound Power Level (PWL)**

**Experiment Implementation**

![Experimental Setup](image)

**Figure 9 Experimental Setup**

The experimental setup is depicted in Figure 9. The controller was implemented using a dSPACE DS1104 rapid prototyping Controller Board together with the MATLAB and SIMULINK software. The sampling frequency was set at 10 KHz. The cut-off frequency of the low pass filter was set at 500 Hz. A Physik Instrumente GmbH E-865.10 piezo-amplifier was used to amplify the filtered signal with a gain of 10 in order to supply necessary voltage for the actuating piezoceramic patches. A Bruel & Kjaer type 4344 accelerometer, a charge amplifier, and a HP 35665A Dynamic Signal Analyser were used to obtain frequency responses of the plate. The developed colocated feedback controllers in previous section were implemented to the plate.

**Vibration Measurement**

Vibration testing was carried out to verify the ability of spatial feedback controller in suppressing disturbance over a plate in frequency band 0-500 Hz (refer to Figure 7). Experiments in this section were performed in the time domain. A random noise (low-pass filtered at 500Hz) with duration of approximately 3 minutes was applied to the disturbance source. The velocity response over the plate was measured with an accelerometer. The velocity response was filtered by a low-pass filter with a cut off frequency of 500 Hz. The input random noise signal and the corresponding velocity measured at each point were all recorded in RMS (root mean squared). The experiments were repeated with control ON and OFF until vibration responses at 21 points (11 horizontal points and 10 vertical points) along the centre of the plate were measured (refer to Figure 4 for the measurement locations.)

For clearer comparison, the normalised simulation result of the $H_2$ norm (Velocity) along the centre X and Y axes of the controlled and uncontrolled plate has also been plotted as shown in the following Figure 10.

![Vibration Energy (Velocity) – Experiment VS Simulation, (a) Along centre X-axis, (b) Along centre Y-axis](image)
Controller achieved more vibration reduction over the plate compared to the Point-wise Controller, as expected from simulation.

**Sound Power Measurement**

![Figure 11. Noise Measurement – Experiment Setup in Anechoic Chamber](image)

The flexible plate was placed in an infinite baffle within the Anechoic Chamber as shown in Figure 11. A B&K sound intensity probe was used to record the radiated sound. For each measurement point, the power spectrum of each microphone and the transfer function between the 2 microphones were recorded. The measurement was repeated for Control ON and OFF cases. A total of 36 points have been measured for each case. The sound power $W$ between the 2 microphones can then be estimated by:

$$ W \approx \sum_{i=0}^{36} I_{x_{ii}} \times (\Delta x \times \Delta y) $$  \hspace{1cm} (11)

where the time averaged intensity in the direction pointing towards the plate between the 2 microphones are:

$$ I_{x_{ii}} = \frac{|P_1||P_2|}{2\rho_0\Delta x_{12}} \sin(\phi_1 - \phi_2) $$  \hspace{1cm} (12)

where $|P_1|$ and $|P_2|$ are the power spectrum of the 1st and 2nd microphones respectively. The phase difference between the two microphones $(\phi_1 - \phi_2)$ was obtained by recording the transfer function between the two. The experiment result of the sound power level is shown in Figure 12.

The significance of these experimental results is that the spatial feedback controller is able to reduce the energy transferred from the disturbance to the structural response across the plate.

Comparing the overall performance of the spatial controller and the point-wise controller, it can be observed that the spatial controller is able to suppress the broadband structural vibration contributed by all in-bandwidth vibration modes; while performance of the point-wise controller strongly depends on the location of the point chosen during the controller design.

**Conclusion**

Experiment results demonstrated that the spatial controller is able to reduce the energy transferred from the disturbance to the structural response across the structure.

This Spatial Control Approach can be used to design an active controller that minimizes the global vibration of a structure. With successful attenuation of the global structural vibration disturbance, the lifetime of the structure would be extended by increasing material durability and fatigue life, and hence save the cost for maintenance or facility down time installation. Furthermore, if the structural radiated noise could indirectly be reduced at the same time, machine operator fatigue would be reduced and ergonomics would be improved.

**Reference**


