Non-resonant response of van der Pol-Duffing oscillator with nonlinear feedback control

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ABSTRACT
Non-resonant bifurcations of codimension two may appear in a controlled van der Pol-Duffing oscillator when the two critical time delays corresponding to two Hopf bifurcations have the same value. In the vicinity of non-resonant Hopf bifurcations, the presence of periodic excitations in the controlled oscillator can induce complicated behaviour of the system. In addition to non-resonant response, some types of resonances including primary, super- and sub-harmonic resonances, additive and difference resonances may appear in the forced response of the controlled system, when the frequency of the external excitation and the frequencies of the Hopf bifurcations satisfy a certain relationship. With the aid of normal form theory and the centre manifold theorem as well as a perturbation method, the non-resonant response of the controlled system at the non-resonant bifurcations of codimension two is investigated by studying the possible solutions and the stability of the four-dimensional ordinary differential equations on the centre manifold. It is shown that the non-resonant response of the oscillator may exhibit quasi-periodic motions on either 2-D or 3-D tori. Illustrative examples are given to show the periodic and quasi-periodic motions. The analytical predictions are found to be in good agreement with the results of numerical integration of the original delay differential equation.

1 INTRODUCTION
Many dynamic problems raised in mechanical engineering can be mathematically modelled by van der Pol-Duffing oscillator, which has been shown to exhibit complex behaviour and thus has received considerable attention in the literature [1]. An externally forced van der Pol-Duffing oscillator under nonlinear feedback control considered in the present paper is of the form

\[\ddot{x} - (\mu - \beta x^2) \dot{x} + \omega^2 x + \alpha x^3 = e_0 \cos(\Omega_0 t) + U(t), \quad (1)\]

where \(x\) is the displacement, \(\omega\) is the natural frequency, \(\alpha\) is the coefficient of nonlinear term, \(\mu > 0, \beta > 0, e_0\) and \(\Omega_0\) represent the amplitude and frequency of the external excitation, respectively, \(U(t)\) is the feedback control input, and an overdot indicates the differentiation with respect to time \(t\). The nonlinear feedback control takes the general form of

\[U(t) = px(t - \tau) + q\dot{x}(t - \tau) + k_1 x(t - \tau) + k_2 \dot{x}(t - \tau) + k_3 x(t - \tau) x^2(t - \tau) + k_4 \dot{x}^2(t - \tau)x(t - \tau), \quad (2)\]

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where \( p \) and \( q \) are the proportional and derivative linear feedback gains, \( k_i \) are the weakly nonlinear feedback gains, and \( \tau \) denotes the time delay occurring in the feedback path. Only one time delay is considered here for the sake of simplicity.

The corresponding autonomous system for which the external excitation is neglected in equation (1), obtained by letting \( e_0 = 0 \) in equation (1), is then given by

\[
\ddot{x} - \mu \dot{x} + \omega^2 x - px(t - \tau) - qx(t - \tau) + \beta k_1 \dot{x} + \alpha x^3 - k_3 \dot{x}^3 (t - \tau) - k_4 \dot{x}^2 (t - \tau) = 0.
\]

(3)

It was shown that the trivial equilibrium of the autonomous system given by equation (3) may lose its stability via either a subcritical or a supercritical Hopf bifurcation and regain its stability via a reverse subcritical or a supercritical Hopf bifurcation as the time delay increases [2]. It was also found that an interaction of two Hopf bifurcations may occur when the two corresponding critical time delays have the same value. In the vicinity of non-resonant Hopf bifurcations, the controlled oscillator without excitations was found to have the initial equilibrium solution, two periodic solutions as well as a quasi-periodic solution on a 2-D torus [3].

The forced response of the controlled system with time delays has received less attention in the available literature. When an external excitation is presented in the controlled system involving time delay, a technical issue that needs to be addressed is the interaction of the external excitation and the periodic (or quasi-periodic) motions of the corresponding autonomous system that results from non-resonant Hopf bifurcations. As will be shown, some types of resonances including primary, super- and sub-harmonic resonances, additive and difference resonances may appear in the forced response of the controlled system, when the forcing frequency and the frequencies of Hopf bifurcations satisfy a certain relationship. The primary objective of the present paper is to study the non-resonant response of the controlled system that results from an interaction of the external excitation and the bifurcation solutions of the corresponding autonomous system at non-resonant Hopf bifurcations.

The remainder of the present paper proceeds as follows. In the next section, the existence of non-resonant Hopf bifurcations is briefly reviewed for the corresponding autonomous system. In Section 3, the reduction of the delay differential equation is briefly discussed. The non-resonant response of the system is analytically studied in Section 4 using the method of multiple time scales. In Section 5, illustrative examples are given to show the non-resonant response of the controlled system. Conclusion is presented in Section 6.

2 EXISTENCE OF NON-RESONANT HOPF BIFURCATIONS

This section reviews background materials on the existence of non-resonant bifurcation of codimension two in the corresponding autonomous system after its trivial equilibrium loses it stability. More details can be found in reference [3].

It was found that if \((q^2 - \mu^2 + 2\omega^2) > 0, (\mu^2 - q^2)(\frac{1}{4}q^2 - \frac{1}{4}\mu^2 + \omega^2) < p^2 < \omega^4\),

the trivial equilibrium of equation (3) may lose its stability via a double Hopf bifurcation. The frequencies of Hopf bifurcations are given by

\[
\delta^2_x = \frac{1}{2}(q^2 - \mu^2 + 2\omega^2 \pm \sqrt{4(p^2 - \omega^4) + (2\omega^2 + q^2 - \mu^2)^2}),
\]

(4)

where \( \delta_+ > \delta_- > 0 \).
Two sets of the critical time delay $\tau_c$ corresponding to the double Hopf bifurcation are given by
\[
\tau_{1c,n} = \frac{s_1}{\delta_+} + \frac{2n\pi}{\delta_+}, \quad n = 0, 1, 2 \ldots
\] (5a)
where $0 \leq s_1 < 2\pi$, $\sin s_1 = \frac{(\mu p + q\omega^2 - q\delta^2)}{p^2 + q^2\delta^2}$, $\cos s_1 = \frac{p\omega^2 - p\delta^2 - \mu q\delta^2}{p^2 + q^2\delta^2}$; and
\[
\tau_{2c,n} = \frac{s_2}{\delta_-} + \frac{2n\pi}{\delta_-},
\] (5b)
where $0 \leq s_2 < 2\pi$, $\sin s_2 = \frac{(\mu p + q\omega^2 - q\delta^2)}{p^2 + q^2\delta^2}$, $\cos s_2 = \frac{p\omega^2 - p\delta^2 - \mu q\delta^2}{p^2 + q^2\delta^2}$.

For clarity, the bifurcations occurring at points ($\delta_+, \tau_{1c,n}$) and ($\delta_-, \tau_{2c,n}$) will be termed the first and second single Hopf bifurcations, respectively. The frequencies of the first and second Hopf bifurcations, namely, $\delta_{01}$ and $\delta_{02}$, are then given by $\delta_{01} = \delta_+$, $\delta_{02} = \delta_-$.

An intersection of the first and second Hopf bifurcations generally occurs in two cases, non-resonant and resonant Hopf-Hopf interactions, depending on the ratio of the frequencies of two Hopf bifurcations. As an illustrative example, consider a specific system with the parameters in equation (3) given by $\mu = 0.1$, $\omega = 1.0$, $p = -0.4$, $\alpha = 0.4$, $\beta = 0.5$, $k_1 = 0.2$, $k_4 = 0.5$, $k_2 = k_3 = 0.0$. It is easy to find from equation (4) that the first and second Hopf bifurcation frequencies are $\delta_{01} = 1.28038$ and $\delta_{02} = 0.71582$. The point of intersection is at $(q, \tau) = (-0.40219, 5.46397)$, where time delay $\tau$ has the same value on the two curves. The intersection points are usually referred to as the points of co-dimension two bifurcations and can be a source of more complicated dynamics.

3 REDUCTION OF THE DELAY DIFFERENTIAL EQUATION

For simplicity, it is assumed that an intersection of non-resonant Hopf bifurcations occurs at point $(p_0, q_0, \tau_0)$. In order to study the periodic solutions in the neighbourhood of the bifurcation point $(p_0, q_0, \tau_0)$, three small perturbation parameters, $\alpha_1$, $\alpha_2$ and $\alpha_3$, are introduced in terms of $p = p_0 + \alpha_1$, $q = q_0 + \alpha_2$, $\tau = \tau_0 + \alpha_3$. These perturbation parameters can conveniently account for the small variations of the linear feedback gains and the critical time delay.

Introducing the three dummy parameters defined above and letting $y_1 = x$, $y_2 = \dot{x}$ in equation (1) yields two first-order equations
\[
y_1' = \tau_0 y_2 + \alpha_1 y_2,
\]
y_2' = -\omega^2 \tau_0 y_1 + \mu \tau_0 y_2 + p_0 \tau_0 y_1(t-1) + q_0 \tau_0 y_2(t-1) - \alpha_1 \omega^2 y_1 + \alpha_2 \mu y_2
+ \alpha_3 p_0 y_1(t-1) + \alpha_3 q_0 y_2(t-1) + (\tau_0 + \alpha_3)[f_a(y(1)) - f_a(y) + f_a(y(1))] + \epsilon \cos(\Omega t),
\] (6)
where $f_a(y) = \beta y^2_1 y_2 + \alpha y^3_1$, $f_a(y(1)) = \alpha y_1(t-1) + \alpha_2 y_2(t-1)$,
\[
f_a(y(1)) = k_1 y_1^3(t-1) + k_2 y_2^3(t-1) + k_3 y_1^2(t-1) y_2(t-1) + k_4 y_1(t-1) y_2^2(t-1),
\]
\[ e = (\tau_0 + \alpha_1)e_0, \quad \Omega = (\tau_0 + \alpha_2)\Omega_0, \] and the time \( t \) has been normalized in terms of \( t = \bar{t}\tau \) with \( \tau = \tau_0 + \alpha_3 \), a prime indicates the differentiation with respect to the new time \( \bar{t} \) and its above bar has been removed here for brevity. The frequencies of Hopf bifurcation in equation (6), namely, \( \delta_1 \) and \( \delta_2 \), are now given by \( \delta_1 = \tau_0\delta_{01}, \delta_2 = \tau_0\delta_{02} \).

Following the normal procedure of the reduction of delay differential equations to ordinary differential equations based on semigroups of transformations and the decomposition theory [4-6], the dynamic behaviour of the solutions of equation (1) in the neighbourhood of non-resonant Hopf bifurcation of co-dimension two can be interpreted by the solutions and their stability of a set of four ordinary differential equations on the centre manifolds. By treating the external excitation in equation (6) as an additional perturbation term and performing the similar algebraic manipulations to those developed in reference [3], the nonlinear equations governing the local flow on the centre manifold are given in component form by

\[ \begin{align*}
\dot{z}_1 &= l_{11}z_1 + (\delta_1 + l_{12})z_2 + l_{13}z_3 + l_{14}z_4 + f_{10}(z) + e_{10}\cos(\Omega t), \\
\dot{z}_2 &= (-\delta_1 + l_{21})z_1 + l_{22}z_2 + l_{23}z_3 + l_{24}z_4 + f_{20}(z) + e_{20}\cos(\Omega t), \\
\dot{z}_3 &= l_{31}z_1 + l_{32}z_2 + l_{33}z_3 + (\delta_2 + l_{34})z_4 + f_{30}(z) + e_{30}\cos(\Omega t), \\
\dot{z}_4 &= l_{41}z_1 + l_{42}z_2 + (-\delta_2 + l_{43})z_3 + l_{44}z_4 + f_{40}(z) + e_{40}\cos(\Omega t),
\end{align*} \tag{7} \]

where \( f_{10}(z) = b_{12}NL T, \quad f_{20}(z) = b_{22}NL T, \quad f_{30}(z) = b_{32}NL T, \quad f_{40}(z) = b_{42}NL T, \) \( e_{10} = b_{12}e, \quad e_{20} = b_{22}e, \quad e_{30} = b_{32}e, \quad e_{40} = b_{42}e, \) the shortening phase \( NLT \) stands for the nonlinear terms consisting of 20 terms of the third order, and the relevant coefficients involved in equation (7) are explicitly given in reference [3].

Depending on the relationship of the two natural frequencies with the forcing frequency, the forced behaviour of system (7) may exhibit non-resonant and resonant response. Primary, sub-harmonic and super-harmonic resonances, as well as additive and difference resonances can occur in the forced response. The non-resonant response of the system will be discussed in the next section using a perturbation method, as the closed form of the solutions of equation (7) cannot be found analytically. The section reviews background materials on the existence of non-resonant bifurcation of codimension two in the corresponding autonomous system after its trivial equilibrium loses its stability. More details can be found in reference [3].

## 4 NON-RESONANT RESPONSE

In this section, the solutions of equation (7) will be approximately obtained using the method of multiple scales [7]. The dynamic behaviour of the system in the neighbourhood of the point of non-resonant codimension two bifurcations will then be studied based on a set of four averaged equations that determine the amplitudes and phases of the bifurcating periodic solutions.

It is assumed that the solutions of equation (7) in the neighbourhood of the trivial equilibrium are represented by an expansion of the form

\[ z_i(t;\epsilon) = \epsilon^{\frac{i}{2}}z_{i0}(T_0, T_1, \cdots) + \epsilon^{\frac{i}{2}}z_{i2}(T_0, T_1, \cdots) + \cdots, \quad (i = 1, 2, 3, 4). \tag{8} \]

where \( \epsilon \) is a non-dimensional small parameter, and the new multiple independent variables of time are introduced according to \( T_k = \epsilon^k t, \quad k = 0, 1, 2, \cdots \).

Substituting the approximate solutions (8) into equation (7) and then balancing the like powers of \( \epsilon \) results in the following ordered perturbation equations.
\[ e^\delta_1: \quad D_0z_{11} = \delta_1z_{21} + e_1\cos(\Omega T_0), \quad D_0z_{21} = -\delta_1z_{11} + e_2\cos(\Omega T_0), \quad (9,10) \]
\[ D_0z_{31} = \delta_2z_{41} + e_3\cos(\Omega T_0), \quad D_0z_{41} = -\delta_2z_{31} + e_4\cos(\Omega T_0), \quad (11,12) \]
\[ e^\delta_2: \quad D_0z_{12} = g_{11}(z_{11},z_{21},z_{31},z_{41}) + \delta_1z_{22} - D_1z_{11} + f_{11}(z_{11},z_{21},z_{31},z_{41}), \quad (13) \]
\[ D_0z_{22} = g_{21}(z_{11},z_{21},z_{31},z_{41}) - \delta_1z_{12} - D_1z_{21} + f_{21}(z_{11},z_{21},z_{31},z_{41}), \quad (14) \]
\[ D_0z_{32} = g_{31}(z_{11},z_{21},z_{31},z_{41}) + \delta_2z_{42} - D_1z_{31} + f_{31}(z_{11},z_{21},z_{31},z_{41}), \quad (15) \]
\[ D_0z_{42} = g_{41}(z_{11},z_{21},z_{31},z_{41}) - \delta_2z_{32} - D_1z_{41} + f_{41}(z_{11},z_{21},z_{31},z_{41}), \quad (16) \]

where \( D_0 = \partial / \partial T_0 \), \( D_1 = \partial / \partial T_1 \), the coefficients of the perturbation linear terms are given in reference [3], and the amplitudes of the excitations in equation (7) have been rescaled in terms of \( e_{10} = e^{\delta_1}e_1, \ e_{20} = e^{\delta_1}e_2, \ e_{30} = e^{\delta_1}e_3, \ e_{40} = e^{\delta_1}e_4 \).

By following the normal procedure of the method of multiple scales, the solutions to equations (9)-(12) can be written in general form as
\[ z_{11} = r_1\cos(\delta_1 T_0 + \phi_1) + A_1\cos(\Omega T_0) + A_2\sin(\Omega T_0), \quad (17) \]
\[ z_{21} = -r_1\sin(\delta_1 T_0 + \phi_1) + B_1\cos(\Omega T_0) + B_2\sin(\Omega T_0), \quad (18) \]
\[ z_{31} = r_2\cos(\delta_2 T_0 + \phi_2) + A_3\cos(\Omega T_0) + A_4\sin(\Omega T_0), \quad (19) \]
\[ z_{41} = -r_2\sin(\delta_2 T_0 + \phi_2) + B_3\cos(\Omega T_0) + B_4\sin(\Omega T_0), \quad (20) \]

where \( r_1, r_2, \phi_1, \phi_2 \) represent, respectively, the amplitudes and phases of the free-oscillation terms, and the coefficients \( A's \) and \( B's \) are not reproduced here.

Solutions (17)-(20) suggest that the non-resonant response of the system consists of the particular solution, which has the same frequency as the excitation, and a freeoscillation term with the frequencies of Hopf bifurcations, which may not decay with time. In seeking the second order solutions, it was found that the right hand sides of equations (13)-(16) may involve 44 terms of trigonometric functions. In addition to four terms that are proportional to \( \sin(\delta_1 t + \phi_1) \), \( \cos(\delta_1 t + \phi_1) \), \( \sin(\delta_2 t + \phi_2) \), and \( \cos(\delta_2 t + \phi_2) \), secular or nearly secular terms may occur whenever the system possesses primary, secondary, additive or difference resonances. In particular, the primary resonances may occur when \( \delta_1 \equiv \Omega \) or \( \delta_2 \equiv \Omega \). The sub-harmonic resonances take place when \( \delta_1 \equiv \frac{1}{3} \Omega \) or \( \delta_2 \equiv \frac{1}{3} \Omega \). In addition to the primary and super-harmonic as well as sub-harmonic resonances, it is noted that for the specific system with \( \delta_1 > \delta_2 \), the system may exhibit a number of combinations of additive and differences resonances when the Hopf bifurcation frequencies and the frequency of excitation satisfy certain relationships. In particular, the additive resonances may appear when either \( \delta_1 + \delta_2 \equiv 2\Omega \), or \( 2\delta_1 + \delta_2 \equiv \Omega \), or \( \delta_1 + 2\delta_2 \equiv \Omega \). The difference resonances can occur when either \( \delta_1 - \delta_2 \equiv 2\Omega \), or \( 2\delta_1 - \delta_2 \equiv \Omega \); or \( 2\delta_2 - \delta_1 \equiv \Omega \). As such, a total of 13 kinds of resonances may take place in the system. Thus in eliminating the terms that produce secular terms, six cases need to be distinguished based on the five types of resonances discussed above, which are (a) primary resonances; (b) sub-harmonic resonances; (c) super-harmonic resonances; (d) additive resonances; (e) difference resonances and; (f) non-resonances when \( \Omega \) is well separated from the above-mentioned resonances.
The amplitudes and phases of the first order approximate solutions of the free-oscillation terms, which are obtained by eliminating the possible secular terms which may appear in the solutions of \( z_{12} \) and \( z_{32} \), are determined by

\[
\begin{align*}
\dot{r}_1 &= -\mu_1 r_1 + s_{11} r_1^3 + s_{12} r_1^2 r_2^2, \\
\dot{r}_2 &= -\mu_2 r_2 + s_{21} r_2^3 r_2 + s_{22} r_2^3, \\
\dot{r}_1 \phi_1 &= r_1 \phi_1 + r_1 \rho_1 + s_{31} r_1^2 + s_{32} r_1^2 r_2^2, \\
\dot{r}_2 \phi_2 &= r_2 \phi_2 + r_2 \rho_2 + s_{41} r_2^2 + s_{42} r_2^3. \tag{21}
\end{align*}
\]

where, the coefficients are not reproduced here for brevity.

The dynamics of equation (21) is understood by finding the fixed points and studying their stability. The fixed points are obtained by setting \( \dot{r}_1 = \dot{r}_2 = 0 \) in the first two equations of system (21). It is easy to note that \((r_1, r_2) = (0,0)\) is always an equilibrium and that up to three other equilibria can appear as follows.

\[
(r_1, r_2) = \left( \frac{\mu_1}{s_{11}} , 0 \right), \text{ for } \frac{\mu_1}{s_{11}} > 0,
\]

\[
(r_1, r_2) = \left( 0, \sqrt{\frac{\mu_2}{s_{22}}} \right), \text{ for } \frac{\mu_2}{s_{22}} > 0,
\]

\[
(r_1, r_2) = \left( \frac{s_{12} \mu_2 - s_{22} \mu_1}{s_{12}^2 s_{21} - s_{11} s_{22}}, \frac{s_{21} \mu_1 - s_{11} \mu_2}{s_{12} s_{21}^2 - s_{11} s_{22}} \right), \text{ for } \frac{s_{12} \mu_2 - s_{22} \mu_1}{s_{12}^2 s_{21} - s_{11} s_{22}} > 0, \frac{s_{21} \mu_1 - s_{11} \mu_2}{s_{12} s_{21}^2 - s_{11} s_{22}} > 0. \tag{22}
\]

For simplicity, the above solutions will be referred to here as solutions S1, S2, S3 and S4, respectively. In particular, solution S1 indicates that the steady state non-resonant response of the system consists of the forced term only. Solutions S2 and S3 suggest that the steady state response of the system comprises both the free-oscillation term and forced term. The free-oscillation terms in solutions (17)-(20) do not decay to zero. For these two cases, the non-resonant response may be quasi-periodic if the frequencies resulting form the first or second Hopf bifurcations of the corresponding autonomous system and the forcing frequency are not commensurate. Solution S4 indicates that the non-resonant response is a quasi-periodic motion on a 3-D torus, which can be viewed as a motion by adding a third periodic motion to the 2-D quasi-periodic motion of S2 or S3. The stability of the solutions can be examined by studying the eigenvalues of the corresponding Jacobian matrix. It is easy to note from above discussion that the possible response of the controlled system with external excitation in the neighbourhood of non-resonant codimension two bifurcation of Hopf-Hopf interactions may be periodic motion or quasi-periodic motions on a 2-D or 3-D tori.

5 ILLUSTRATIVE EXAMPLES

For a specific system with the parameters given by \( q = -0.402189 \) and \( \Omega_0 = 0.316 \), equation (21) becomes

\[
\begin{align*}
\dot{r}_1 &= -(1.4878 \alpha_1 + 0.4543 \alpha_2 + 0.3989 e_0^3) r_1 + 0.2335 r_1^3 - 0.0478 r_1 r_2^2, \\
\dot{r}_2 &= -(0.4483 \alpha_1 + 1.3858 \alpha_2 + 0.6660 e_0^3) r_2 + 1.5127 r_1^2 r_2 - 0.6932 r_2^3, \\
r_1 \phi_1 &= (6.9959 + 0.3548 \alpha_1 - 1.9049 \alpha_2 + 1.8666 e_0^3) r_1 + 0.5545 r_1^3 + 1.0091 r_1 r_2^2, \\
r_2 \phi_2 &= r_2 \phi_2 + 0.5545 r_2^3 + 1.0091 r_1 r_2^2.
\end{align*}
\]

\]
\[ r_2 \dot{\alpha}_2 = (3.9112 + 1.9359\alpha_1 - 0.3209\alpha_2 + 1.7932e_0) r_2 + 1.7226r_1^2 r_2 + 0.5886r_2^3, \]  

(23)

here the third dummy parameter has been set that \( \alpha_3 = 0.0 \).

The steady state solutions of equation (23) can be easily found as follows:

Solution S1, the initial equilibrium solution, \( r_1 = r_2 = 0.0 \).  

(24)

Solution S2, the first Hopf bifurcation solution with frequency

\[ r_1 = 2.92672 \sqrt{-0.743886\alpha_1 - 0.22715\alpha_2 - 0.19944e^2}, \quad r_2 = 0, \]
\[ \omega_{\text{h1}} = 6.99593 - 3.17862\alpha_1 - 2.98387\alpha_2 + 0.919236e^2, \]  

(25)

Solution S3, the second Hopf bifurcation solution with frequency

\[ r_1 = 0, \quad r_2 = 1.69862 \sqrt{-0.224145\alpha_1 - 0.692885\alpha_2 - 0.330278e^2}, \]
\[ \omega_{\text{h2}} = 3.9112 + 1.55528\alpha_1 - 1.49753\alpha_2 + 1.23236e^2. \]  

(26)

Solution S4, the quasi-periodic solution on a 2-D torus with frequencies

\[ r_1 = 6.68465 \sqrt{-0.252459\alpha_1 - 0.0621609\alpha_2 - 0.0612249e^2}, \]
\[ r_2 = 1.49473 \sqrt{10.7289\alpha_1 + 1.81818\alpha_2 + 2.2456e^2}, \]
\[ \omega_{\text{h1}} = 6.99593 + 18.2884\alpha_1 + 0.65405\alpha_2 + 5.41238e^2, \]
\[ \omega_{\text{h2}} = 3.9112 - 3.38795\alpha_1 - 2.71464\alpha_2 + 0.0335653e^2. \]  

(27)

Figure 1. Bifurcation diagram in the perturbation parameters \( (\alpha_1, \alpha_2) \) plane for non-resonant response of the controlled system in the neighbourhood of non-resonant Hopf bifurcation of co-dimension two.

Careful check on the existence of solutions and their stability conditions indicates that the \( (\alpha_1, \alpha_2) \) parameter space can be divided into four regions. The boundaries of these regions are defined by four critical lines, namely, \( B_1, B_2, B_3 \) and \( B_4 \), which are described by

\[ B_1: \alpha_2 = -0.3235\alpha_1 - 0.4767e^2, \quad \alpha_1 > (-0.13598e^2), \]
$B_2: \alpha_2 = -3.2748\alpha_1 - 0.878e^2, \alpha_1 < (-0.13598e^2),$

$B_3: \alpha_2 = -4.0614\alpha_1 - 0.9849e^2, \alpha_1 > (-0.13598e^2),$

$B_4: \alpha_2 = -5.901\alpha_1 - 1.2351e^2, \alpha_1 > (-0.13598e^2).$ (28)

The critical bifurcation lines are illustrated in Figure 1. Stable solution S1 exists in the region between the lines $B_1$ and $B_2$. Crossing these two critical lines leads to solutions either S2 or S3, which bifurcates from the solution S1 via appearance of a zero eigenvalue. Stable solution S2 exists in the region between the lines $B_2$ and $B_4$, and stable solution S3 exists in the region between two lines $B_1$ and $B_3$. Along the critical line $B_3$, a second Hopf bifurcation solution with frequency $\omega_{h1}$ takes place from the solution S3, which leads to a 3-D torus described by equation (27). Similarly, a secondary Hopf bifurcation solution with frequency $\omega_{h2}$ occurs from the solution S2 along the critical line $B_4$, which also gives rise to a 3-D torus solution.

The analytical predictions can be easily validated by numerical results. Figure 2 shows the time histories and phase portraits of a periodic solution for $\alpha_1 = 0.001$, $\alpha_2 = 0.1$ and $e_0 = 0.06$, a 2-D torus solution for $\alpha_1 = 0.001$, $\alpha_2 = -0.0035$ and $e_0 = 0.04$, as well as a 3-D torus solution for $\alpha_1 = 0.001$, $\alpha_2 = 0.0053$, and $e_0 = 0.016$. It can be concluded that the analytical predictions of the amplitudes and frequencies are good representatives of the numerical results.
6 CONCLUSION

A controlled van der Pol-Duffing oscillator with time delay involved in the nonlinear feedback control has been studied in detail to explore a rich dynamic behaviour of the system in the vicinity of nonresonant Hopf bifurcations. The nonresonant response of the system may exhibit the periodic solutions, and quasi-periodic solutions on 2-D torus as well as on 3-D torus, depending on the dummy unfolding parameters and system parameters as well as excitation parameters. Numerical results have been given to illustrate an interaction of the excitation and the solutions that result from supercritical-supercritical Hopf bifurcations occurring in a specific system.

7 ACKNOWLEDGEMENTS

JC is grateful for the financial support from the ECMS faculty at the University of Adelaide through ECMS Category 1 Scheme.

8 REFERENCES