Zero stiffness magnetic supports

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ABSTRACT

Active vibration isolation of large load structures typically requires significant actuator energy. With a zero stiffness support, which can be realised with a permanent magnet system, reactive forces may be applied to the structure with comparatively little effort. This paper presents results for such a magnetic spring arrangement which is stabilised by a non-linear controller. Transmissibility less than unity is demonstrated over the entire frequency spectrum.

1 INTRODUCTION

In a large load vibration isolation system, significant forces are required to counter the effects of gravity. In order to bear this load, a conventional linear isolator requires either a high stiffness ($k$) if the allowable static displacements ($x$) are small, or large displacement if the stiffness must be low, via force $F = -k(x - x_0)$ with $x_0$ the static deflection.

While low stiffness is advantageous for vibration isolation, there is a practical lower limit on obtaining a stiffness to support a large load. As a spring compresses, its stiffness tends to increase, which has the disadvantage of increasing the system’s resonant frequency $\omega_n = \sqrt{k/m}$, for mass $m$. A high resonant frequency has the general disadvantage of poor passive vibration isolation, as attenuation occurs only above $\sqrt{2} \cdot \omega_n$.

With these points in mind, a method of supporting large loads while keeping the stiffness low is desirable. For an active system, further advantage is gained from reduced control effort in actuating the device.

In this paper, a permanent magnetic configuration is demonstrated that can reduce the passive stiffness to zero at unstable equilibrium while still providing a supporting force. The magnetic design is scalable, which provides for the capability of large load bearing. Non-linear control laws are proposed to stabilise the system and are demonstrated via simulation. Finally, implementation issues are discussed for a practical system.

2 SIMPLE MAGNET LOAD-BEARING

The simplest form of magnetic suspension is the vertically attractive pair shown in Figure 1a, in which a fixed upper magnet supports the lower in an unstable manner. Due to the inherent instability resulting from negative stiffness and the non-linear forces involved, this system is often used for demonstrations of the efficacy of active control techniques.

While all permanent magnet levitations are unstable by nature \cite{1, 2}, instability in a direction other than the supporting direction is desired for a completely passive spring (for example, see the spring of Puppin and Fratello \cite{3}). A similarly simple (in terms of geometry) configuration is a vertical pair of magnets in repulsion, with the lower magnet

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Figure 1: Vertically oriented magnetic springs. The shaded magnet in each is supported against gravity ($F_g$) by the respective magnetic forces ($F_m$). Arrows within the magnets indicate their directions of magnetisation.

used as the support as shown in Figure 1b. The stiffness in the load bearing direction is positive, and the system is suitable for bearing loads of varying amounts.

Zero stiffness structures have been examined by Nijsse [4], who introduced the magnet arrangement shown in Figure 1c, a combination of the two aforementioned magnetic springs. Xing et al. [5] have examined the general solution for feedback control systems achieving zero and infinite stiffness. The latter has been demonstrated by Mizuno et al. [6], in which the series combination of a conventional positive spring and a negative magnetic spring results in theoretically infinite stiffness in total:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}, \quad \therefore k = \frac{k_1k_2}{k_1 + k_2} = \infty, \text{ if } k_1 + k_2 = 0.$$

This paper further develops the concept of zero stiffness through the parallel combination of two magnetic springs, $k = k_1 + k_2 = 0$, via theoretical modelling and simulated non-linear control.

3 ZERO STIFFNESS

The term ‘zero stiffness’ strictly denotes a decoupling between displacement and force for two disparate objects in space;

$$F(x) = \lim_{k \to 0} -kx = 0. \quad (1)$$

However, for the proposed zero stiffness magnetic system the stiffness varies by position, so perturbations from the equilibrium point eliminate the zero stiffness property. On the other hand, small perturbations lead only to small changes in the stiffness, which will therefore remain close to zero.

Taking the inverse of Equation 1 naïvely implies that for a totally zero stiffness system, applied force will effect a displacement without bound: $x(F) = \lim_{k \to 0} -F/k = \infty$. In a more realistic case, the lower the stiffness the more easily the system can be actuated.
The advantage of the proposed magnetic configuration is that a reasonable load may be supported passively by the permanent magnetic force, while an active system may apply control forces to the structure with small comparative effort with respect to a conventional (i.e., positive stiffness) system.

4 TRANSLATORY STABILITY OF SIMPLE MAGNETIC SPRINGS

In this section, the stabilities through displacement for the three springs shown in Figure 1 are analysed.

4.1 Mathematical models

Expressions for the forces between magnets follow directly from Maxwell’s equations, and may be derived in a number of ways. For cuboid-shaped magnets with parallel/anti-parallel magnetisations that experience translation in three degrees of freedom but no rotation, a relatively concise analytical solution for the forces has been known for over twenty years [7]:

\[ F = \frac{JJ'}{4\pi\mu_0} \sum_{(i,j,k,l,p,q) \in \{0,1\}^6} f[u_{ij}, v_{kl}, w_{pq}] \cdot (-1)^{i+j+k+l+p+q}, \]  

(2)

where

\[ f_x = \frac{1}{2} (v^2 - u^2) \ln[r - u] + uv \ln r - v + uv \arctan \left( \frac{uv}{rw} \right) + \frac{1}{2} ru, \]
\[ f_y = \frac{1}{2} (u^2 - w^2) \ln[r - v] + uv \ln r - u + uv \arctan \left( \frac{uv}{rw} \right) + \frac{1}{2} rv, \]
\[ f_z = -uv \ln[r - u] - vw \ln[r - v] + uv \arctan \left( \frac{uv}{rw} \right) - rv, \]

and

\[ u_{ij} = \alpha - a(-1)^i + A(-1)^j, \]
\[ v_{kl} = \beta - b(-1)^k + B(-1)^l, \]
\[ w_{pq} = \gamma - c(-1)^p + C(-1)^q, \]
\[ r = \sqrt{u_{ij}^2 + v_{kl}^2 + w_{pq}^2}, \]

where \((a, b, c)\) and \((A, B, C)\) are the half dimensions of the fixed and floating magnets, respectively, and \((\alpha, \beta, \gamma)\) is the distance between their centres; \(J\) and \(J'\) are the magnetisations, and \(\mu_0\) is the permeability of free space. More recently, equivalent methods have been demonstrated that provide more abstract methods for dealing with more complex geometries [8].

The solution shown in Equation 2, and others like it, provide a convenient way to analyse the behaviour of any simple permanent magnet configuration. Since the coercivity of rare earth magnetic material is great enough to ensure that nearby magnets will not demagnetise each other, the forces follow the principle of superposition. That is, referring back to Figure 1, \(F_{m3} = F_{m1} + F_{m2}\).

Equation 2 can be differentiated to obtain the stiffnesses, \(K = -\nabla_{xyz} F\):

\[ K = \frac{JJ'}{4\pi\mu_0} \sum_{(i,j,k,l,p,q) \in \{0,1\}^6} k[u_{ij}, v_{kl}, w_{pq}] \cdot (-1)^{i+j+k+l+p+q}, \]  

(3)
where
\[ k_x = \kappa[u, v, w], \quad k_y = \kappa[v, u, w], \quad k_z = -k_x - k_y, \]
\[ \kappa[u, v, w] = -\frac{vu^2}{u^2 + w^2} - r - v \log[r - v], \]
and the parameters \( u, v, w, \) and \( r \) are as given in Equation 2. Note that the result \( k_x + k_y + k_z = 0 \) follows from Earnshaw’s theorem [1].

### 4.2 Stability analyses
The stability of a magnetic configuration may be investigated by examination of the stiffnesses in each direction. Stability exists for negative force gradients (positive stiffness), where the reaction forces act to oppose perturbatory displacements. In the analysis presented in this paper, cube magnets of side length 20 mm are used with magnetisations of 1 T, oriented as depicted in Figure 1.

The stiffness in each translatory direction for the vertical attracting spring is shown in Figure 2. It can be seen that this spring is unstable in the vertical direction, but stable in both horizontal directions.

A similar analysis has been performed on the vertical repelling spring, for which the results are displayed in Figure 3. These show opposite tendencies to the previous spring: both horizontal directions are unstable, but the vertical, load-bearing, direction is stable.

When these two springs are combined to create the spring shown in Figure 1c, the stiffnesses in each bearing direction go through an inflexion point, as shown in Figure 4. This inflexion, at the point of equidistance between the two magnets, is the point of zero stiffness. The supporting force at this point is a function of the gap. This relationship is shown in Figure 5a, which plots vertical force vs. vertical displacement for a zero stiffness spring with varying gaps.

The curves shown in Figure 5b demonstrate the nominal load-bearing force of a zero stiffness spring with increasing cube magnet side-lengths. This figure indicates that a large design space is possible with appropriately chosen parameters.

### 4.3 Development of more advanced magnetic geometries
The results in Figure 3 show that the vertically repelling spring is unstable in both horizontal directions. Since \( k_x + k_y + k_z = 0 \), there are two options for the behaviour of the horizontal dynamics. Stabilisation in one of these directions is possible through the addition of horizontally located magnets [9]. In this case, at rest, \( k_z = 0 \) and \( k_y = -k_x > 0 \); passive stabilisation in one horizontal degree of freedom is achieved.

Alternatively, with the proposed combination spring, the horizontal stiffnesses in Figure 4b are also zero at the midpoint: \( k_x = k_y = k_z = 0 \). With stabilising controllers (equivalent to those shown in this paper for the vertical direction) in every degree of freedom, an object may be supported with zero stiffness in each translatory direction.

For increasing the load bearing ability of a magnetic spring, simple scaling of the magnetic volume is inefficient [10]. Greater forces over shorter distances may be achieved with multipole (‘Halbach’) arrays, which are an approximation of a sinusoidally polarised magnet [9]. The tenet of zero stiffness for such structures remains the same as for homogeneous magnets, however, and these structures will not be analysed further here.
Figure 2: Stiffnesses of the vertical attracting spring (Figure 1a).

Figure 3: Stiffnesses of the vertical repelling spring (Figure 1b).

Figure 4: Stiffnesses of the zero stiffness spring (Figure 1c).
(a) Force/displacement curves for the magnetic arrangement shown in Figure 1c. 20 mm cube magnets are used with various gaps.

(b) Supporting force/gap curves for various magnet sizes at zero displacement. In each case, the gap range is 1–2 magnet dimensions.

Figure 5: Dependence of the supporting force on geometry of the zero stiffness spring.
5 CONTROL STRATEGIES

The basic zero stiffness configuration that is under examination is marginally stable at its operating point. The stable effects of the lower magnet and the unstable effects of the upper magnet combine to produce a unique force/displacement characteristic as seen in Figure 5a in the previous section.

To a good approximation, this relationship may be modelled as quadratic: \( F_m = Kx^2 + F_0 \), for displacement \( x \) from the rest position at the point of zero stiffness, midway between the two fixed magnets, with a supporting force \( F_0 \). When summed with the load due to gravity, this simplifies to \( F = F_m - mg = Kx^2 \).

For this stage of the analysis, the damping of the system is neglected, as its small effect will not induce instability. The dynamic equation of motion of the zero stiffness spring is therefore

\[
m\ddot{x} - Kx^2 = 0.
\]  

A controller for this unstable system may now be designed for the purposes of stable operation at the zero stiffness position. The design for a linear controller might proceed from here by linearisation around the operating point; linear stiffness \( k = \frac{\partial F}{\partial x}\big|_{x=0} \).

The linearised system is now expressed as

\[
\begin{bmatrix}
\dot{x} \\
\ddot{x}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
x
\end{bmatrix} + \begin{bmatrix}
1/m \\
0
\end{bmatrix} u. 
\]  

A standard linear controller designed around this model fails to stabilise the actual non-linear system, and more complex control strategies are required. A simple backstepping controller \([11]\) is used as an example. In order to compare the results from this system to a more conventional spring, the same analysis is performed on the vertically attracting spring. For the purposes of vibration isolation, it is desired to examine the behaviour with indirect excitation through the (previously assumed fixed) base, representing undesirable ground vibration. Schematics for the two systems are shown in Figure 6, which defines \( U_1 \) and \( U_2 \) as inputs to the single magnet and zero stiffness springs, respectively.

The dynamic equation for these systems, assuming some linear viscous damping, \( d \), and a non-linear stiffness force \( F_k[\cdot] \), is

\[
m\ddot{x} + d(\dot{x} - \dot{y}) - F_k[x - y] = U. 
\]  

The forces \( F_k[x - y] \) may be calculated via application of Equation 2. From such numerical results, approximations can be derived as follows for use in the control laws:

\[
\begin{align*}
F_{k_1}[\bar{x}] &\approx K_3\bar{x}^3 + K_2\bar{x}^2 + K_1\bar{x}, & \text{(single magnet)} \\
F_{k_2}[\bar{x}] &\approx K\bar{x}^2, & \text{(zero stiffness)}
\end{align*}
\]  

in which cancelation of the static force with the gravity load has been taken into account.
5.1 Zero stiffness controller derivation

The approximate non-linear system dynamics for the zero stiffness spring may be written in the following form:

\begin{align}
\dot{x} &= v, \\
\dot{v} &= k(x - y)^2 + u, 
\end{align}

where \(x\) is the displacement state, \(v\) is the velocity state, \(k = K/m\) from Eqs. 4/8, and \(u = U_2/m\) is the normalised input force. A virtual control \(\xi\) is defined for Equation 9, and the virtual state \(z\) introduced to track the difference between this virtual input and \(v\).

\[ z \overset{\text{def}}{=} v - \xi, \quad \dot{z} = \dot{v} - \dot{\xi} = k(x - y)^2 + u - \dot{\xi}. \]  

A basic Lyapunov function is chosen to design the stability of the system:

\[ V_1 = \frac{1}{2}x^2, \quad \dot{V}_1 = x\dot{x} = xv = xz + x\xi. \]

Control parameters may now be chosen to stabilise the system by ensuring Equation 12 is negative (semi-)definite. The cross term \(xz\) cannot be stabilised yet, so \(\xi\) is chosen to deal with the second term only.

\[ \xi \overset{\text{def}}{=} -c_1x, \quad c_1 \in \mathbb{R}^+, \quad \dot{\xi} = -c_1\dot{x} = -c_1v. \]

\(z\) can now be expressed as

\[ z = c_1x + v, \quad \dot{z} = c_1v + k(x - y)^2 + u. \]

A new Lyapunov function is defined

\[ V_2 = V_1 + \frac{1}{2}z^2, \quad \dot{V}_2 = \dot{V}_1 + \dot{z}z = (xz + x\xi) + z(u + k(x - y)^2 + c_1v), \]

\[ = -c_1x^2 + z(u + x + k(x - y)^2 + c_1v). \]
The simplest route to stability is taken when the non-linearities are simply cancelled by the controller:

\[ u \overset{\text{def}}{=} -c_2 z - x - k(x - y)^2 - c_1 v, \quad c_2 \in \mathbb{R}^+ \]
\[ = -x(c_1 c_2 + 1) - v(c_1 + c_2) - k(x - y)^2, \]  
\[ \Rightarrow \dot{V}_2 = -c_1 x^2 - c_2 z^2 \leq 0 \forall x, z \]  
(17) (18)

For an ideal system, global asymptotic stability is proven by Equation 18.

This controller design is equivalent to feedback linearisation since the choice for \( u \) simply cancels the nonlinearity and applies controller gains on the states \( x \) and \( v \). It is effective for the purposes of demonstrating stabilisation control of the zero stiffness spring, although more complex controllers can be devised. The final result is the closed loop system, for some controller gains \( c_1 \) and \( c_2 \).

\[ \ddot{x} = -x(c_1 c_2 + 1) - \dot{x}(c_1 + c_2). \]  
(19)

Note that these closed loop dynamics are independent of the disturbance \( y \). For an ideal system, attenuation is therefore infinite. In practice, however, errors will occur as the true dynamics deviate from the approximation. These errors lead to instability for low controller gains \( c_1 \) and \( c_2 \), but this instability can be compensated for by sufficiently fast (high) controller gains.

An equivalent controller can be designed for the single magnet spring, resulting in identical closed loop dynamics. Referring to Figure 6, the two controllers are:

\[ U_2 = -m x(c_1 c_2 + 1) - m \dot{x}(c_1 + c_2) - K(x - y)^2, \]  
\[ U_1 = -m x(c_1 c_2 + 1) - m \dot{x}(c_1 + c_2) - K_3(x - y)^3 - K_2(x - y)^2 - K_1(x - y). \]  
(20) (21)

6 SIMULATION RESULTS

Simulation results for the zero stiffness spring and the more conventional vertical repelling (single magnet) spring with the above controllers are shown in Figures 7–9, which show time traces of the displacements and forces experienced by the controlled springs and frequency plots of the transmissibilities. The transmissibilities are calculated as the transfer function between the output displacement and input disturbance.

The masses of the suspended magnets are equal and chosen to match the nominal force of the zero stiffness spring with a gap of 20 mm. The disturbance input is assumed to be Gaussian-distributed random displacement with a standard deviation of 1 mm; this is numerically differentiated to obtain the input velocities for a linear damping of 5%.

Although the ideal closed loop dynamics should equal each other (if the controller cancellations were perfect) and the displacements of the suspended magnet should tend towards zero, the mismatch between the simulated and actual dynamics shows closer to real world behaviour. Table 1 shows the RMS values of displacement and force for the two systems. It can be clearly seen that the zero stiffness spring has a much lower control effort to achieve vibration isolation.

Due to inaccuracies in the models used in the controllers, the true dynamics are not entirely compensated. This can be clearly seen for the single magnet spring which has a
Figure 7: Displacement traces for the actively controlled springs.

Figure 8: Force traces for the actively controlled springs. Note that the force axes are not scaled equally due to the large discrepancy.

Table 1: RMS values of the displacements, total forces, and control forces.

<table>
<thead>
<tr>
<th>System</th>
<th>RMS Disp. (mm)</th>
<th>RMS Force (N)</th>
<th>RMS Control (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single magnet</td>
<td>0.351</td>
<td>2.01</td>
<td>0.666</td>
</tr>
<tr>
<td>Zero stiffness</td>
<td>0.090</td>
<td>0.163</td>
<td>0.0683</td>
</tr>
</tbody>
</table>
(small) resonance peak at approximately 5 Hz. The zero stiffness spring, which does not have a resonance peak by its nature, does not exhibit this behaviour; indeed, its transmissibility is a marked improvement over the other. The zero stiffness spring provides two advantages over a more conventional one: lower control effort and greater transmissibility reduction.

7 INHERENT ASSUMPTIONS
Many assumptions have been made to simplify the analyses in this paper. These assumptions will be lifted in the practical development of the vibration isolator. The zero stiffness property only holds when the magnet distances are tuned to support the mass of the isolator in the region of local force minima. Variations and wind-up of the load would require a variable magnet separation distance, which could be effected with an actively controlled screw drive for (slow) online tuning.

The effects of the actuator dynamics have not been modelled, nor have problems relating to achieving torque-free forces been addressed. With careful placement of zero stiffness Lorentz (voice-coil) actuators [4, §5.1.2], these problems should not be significant in practice.

8 SUMMARY
The unique force characteristic of a magnetic configuration combining vertically attracting and vertically repelling springs allows non-contact load bearing with zero stiffness properties. This is different from a classical spring that has a lower bound on its stiffness and hence a lower limit on its vibration isolation capabilities.

The non-linear force-displacement relationship of this combination spring required a non-linear controller, which was developed using a simple backstepping technique. Simulation results based on this controller displayed the zero stiffness tendency anticipated;
smaller than unity transmissibility was achieved over the entire frequency spectrum.

The use of a zero stiffness spring could provide significant benefits to the application of a practical large load vibration isolator mount, which will be further investigated in future research.

REFERENCES