High frequency spatial vibration control using $\mathcal{H}_\infty$ method

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Abstract

A general feedback control approach has been developed to reduce the amplitude of high frequency structural vibration modes without spending any control effort on reducing the vibration level of low frequency modes. A procedure is outlined for developing a truncated structural model, which only includes the vibration modes of interest and takes account of the contributions of lower and higher order modes in the frequency band of interest. Procedures for calculating the lower and higher order mode contributions are outlined, and a better model for estimating the contributions from the lower order modes is developed by introducing a second order term. The new controller so obtained has a lower order than standard feedback controllers and is also able to concentrate all the control energy on the modes resonant in the frequency band of interest. The control approach is further developed by using spatial output control to achieve a global attenuation of the structure for the desired frequencies and to allow the distribution of control energy among the controlled frequencies in any desired way. The general theory developed in this paper is validated experimentally using a cantilevered beam as a representative structure and controlling the 4th to the 7th order vibration modes over a frequency range from 342Hz to 1125Hz. Piezoelectric patches are used as a sensor and actuator and the global attenuation is measured using a scanning laser vibrometer.

Key words: vibration control, flexible structures, $\mathcal{H}_\infty$ control, spatial output control, piezoelectric sensor/actuator

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1 Introduction

Many methods have been investigated by previous researchers to efficiently attenuate vibration from flexible structures. Classical control theory approaches suffered from stability problems caused by the inability to incorporate the spillover of unmodelled vibration modes in the controller design. Modern control theory provided tools that allow uncertainty problems that arise due to the truncation, or the inaccuracy, of the model to be dealt with more reliably. Modern control theory also provides a framework for reducing the influence of undesired perturbation and, as will be shown, it can be used to develop a controller design to concentrate the controller energy in a selected frequency band. Extending modern control theory using $\mathcal{H}_\infty$ robust control theory, the uncertainties between the theoretical model and the actual structure can be analyzed by means of numerical computation and experimentation, which lays the foundation of robust control design [1–4]. One of the issues with traditional vibration attenuation control design is that the controller has only a local effect and global structural vibration control requires a large number of actuators. Recently, with the introduction of the spatial norm concept [5], vibration reduction over an entire structure has been shown to be possible with just a few control actuators. Although the underlying theory describing this approach is well documented, the control approach mainly targets the first few structural resonances within a particular bandwidth. However, in some practical cases, it may be necessary to target resonances at higher frequencies such as for reducing the sound radiation from a structure.

The purpose of the current paper is twofold: first, to provide a procedure for the generalized robust design of a practical control system for high frequency, spatial vibration control; and second to apply the procedure to the spatial vibration attenuation of a cantilever beam, for frequencies between the 4th resonance frequency (342Hz) and the 7th resonance frequency (1125Hz) using one piezoelectric sensor and one piezoelectric actuator subjected to a point-wise, broadband perturbation. This example has the advantage of simplicity because of its uni-dimensionality while still being a complete application. The performance of the spatial control system is measured using a scanning laser vibrometer, which measures vibration velocities at discrete points according to a predefined structural mesh. Application to a specific case allows the illustration of each step in the process.

In the following sections, the first matter discussed is an adjustment to the truncated model in order to achieve a better approximation of the real system. This is followed by an explanation of how a $\mathcal{H}_\infty$ controller is designed for high frequency control with the use of the spatial concept and appropriate weight functions. The last section is dedicated to the validation of this theory through a cantilever beam vibration control experiment using a piezoelectric sensor and actuator.
2 Truncated model design for high frequency control

2.1 Modelling of flexible structure systems

The analytical model for axial, torsional and flexural vibration is derived using standard partial derivative equation methods, which can be found in [6,7]. For spatially distributed systems:

\[
\mathcal{L}\{y(t, r)\} + \mathcal{M}\left\{\frac{\partial y(t, r)}{\partial t}\right\} + \mathcal{C}\left\{\frac{\partial^2 y(t, r)}{\partial t^2}\right\} = f(t, r)
\]

(1)

where \(y\) is the structural displacement at the location \(r\) along the structure. \(\mathcal{L}\) and \(\mathcal{M}\) are linear homogeneous differential operators and \(\mathcal{C}\) is the damping operator. In this work it is assumed that the modes are not coupled through the damping and the damping is proportional, as is commonly used in modal theory. \(\mathcal{C}\) is then equal to \(c_1 \mathcal{L} + c_2 \mathcal{M}\), where \(c_1\) and \(c_2\) are non-negative constants. Finally, a general excitation force is denoted by \(f\).

In modal analysis, the solution for \(y(t, r)\) in (1) can be assumed to be in a separable form, consisting of contributions from an infinite number of modes:

\[
y(t, r) = \sum_{i=1}^{\infty} \phi_i(r)q_i(t)
\]

(2)

where \(q_i(t)\) is the temporal function of the system, \(\phi_i(r)\) is the structural eigenfunction obtained by solving the associated eigenvalue problem:

\[
\mathcal{L}\{\phi_i(r)\} = \lambda_i \mathcal{M}\{\phi_i(r)\}
\]

(3)

with \(\lambda_i\) related to the natural frequency of mode \(i\) (\(\lambda_i = \omega_i^2\)).

The eigenfunction mode shapes are orthogonal and are normalized via the following orthogonality conditions:

\[
\int_{\mathcal{R}} \phi_i(r)\mathcal{L}\{\phi_j(r)\}dr = \delta_{ij}\omega_i^2
\]

(4a)

\[
\int_{\mathcal{R}} \phi_i(r)\mathcal{M}\{\phi_j(r)\}dr = \delta_{ij}
\]

(4b)

\[
\int_{\mathcal{R}} \phi_i(r)\mathcal{C}\{\phi_j(r)\}dr = 2\delta_{ij}\zeta_i\omega_i
\]

(4c)

where \(\delta_{ij}\) is the Kronecker delta function, \(\omega_i\) the \(i^{th}\) natural frequency, \(\zeta_i\) the \(i^{th}\) damping ratio (\(\zeta_i = \frac{c_1\omega_i^2 + c_2}{2\omega_i}\)) and \(\mathcal{R}\) is the domain of the structure where \(r \in \mathcal{R}\).
Substituting (2) into (1), the following is obtained:

\[
\mathcal{L}\left\{\sum_{i=1}^{\infty} \phi_i(r)q_i(t)\right\} + \mathcal{M}\left\{\frac{\partial}{\partial t} \sum_{i=1}^{\infty} \phi_i(r)q_i(t)\right\} + \mathcal{C}\left\{\frac{\partial^2}{\partial t^2} \sum_{i=1}^{\infty} \phi_i(r)q_i(t)\right\} = f(t, r).
\]

(5)

Multiplying (5) by \(\phi_j(r)\), integrating over its domain \((\mathcal{R})\) and using the orthogonality conditions (4a),(4b) and (4c), then following is obtained:

\[
\ddot{q}_i(t) + 2\zeta_i\omega_i\dot{q}_i(t) + \omega_i^2q_i(t) = F_i(t), \quad i = 1, 2, \ldots \text{ and } F_i = \int_{\mathcal{R}} \phi_i(r)f(t, r)dr.
\]

(6)

The transfer function between the applied force \(f(t)\) and the displacement \(y(t, r)\) can then be expressed by taking the Laplace transform of (6):

\[
G(s, r) = \sum_{i=1}^{\infty} \frac{\phi_i(r)F_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}
\]

(7)

with \(F_i\) the modal amplitude of the applied force.

The case considered here is the flexural vibration attenuation of a cantilever Bernoulli-Euler beam. Hence \(\mathcal{L} = \frac{\partial^2}{\partial r^2} \left(EI(r)\frac{\partial^2}{\partial r^2}\right)\) and \(\mathcal{M} = \rho A\) with \(I(r)\) the inertial moment, \(E\) the Young’s Modulus of the beam, \(\rho\) the beam density and \(A\) the beam cross sectional area.

The piezoelectric actuator/sensor model can be deduced using the same approach. As previously mentioned, the chosen example system is a cantilever beam; hence the piezoelectric actuator/sensor model is derived based on Kirchhoff’s hypothesis of flat isotropic plate theory. According to [8], the modal force contribution by piezoelectric actuators can be written as:

\[
\ddot{q}_i(t) + 2\zeta_i\omega_i\dot{q}_i(t) + \omega_i^2q_i(t) = \sum_{j=1}^{N_a} \kappa_j \Psi_{ij} v_{aj}(t), \quad i = 1, 2, \ldots
\]

(8)

where \(v_{aj}\) is the voltage applied to the \(j^{th}\) actuator, \(N_a\) is the number of actuators used and if all piezoelectric actuators are considered to be identical:

\[
\kappa_j = \kappa = \frac{Ed_{31}h^3w_a}{12h_a} \left(\frac{12E_a h_a(h_a + h)}{2Eh^3 + E_a((h + 2h_a)^3 - h^3)}\right)
\]

with \(E_a\) the piezoelectric Young Modulus, \(d_{31}\) the charge constant, \(h\) beam height, \(h_a\) piezoelectric height and \(w_a\) piezoelectric width.

\[
\Psi_{ij} = \frac{\partial \phi_i(r_{ej})}{\partial r} - \frac{\partial \phi_i(r_{b})}{\partial r}
\]

(9)
and \( r_b \) the location of one end of the piezoelectric patch, \( r_e \) the the location of the other end, the subscript \( i \) indicates the mode and the subscript \( j \) indicates the actuator. The Multi-Input, Multi-Output (MIMO) transfer function from the actuator voltage \( v_a = [v_{a1} \cdots v_{aN_a}]^T \) to the beam deflection \( y(s) \) at location \( r \) is:

\[
y(s, r) = \sum_{i=1}^{\infty} \frac{\phi_i(r)}{s^2 + 2\zeta_i\omega_is + \omega_i^2} \sum_{j=1}^{N_a} \kappa_{ij} \psi_{ij}(s) \equiv G_a(s, r) = \sum_{i=1}^{\infty} \frac{\phi_i(r) P_i}{s^2 + 2\zeta_i\omega_is + \omega_i^2}
\]

(10)

where \( P_i = \frac{1}{\rho A} [\kappa_i \psi_{i1} \cdots \kappa_{Ni} \psi_{iN_i}] \).

Following the procedure described in [9], the sensor voltage \( v_s \) can be related to the beam deflection \( y(s, r) \) as:

\[
v_{sk}(t) = \Omega_k \sum_{i=1}^{\infty} \Psi_{ik} q_i(t)
\]

(11)

with \( \Omega_k = \frac{k_{31}^2 w_s}{C_k g_{31}} \), \( k_{31} \) the electromechanical coupling factor, \( w_s \) the piezoelectric patch width, \( C_k \) the capacitance of the \( k^{th} \) sensor, \( g_{31} \) voltage constant and \( h_s \) piezoelectric actuator thickness.

The Bernoulli-Euler beam equation becomes:

\[
v_{sk}(s) \sum_{i=1}^{\infty} \left( s^2 + 2\zeta_i\omega_is + \omega_i^2 \right) = \Omega_k \sum_{i=1}^{\infty} \Psi_{ik} f(t) \phi_i(r_f)
\]

\[
\equiv \frac{v_{sk}(s)}{f(s)} = \Omega_k \sum_{i=1}^{\infty} \frac{\Psi_{ik} \phi_i(r_f)}{s^2 + 2\zeta_i\omega_is + \omega_i^2}
\]

\[
\equiv G_s(s, r) = \sum_{i=1}^{\infty} \frac{\Upsilon_i \phi_i(r_f)}{s^2 + 2\zeta_i\omega_is + \omega_i^2}
\]

(12)

where \( \Upsilon_i = [\Omega_i \psi_{i1} \cdots \Omega_i \psi_{iN_i}]^T \), \( N_i \) the number of sensors and \( v_s = [v_{s1} \cdots v_{sN_s}]^T \) for the case of a MIMO system. Manipulating (10) and (11), we can obtain the transfer function between the \( j^{th} \) actuator and the \( k^{th} \) sensor.

\[
G_{sk,a_j}(s) = \frac{v_{sk}(s)}{v_{aj}(s)} = \Omega_k K_j \sum_{i=1}^{\infty} \frac{\Psi_{ik} \Psi_{ij}}{s^2 + 2\zeta_i\omega_is + \omega_i^2}
\]

\[
\equiv G_{sa}(s, r) = \sum_{i=1}^{\infty} \frac{\Upsilon_i P_i}{s^2 + 2\zeta_i\omega_is + \omega_i^2}
\]

(13)
2.2 The optimal truncated model

For high frequency mode control, a particular frequency band is usually of interest, but in standard controller design approaches, all lower frequency modes are normally included, even if their resonance frequencies are below the region of interest. This may make the control design complicated since the states contributed by the lower frequency modes also need to be included, resulting to a controller that unnecessarily attempts those lower frequency modes. In this work, however, a control approach will be presented that will minimize the unnecessary control effort being spent on those modes, producing a lower order controller.

As the model has to be truncated, the reduced model, \( G_r(s) \), and the way in which it is reduced has a direct consequence on the controller performance. In order to maintain high controller performance, the residual dynamic, \( G_d(s) \), due to the higher frequency modes has to be considered and taken into account in the model. This was done by [10] who shows the poles and zeros localization alteration due to the model reduction. However in the analysis of [10], all the lower modes are included. As mentioned previously, when the focus is only on a specific frequency band, the modes above and below the band need to be truncated in order to ensure maximum control efficiency in the band of interest. However, in the truncated model, it is necessary to account for the lower order modes (as well as the higher order modes) by using the low frequency residual dynamic, \( G_l(s) \), in addition to the high frequency dynamic, \( G_d(s) \), to account for the altered poles and zeros as a result of the truncation. Consider a general transfer function, similar to the one shown in (7). Suppose one is interested in controlling broadband vibration between the the frequencies of the \( m_1^{th} \) and \( m_2^{th} \) vibration modes.

\[
G(s) = \sum_{i=1}^{\infty} \frac{F_i}{s^2 + 2\zeta_i\omega_is + \omega_i^2} = \sum_{i=1}^{m_1-1} \frac{F_i}{s^2 + 2\zeta_i\omega_is + \omega_i^2} + \sum_{i=m_1}^{m_2} \frac{F_i}{s^2 + 2\zeta_i\omega_is + \omega_i^2} + \sum_{i=m_2+1}^{\infty} \frac{F_i}{s^2 + 2\zeta_i\omega_is + \omega_i^2} = G_l(s) + G_r(s) + G_d(s)
\]

(14)

where \( m_1 \) is the mode number of the first vibration mode of interest, \( m_2 \) the last one and \( F_i \) matrix of the external forces.

Using a similar approach to that used in modal analysis [11] (Fig. 1), it can be seen that the contribution of the modes with resonance frequencies above and below the frequency band of interest can be approximated by a relatively simple function. Hence, the full model, \( G(s) \) can be approximated by \( G(s) \), which is a function containing \( G_r(s) \) (including the modes in the frequency band of interest) plus a zero order parameter \( K_d \), representing the contribution
Fig. 1. Mode contributions to energy inside and outside the frequency band of interest: a) low frequencies, b) frequencies of interest and c) high frequencies, from [11].

of the higher frequency modes to the frequency band of interest, and second order parameter $K_l/\omega^2$, representing the contribution of the lower frequency modes ($s = j\omega$) to the frequency band of interest:

$$\tilde{G}(\omega) = \frac{K_l}{\omega^2} + G_r(\omega) + K_d$$  \hspace{1cm} (15)

The idea is to try to evaluate the optimal $K_d$ and $K_l$ by minimizing the $H_2$ norm of the following cost function $J$:

$$J = \|W(\omega) \left( G(\omega) - \tilde{G}(\omega) \right) \|^2_2$$  \hspace{1cm} (16)

where $W(\omega)$ is a perfect band pass filter that has a unit value in $[-\omega_c, -\omega_a]$ and $[\omega_a, \omega_c]$ where $\omega_c = \frac{\omega_{m_1}^{m_2} + \omega_{m_1}^{m_1-1}}{2}$ and $\omega_a = \frac{\omega_{m_1}^{m_1} + \omega_{m_{1-1}}^{m_1}}{2}$.

The optimum values of $K_d$ and $K_l$ can be found by differentiating $J$ with respect to $K_d$ and $K_l$. Since the damping is usually small for flexible structure
system, the following derivation assumes $\zeta_i \to 0$.

$$J = \frac{1}{2\pi} \int_{-\omega_n}^{\omega_n} \text{tr} \left\{ \left( W(\omega) (G(\omega) - \tilde{G}(\omega)) \right)^* \left( W(\omega) (G(\omega) - \tilde{G}(\omega)) \right) \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_n}^{\omega_n} \text{tr} \left\{ \left( \sum_{i=1}^{m_1-1} \frac{F_i}{\omega_i^2 - \omega^2} + \sum_{i=m_2+1}^{\infty} \frac{F_i}{\omega_i^2 - \omega^2} - K_d - \frac{K_l}{\omega^2} \right)^* \left( \sum_{i=1}^{m_1-1} \frac{F_i}{\omega_i^2 - \omega^2} + \sum_{i=m_2+1}^{\infty} \frac{F_i}{\omega_i^2 - \omega^2} - K_d - \frac{K_l}{\omega^2} \right) \right\} d\omega$$

$$+ \frac{1}{2\pi} \int_{\omega_c}^{\infty} \text{tr} \left\{ \left( \sum_{i=1}^{m_1-1} \frac{F_i}{\omega_i^2 - \omega^2} + \sum_{i=m_2+1}^{\infty} \frac{F_i}{\omega_i^2 - \omega^2} - K_d - \frac{K_l}{\omega^2} \right) \right\} d\omega$$

$$= \frac{1}{\pi} \int_{\omega_n}^{\omega_c} \text{tr} \left\{ \sum_{i=1}^{\infty} \frac{F_i^*}{\omega_i^2 - \omega^2} \sum_{i \in [m_1,m_2]} \frac{F_i}{\omega_i^2 - \omega^2} \right\} - 2 \text{Re} \left\{ \sum_{i \in [m_1,m_2]} \frac{F_i^*}{\omega_i^2 - \omega^2} K_d \right\} + \text{tr} \left\{ K_d^T K_d \right\}$$

$$- 2 \text{Re} \left\{ \sum_{i \in [m_1,m_2]} \frac{F_i^*}{\omega_i^2 - \omega^2} K_l \right\} + \text{tr} \left\{ K_l^T K_l \right\} + 2 \text{tr} \left\{ \frac{K_l^T K_l}{\omega^2} \right\} d\omega$$

(17)

where $\text{tr}\{F\}$ represents the trace of a matrix $F$.

Differentiating $J$ with respect to $K_l$ and setting it to zero:

$$\frac{\partial J}{\partial K_l} = \frac{2}{\pi} \int_{\omega_n}^{\omega_c} \left\{ \frac{K_l}{\omega^4} + \frac{K_d}{\omega^2} - \frac{1}{\omega^2} \sum_{i \in [m_1,m_2]} \frac{F_i}{\omega_i^2 - \omega^2} \right\} d\omega = 0. \quad (18)$$

The first term of the above equation can be expressed as follows:

$$\int_{\omega_n}^{\omega_c} \frac{K_l}{\omega^4} d\omega = \left( \frac{1}{3} \frac{\omega_c^3 - \omega_n^3}{\omega_c^3 \omega_n^3} \right) K_l = \beta K_l. \quad (19)$$

The second term is expressed as:
\[ \int_{\omega_a}^{\omega_c} \frac{-K_d}{\omega^2} d\omega = - \left( \frac{\omega_c - \omega_a}{\omega_a \omega_c} \right) K_d = -\varpi K_d. \quad (20) \]

Similarly for the third term where \[ \sum_{i=1}^{\infty} \sum_{\substack{i \notin [m_1, m_2]}} = \sum_{\varpi} \] is used to simplify the notation:

\[ \int_{\omega_a}^{\omega_c} \frac{1}{\omega^2} \sum_{\varpi} \frac{F_i}{\omega_i^2 - \omega^2} d\omega = \sum_{\varpi} \frac{F_i}{\omega_i^2} \left( \frac{1}{\omega_i^2 - \omega^2} + \frac{1}{\omega^2} \right) d\omega \]
\[ = \sum_{\varpi} \frac{F_i}{\omega_i^2} \left[ \frac{\omega_c - \omega_a}{\omega_i \omega_a} + \frac{1}{\omega_i^2} \ln \left\{ \frac{(\omega_c + \omega_i)(\omega_a - \omega_i)}{|\omega_c - \omega_i|} \right\} \right] \]
\[ = \sum_{\varpi} \frac{\chi_i}{\omega_i^2} + \varpi \sum_{\varpi} \frac{F_i}{\omega_i^2} \quad (21) \]

with \[ \chi_i = \frac{F_i}{\omega_i^2} \ln \left\{ \frac{(\omega_c + \omega_i)(\omega_a - \omega_i)}{|\omega_c - \omega_i|} \right\} \]. Thus, the optimal \( K_t \) can be obtained using (19), (20) and (21):

\[ K_t = \frac{1}{\beta} \left( \sum_{\varpi} \frac{\chi_i}{\omega_i^2} + \varpi \sum_{\varpi} \frac{F_i}{\omega_i^2} - \varpi K_d \right) = \frac{1}{\beta} (\Gamma_i - \varpi K_d) \quad (22) \]

with \[ \Gamma_i = \sum_{\varpi} \frac{\chi_i}{\omega_i^2} + \varpi \sum_{\varpi} \frac{F_i}{\omega_i^2}. \]

Using a similar approach, the optimal \( K_d \) is found by differentiating \( J \) in (17) with respect to \( K_d \). Substituting (22) into (17) gives:

\[ \frac{\partial J}{\partial K_d} = \frac{2}{\varpi} \int_{\omega_a}^{\omega_c} \left\{ - \sum_{\varpi} \frac{F_i}{\omega_i^2 - \omega^2} + \frac{K_t}{\omega^2} + K_d \right\} d\omega \]
\[ = \int_{\omega_a}^{\omega_c} \left\{ K_d + \frac{1}{\omega^2} \left[ \frac{1}{\beta} (\Gamma_i - \varpi K_d) \right] - \sum_{\varpi} \frac{F_i}{\omega_i^2 - \omega^2} \right\} d\omega \]
\[ = \int_{\omega_a}^{\omega_c} \left\{ K_d \left(1 - \frac{\varpi}{\beta \omega^2} \right) + \frac{\Gamma_i}{\beta \omega^2} - \sum_{\varpi} \frac{F_i}{\omega_i^2 - \omega^2} \right\} d\omega = 0. \quad (23) \]

The first term of (23) is equal to:

\[ \int_{\omega_a}^{\omega_c} K_d \left(1 - \frac{\varpi}{\beta \omega^2} \right) d\omega = \left( \omega_c - \omega_a - \frac{\varpi^2}{\beta} \right) K_d = \gamma K_d, \quad (24) \]
while the second term is:

\[ \int_{\omega_a}^{\omega_c} \frac{\Gamma_i}{\beta \omega^2} d\omega = \frac{\omega_c - \omega_a}{\beta} \Gamma_i, \]

and the third term is:

\[ -\int_{\omega_a}^{\omega_c} \sum_{\notin \mathcal{F}_i} \frac{F_i}{\omega_i^2 - \omega^2} d\omega = -\sum_{\notin \mathcal{F}_i} \chi_i. \]

Substituting (24), (25), (26) into (23), the optimal \( K_d \) is found to be

\[ K_d = \frac{1}{\gamma} \left( \sum_{\notin \mathcal{F}_i} \chi_i - \frac{\omega_c - \omega_a}{\beta} \Gamma_i \right). \]

A particular example of a cantilevered beam whose properties will be described in section 4 is now considered. The significance of this correction method can be seen from the frequency response between the beam’s tip displacement \( y(\omega) \) and the applied point force \( f(\omega) \), as shown in Fig 2. Fig. 3 shows the effect of the model reduction without any correction term, \( G_r(s) \) for the following cases: when the only term taken into account in the objective function \( J \) is the zeroth order term of \( K_d \) (in this case \( \gamma = \omega_c - \omega_a \) and \( \Gamma_i = 0 \)), \( G_r(s) + K_{do} \); when both terms \( K_l \) and \( K_d \) are taken into account; and when the full model using the first 30 modes, \( G(s) \), is used. As mentioned in the introduction, the bandwidth of interest [330Hz to 1150Hz] lies between the two vertical thick black lines in Fig. 3. The frequency response in this range is expanded for clarity and shown in Fig. 4. The effect on the optimization of using just the two terms described above can also be seen by comparing the zeros (or anti-resonance frequencies) of the full system with the truncated ones (knowing that the poles remain identical) as shown in Fig. 5:
Fig. 3. Frequency response $\frac{y(\omega)}{f(\omega)}$ due to model reduction and corrections $G(s) =$ the full model using 30 modes, $G_r(s) =$ the truncated model without any correction term, $G_r(s) + K_{do} =$ the truncated model with the optimal zero order term of $K_d$ and $G_r(s) + K_d + K_l =$ the truncated model with both optimal terms $K_l$ and $K_d$.

Fig. 5. Location of zeros for various models.
For the zeros within the frequency band of interest, the reduced model (square symbols) with $K_l$ and $K_d$ is the closest to the full model $G(s)$ (diamond symbols). The next step is to improve the traditional state space model by incorporating the additional terms to account for the effect of modes outside the frequency band of interest. First the corrected truncated model of a flexible structure using a standard method with a zeroth order/feedthrough term is considered. Using $x(s) = [q(s) \dot{q}(s)]^T$ as the state variables with $q(s) = [q_1(s) \cdots q_N(s)]^T$, the transfer functions (10), (12), (13) and the correcting coefficients (see Appendix A) which are derived on the basis of work described in [5], the following expressions are obtained:

$$\dot{x}(s) = Ax(s) + B_1(r)f(s) + B_2v_a(s)$$  \hspace{1cm} (28a)

$$y(s, r) = C_1(r)x(s) + D_{11}(r)f(s) + D_{12}(r)v_a(s)$$  \hspace{1cm} (28b)

$$v_s(s) = C_2x(s) + D_{21}f(s) + D_{22}v_a(s)$$  \hspace{1cm} (28c)

where $A$, $B_1$, $B_2$, $C_1$ and $C_2$ are described in Appendix B and $D_{11}$, $D_{12}$, $D_{11}$ and $D_{22}$ in Appendix A. Here, $v_a(s)$ as mentioned before is the control actuator voltage, $f(s)$ is the disturbance force, $y(s, r)$ is the displacement at the location $r$ and $v_s(s)$ is the sensor voltage.

Now, consider correcting the truncated model even further by adding a second
order term as described in (15). The new state space model is based on the previous model with the additional second order terms:

\[
\begin{align*}
\dot{x}_c(s) &= A_c x_c(s) + B_{1c}(r) f(s) + B_{2a} v_a(s) \\
y(s, r) &= C_{1c}(r) x_c(s) + D_{11c}(r) f(s) + D_{12c}(r) v_a(s) \\
v_a(s) &= C_{2a} x_c(s) + D_{21c} f(s) + D_{22a} v_a(s)
\end{align*}
\]

(29a) (29b) (29c)

\[
\begin{align*}
A_c &= \begin{bmatrix} A & 0_{2(N+N_f+N_a)} \\
0_{2(N_f+N_a)} & A_1 \end{bmatrix} \\
A_1 &= \begin{bmatrix} B_1 & 0_{(N_f+N_a) \times N_f} & I_{[N_f \times N_f]} & 0_{[N_a \times N_a]} \\
0_{(N_f+N_a) \times (N_f+N_a)} & I_{[N_f \times N_f]} & 0_{[N_a \times N_a]} \end{bmatrix} \\
B_{1c} &= \begin{bmatrix} B_1 & 0_{(N_f+N_a) \times N_f} & I_{[N_f \times N_f]} & 0_{[N_a \times N_a]} \\
0_{(N_f+N_a) \times (N_f+N_a)} & I_{[N_f \times N_f]} & 0_{[N_a \times N_a]} \end{bmatrix} \\
B_{2c} &= \begin{bmatrix} B_2 & 0_{(N_f+N_a) \times N_a} & I_{[N_f \times N_a]} & 0_{[N_a \times N_a]} \\
0_{(N_f+N_a) \times (N_f+N_a)} & I_{[N_f \times N_a]} & 0_{[N_a \times N_a]} \end{bmatrix}
\end{align*}
\]

(30a) (30b) (30c) (30d)

where \( F_{[r \times c]} \) denotes the number of rows \( r \), and columns \( c \), of matrix \( F \). The correction terms, \( K_i \) and \( K_d \), are calculated for each transfer function considering the combination of any pair of the sensor, actuator or disturbance force. The remaining matrices in (29) are:

\[
\begin{align*}
C_{1c} &= \begin{bmatrix} C_1 & K_{yf} & K_{ys} & 0_{[N_y \times (N_f+N_a)]} \end{bmatrix} \\
C_{2c} &= \begin{bmatrix} C_2 & K_{yf} & K_{ys} & 0_{[N_y \times (N_f+N_a)]} \end{bmatrix} \\
D_{11} &= K_{fy} \\
D_{12} &= K_{fy} \\
D_{21} &= K_{fs} \\
D_{22} &= K_{fas}
\end{align*}
\]

(31a) (31b) (31c) (31d) (31e) (31f)

Subscripts \( a, f, s \) and \( y \) denote the terms associated with the control actuator,
the disturbance force, the sensor and the displacement respectively. \( N \) is the number of vibration modes taken into account \( (N = m_2 - m_1 + 1) \) and \( N_f \) is the number of point-wise disturbance sources.

For the design of optimal \( \mathcal{H}_2 \) or \( \mathcal{H}_\infty \) controllers, it is discussed in [12] that the following associated matrix \( H \) described in (32) has to have a full column rank:

\[
H = \begin{bmatrix}
A_c - j\omega I & B_{2c} \\
C_{1c} & D_{12}
\end{bmatrix}.
\]  (32)

However since \( A_c \) in (29a) depends on \( A_l \) (30b), the associated matrix \( H \) will not have full column rank. One alternative to solve this problem is to use the following adjustment. The truncated model with this adjustment remains the best approximation compared to the previous approach that uses only a zero order term.

\[
A_c = \begin{bmatrix}
A & O_{[2N \times 2(N_f + N_a)]} \\
O_{[2(N_f + N_a) \times 2N]} & A_{adj}
\end{bmatrix}
\]  (33)

where

\[
A_{adj} = \begin{bmatrix}
O_{[N_f + N_a] \times (N_f + N_a)} & I_{[(N_f + N_a) \times (N_f + N_a)]} \\
-\omega_1^2 I_{[(N_f + N_a) \times (N_f + N_a)]} & -2\xi_1\omega_1 I_{[(N_f + N_a) \times (N_f + N_a)]}
\end{bmatrix}.
\]  (34)

This adjustment comes from (6) for \( i = 1 \) and \( F_i(t) = 0 \) which is the contribution of the first mode to the truncated model. The transfer function \( \frac{1}{s^2 + 2\xi_1\omega_1 s + \omega_1^2} \) multiplied by \( K_l \) gives an approximate contribution of all the lowest modes to the truncated model, based on the first mode transfer function.

3 High frequency \( \mathcal{H}_\infty \) control design

Having obtained the reduced order model of the system to be controlled, the next step is to design a controller to achieve global structural vibration control within the prescribed bandwidth. To obtain vibration control across the entire structure, a spatial \( \mathcal{H}_\infty \) control concept [5,13] is used in this work. The control design is the same as any other \( \mathcal{H}_\infty \) controller [1], as shown in Fig. 6, but the weight functions are changed according to the frequency band of interest.
3.1 Spatial $\mathcal{H}_\infty$ control

Here, the spatial control concept is extended for application to a specific high frequency band. The objective of spatial $\mathcal{H}_\infty$ output control is to obtain a controller that minimizes the following cost function [5]:

$$ J_\infty = \int_0^\infty \int_\mathcal{R} y(t,r)^T Q(r) y(t,r) dr dt $$

$$ J_\infty = \int_0^\infty \int_\mathcal{R} f(t)^T f(t) dt $$

where $\mathcal{R}$ is the structural domain of interest which in this case is the length of the beam $L$ and $Q(r)$ is the spatial weighting function describing the structural region of interest.

The cost function thus represents the spatial output energy over the entire structure. It can be shown that spatial $\mathcal{H}_\infty$ output control requires the change described by (29b), which makes the controller independent of the localization of the displacement, $r$:

$$ \tilde{y}(s) = \Pi_c x_c(s) + \Theta_{11c} f(s) + \Theta_{12c} v_a(s) $$

with matrices $\Pi_c$, $\Theta_{11c}$ and $\Theta_{12c}$ determined from $\Gamma = [\Pi_c \Theta_{11c} \Theta_{12c}]$ such that:

$$ \Gamma^T \Gamma = \int_\mathcal{R} \begin{bmatrix} C_{11c}^T(r) \\ D_{11}^T(r) \\ D_{12}^T(r) \end{bmatrix} Q(r) \begin{bmatrix} C_{1c}(r) D_{11}(r) D_{12}(r) \end{bmatrix} dr. $$

In this work, the aim is to achieve global vibration attenuation of the entire structure, so a uniform spatial weighting $Q(r) = 1$ is used. Hence, using $C_{1c}(r)$, $D_{11}(r)$ and $D_{12}(r)$ described earlier and using orthogonality property for the beam:

$$ \int_0^L \phi_i(r) \phi_j(r) dr = \Phi^2 \delta_{ij} $$

with $\phi_i(r)$ is the $i^{th}$ eigenfunction of the structure, the following is obtained:

$$ \Pi_c = \begin{bmatrix} \Pi & 0_{[2N \times (N_f + Na)]} & 0_{[2N \times (N_f + Na)]} \\ 0_{[(N_f + Na) \times 2N]} & \Xi & 0_{[(N_f + Na) \times (N_f + Na)]} \\ 0_{[(N_f + Na) \times 2N]} & 0_{[(N_f + Na) \times (N_f + Na)]} & 0_{[(N_f + Na) \times (N_f + Na)]} \end{bmatrix}, $$

15
\[ \Pi = \text{diag}(\Phi_1, \ldots, \Phi_N, 0_{1 \times N}) \] 
\[ \Xi^T \Xi = \begin{bmatrix} K_{dy}^T K_{dy} & K_{dy}^T K_{day} \\ K_{day}^T K_{dy} & K_{day}^T K_{day} \end{bmatrix} \] (39)

and

\[ \Theta_{11c} = \begin{bmatrix} \Theta_{(2N_f+N_a+N_f) \times N_f} \\ \Theta_f \\ 0_{[N_a \times N_f]} \end{bmatrix} \] (40)

with

\[ \Theta_f = \frac{1}{\gamma} \sum_{i \notin F} F_i^2 \left[ \frac{1}{\omega_i^2} \ln \left( \frac{(\omega_c + \omega_i)(\omega_a - \omega_i)}{[\omega_c - \omega_i](\omega_a + \omega_i)} \right) \left( 1 - \frac{\omega}{\beta \omega_i^2} \right) - \frac{\omega^2}{\beta \omega_i^2} \right]^2 \] (41)

and here

\[ F_i^2_{[N_f \times N_f]} = \Phi_i^2 \phi_i(r_{f_f}) \ldots \phi_i(r_{f_{N_f}}) [\phi_i(r_{f_f}) \ldots \phi_i(r_{f_{N_f}})]^T \]

In this work, \( N_f = 1 \) since only a single point-wise disturbance source is considered.

\[ \Theta_{12c} = \begin{bmatrix} \Theta_{(2N_f+N_a+N_f) \times N_f} \\ \Theta_{[N_f \times N_a]} \\ \Theta_a \end{bmatrix} \] (41)

where \( \Theta_a \) is the same expression as \( \Theta_f \) with the following substitution:

\[ F_i^2_{[N_a \times N_a]} = \Phi_i^2 P_i^T P_i. \]

Hence spatial control can be performed for a specific bandwidth, but it should be noted that matrix \( \Xi \) is not easy to find, even if it is symmetrical. However, it will be shown later that \( \Xi \), \( \Theta_{12c} \) and \( \Theta_{11c} \) may not be necessary to be included in control design.

The first advantage of the proposed approach is that the order of the plant used (\( 2(N_f + N_a + N) \) states) is lower than that of the one commonly used in control design (\( 2m_2 \) states). For example, if one wants to control the vibration of the 5\(^{th}\) and 6\(^{th}\) modes due to a single disturbance input using a single actuator, then \( N_f = 1, N = 2, N_a = 1 \) and \( m_2 = 6 \). Thus, the standard model will require 12 states, while the proposed model only needs 8 states. Since the order of the controller will consist of the sum of the order of the plant and the order of the weighting function used, the proposed method will generate a lower order controller. The second advantage of the proposed approach is that this controller allows control of specific modes within a bandwidth without spending control effort on the lower order modes which are not of interest, something that can not be achieved with the standard approach because the lower order modes are included in the controller design. Even though an or-
der reduction procedure can be performed on the higher order controller, the stability of the closed loop system may be negatively affected.

3.2 Weight functions

It can be seen that the first $N$ terms of the output $\tilde{y}$ in (36) represent the modal contribution of each mode. In the application discussed here, weight coefficients (as illustrated in Fig. 6) are placed on these outputs in order to distribute the controller energy in the most desirable way. This a direct benefit of using the spatial output control approach. The other outputs from the $(N + 1)^{th}$ term to the $(2N + N_f)^{th}$ term of the output $\tilde{y}$ are the the residual outputs, whose contribution to the overall control design is not as significant as that of the previous outputs (ie. the first $N$ terms of $\tilde{y}$). This is why in this control design, $Ξ$, $Θ_{11c}$ and $Θ_{12c}$ can be ignored without sacrificing the control performance. The model in use is derived on the basis of a deliberate truncation of the system, which results in the exclusion of some dynamic effects, which can be represented by uncertainty functions. In the case considered here, the unmodelled dynamic is represented in the form of additional uncertainty, which can be expressed as $G(s) ≃ G_r(s) + Δ_a W_a(s)$, $G_r(s)$ the truncated plant, $||Δ_a||_∞ ≤ 1$, $G(s)$ the full model and $W_a(s) = \text{diag}(W_{a1}, ..., W_{aN_a})$.

For a MIMO system, the objective of $W_{ai}(s)$ is to encompass the unmodelled dynamics $E_{ai}(s) = G_i(s) - G_r_i(s)$ between the $i^{th}$ actuator and the outputs due to the truncation as shown in Fig. 7. The largest singular value of $E_{ai}(s)$ is measured to obtain the shape of $W_{ai}(s)$, $|W_{ai}(ω)| ≥ \hat{σ}(E_{ai}(ω))$. In our case, as $N_a = 1$ and $N_s = 1$, $W_a(s) = W_{a1}(s) = W_a(s)$ and $E_a(s) = E_{a1}(s)$.
Fig. 7. Weight functions: (1) $W_a$ weight function encompassing the spillover, (2) *Elliptic* first step of the passband filter construction, (3) $c_{\text{bottom}}$ constant to flatten the ripples, (4) $\text{Elliptic} + c_{\text{bottom}}$ minimum-phase passband filter, (5) $\bar{\sigma}(E_a(s))$, (6) $H_\infty$ norm of the error and (7) $W_p$ performance weight function equal at $c_p(\text{Elliptic} + c_{\text{bottom}})$

The unmodelled uncertainty function $W_a(s)$ due to truncation of the model is introduced in order to suppress the spillover. As the modes of interest are from the 4th to the 7th (between 342 Hz and 1125 Hz), $W_a(s)$ describes a stopband filter encompassing the lowest and highest order modes of interest. In this work, stopband filter is obtained by utilizing the Matlab signal processing toolbox which offers various filter design such as Butterworth, Chebyshev or elliptic (Cauer) filter design, the last one has been used to shape the weight function. The approach which consist of having directly the stopband filter from the Matlab functions is not used here because the stopband filter obtained is not very adjustable in term of magnitude of the gain over the stopband. So an alternative approach was used which involved generating a minimum phase passband filter which was smoothed and then inverted to produce a stopband filter.

The following describes the method used to obtain the stopband filter in this work. The order of the filter defines the sharpness of the passband and a 4th order filter has been chosen as a good compromise between the order of the filter and the required filter shape to encompass the error shape. The passband
corner frequencies are set at the 4th and the 7th resonance frequencies of 342Hz and 1125Hz respectively. The stopband corner frequencies are set at 332Hz and 1325Hz. Here, a 1dB ripple in the passband and a stopband at 70dB lower than the peak value in the passband have been chosen for this filter design, see curve Elliptic in Fig. 7. As the passband filter so obtained has ripples in the stopband part, a constant $c_{\text{bottom}}$ (see Fig. 7) was introduced to flatten the ripples, which was determined by trial and error to be $8 \times 10^{-4}$. From the passband filter thus found, a minimum phase filter is extracted (curve Elliptic + $c_{\text{bottom}}$ in Fig. 7) and this filter is then inverted to obtained the desired stopband.

A gain $c_{\text{tune}} = 2 \times 10^{-3}$ was determined by trial and error so that the final filter $W_a$ could enclose the error shape $\tilde{\sigma}(E_a(s))$ (see $W_a$ in Fig. 7). It should be noted that a compromise was taken so that the filter $W_a$ did not encompass $\tilde{\sigma}(E_a(s))$ at the first resonance frequency, as can be seen in Fig 7. This allows $W_a$ to be shaped for achieving good spillover performance at most frequencies without unnecessary use of higher order filtering.

The impact of the first resonance on the spillover is minimized by the inclusion of the performance filter $W_p$ which minimizes control energy at low frequencies. It will be shown in the experiment that this control design approach allows good control performance without significant spillover.

The same minimum-phase passband filter (curve Elliptic + $c_{\text{bottom}}$) is multiplied by a gain $c_p$ to define the weight function $W_p(s)$ which determines the controller performance. Trial and error found the optimum $c_p = 8 \times 10^{-2}$, see curve $W_p$ in Fig. 7.

4 Experiment

In this section, the implementation of the proposed high frequency control method is described. The objective is to be able to control broadband structural vibration within a specific bandwidth.

4.1 Setup

The experiment was performed on a cantilever beam shown in Fig. 8. The beam used is aluminium with features described in Table 1:
An active controller was designed to control the $4^{th}$, $5^{th}$, $6^{th}$ and $7^{th}$ bending modes within a bandwidth of [330Hz, 1.15kHz]. It is important to note that the objective was not the control of the highest frequency possible with the available equipment.

![A clamped beam with the mesh used by the scanning laser vibrometer.](image)

The properties of the piezoelectric sensor and the actuator used for the controller input and output are described in Table 2:

<table>
<thead>
<tr>
<th>Properties</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ($l = r_e - r_b$)</td>
<td>50 mm</td>
</tr>
<tr>
<td>Width ($w$)</td>
<td>25 mm</td>
</tr>
<tr>
<td>Thickness ($h$)</td>
<td>0.25 mm</td>
</tr>
<tr>
<td>Young’s Modulus ($E$)</td>
<td>$63 \times 10^9$ N/m$^2$</td>
</tr>
<tr>
<td>Poisson ratio ($v$)</td>
<td>0.3</td>
</tr>
<tr>
<td>Charge constant ($d_{31}$)</td>
<td>$-1.66 \times 10^{-10}$ m/V</td>
</tr>
<tr>
<td>Voltage constant ($g_{31}$)</td>
<td>$-1.15 \times 10^{-2}$ Vm/N</td>
</tr>
<tr>
<td>Capacitance ($C$)</td>
<td>$1.05 \times 10^{-7}$ F</td>
</tr>
<tr>
<td>Electromechanical coupling factor ($k_{31}$)</td>
<td>0.34</td>
</tr>
</tbody>
</table>

It is well known that for a cantilever beam, the optimal location for the piezoelectric actuator patch is close to the clamped edge due to it being a high
strain region [14,13,15]. Thus, the piezoelectric actuator and sensor patches were attached near the clamped edge on either side of the beam as shown in Fig. 8.

The vibration signal from the piezoelectric sensor was accessed using a National Instruments PCI 6023E acquisition card. A power amplifier with a gain set to 10 for the piezoelectric actuator was specifically designed to provide a DC voltage of 100V to allow the piezoelectric actuator to have a larger dynamic range without risk of cracking. As the design of such an amplifier was not a trivial exercise and as suitable low cost amplifiers are not commercially available, the circuit diagram is provided as a service to readers in Appendix C which is based on information provided in [16]. This power amplifier is fed by a National Instruments PCI 6713 output card.

The real time process for implementing the controller was driven by RT-LAB software which provides tools for running and monitoring simulations and controls on various runtime targets [17]. A point-wise disturbance was applied at a discrete location of the beam (0.2m from the clamp) by using an electromagnetic shaker coupled with a magnet of negligible weight attached to the beam was used. This electromagnetic shaker was used so a non-contact disturbance excitation can be generated on the structure.

Vibration velocities were measured at 29 discrete points along the the beam as shown in Fig. 8, using a Polytech PSV 400 scanning laser vibrometer. It is known that control is always more efficient at the location of the sensor/actuator, and so this part of the beam was not taken into account in the determination of global vibration attenuation. If this were not done, it could have been argued that the vibration attenuation at the sensor locations affected the displacement average too much).

4.2 Experimental results

If a homogenous distribution of controller energy across the four modes of interest is desired, the weight functions at the output of the spatial system were set to unity, \( w_1 = w_2 = w_3 = w_4 = 1 \).

The vibration level at 29 points across the beam was measured using the laser vibrometer for the case with and without control. Fig. 9 and 10 show the average spectrum of the displacements at 29 points:
Fig. 9. Displacement average spectrum of the beam without control.

Fig. 10. Displacement average spectrum of the beam with control.

From these figures, it can be seen that the proposed control managed to reduce the beam’s vibration caused by the 4th, 5th, 6th and 7th bending modes. In contrast, other bending modes were not affected significantly by the controller. There are also several torsional modes (identified as (T) on the figures) that were observed. However, the torsional modes were not affected by the con-
controller except for a slight effect on the one inside the controlled frequency band. Table 3 describes the vibration attenuation achieved at each modal resonance frequency, rounded to the nearest 0.5 dB. The vibration attenuation of the 6th and 7th modes are not as high as that of the 4th and 5th modes since the 6th and 7th modes are found to be less controllable and observable from the simulation and the experiment.

Fig. 11 shows the effect of the controller on the maximum displacement amplitude of the beam for the four modes resonant in the frequency band of interest. For the remainder of the modes, there is no significant difference between the controlled and un-controlled beam.

<table>
<thead>
<tr>
<th>mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>T</th>
<th>4</th>
<th>5</th>
<th>T</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>attenuation [dB]</td>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
<td>9.0</td>
<td>9.0</td>
<td>1.0</td>
<td>5.0</td>
<td>3.0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Fig. 11. Displacement of the beam at the 4 controlled frequencies, with and without control.
5 Conclusions

A new control design has been presented for high frequency control over a specified bandwidth using a reduced order plant with corrective terms. The proposed method allows a lower order plant to be used, which consequently lowers the order of the obtained optimal controller. This new control design also allows the ability to concentrate the controller energy in a specific bandwidth. Experiments on a cantilever beam demonstrated the effectiveness of the proposed approach in controlling vibration due to vibration modes within the high frequency bandwidth of interest. Future work will attempt to implement the control method on more complex structures.

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A Correction terms in the state space domain representing the contribution of the high unmodelled frequency modes

The correction terms were developed by [10] who shows the alteration in the localization of poles and zeros due to the model reduction. The correction terms for the truncated model used in this article are described below:

\[
D_{22} = \frac{1}{2\omega_c} \sum_{i=N+1}^{\infty} \frac{\Upsilon_i P_i}{\omega_i} \ln \left( \frac{\omega_i + \omega_c}{\omega_i - \omega_c} \right) \tag{A.1a}
\]

\[
D_{11} = \frac{1}{2\omega_c} \sum_{i=N+1}^{\infty} \frac{\phi_i(r) \phi_i(r_f)}{\omega_i} \ln \left( \frac{\omega_i + \omega_c}{\omega_i - \omega_c} \right) \tag{A.1b}
\]

\[
D_{12} = \frac{1}{2\omega_c} \sum_{i=N+1}^{\infty} \frac{\phi_i(r) P_i}{\omega_i} \ln \left( \frac{\omega_i + \omega_c}{\omega_i - \omega_c} \right) \tag{A.1c}
\]

\[
D_{21} = \frac{1}{2\omega_c} \sum_{i=N+1}^{\infty} \frac{\phi_i(r_f) \Upsilon_i}{\omega_i} \ln \left( \frac{\omega_i + \omega_c}{\omega_i - \omega_c} \right) \tag{A.1d}
\]

The variables a defined in section 2.1.
B State space matrices of a cantilever beam with piezoelectric actuators/sensors and a point wise disturbance

\[
A^{[2N \times 2N]} = \begin{bmatrix}
0_{[N \times N]} & I_{[N \times N]} \\
-\text{diag}(\omega_1^2, \ldots, \omega_N^2) & -2 \text{diag}(\zeta_1 \omega_1, \ldots, \zeta_N \omega_N)
\end{bmatrix}
\]

\[
B_1^{[2N \times N_f]} = \begin{bmatrix}
\phi_1(r_{f_1}) & \cdots & \phi_1(r_{f_{N_f}}) \\
\vdots & \ddots & \vdots \\
\phi_N(r_{f_1}) & \cdots & \phi_N(r_{f_{N_f}})
\end{bmatrix}
\]

\[
B_2^{[2N \times N_a]} = \frac{1}{\rho A} \begin{bmatrix}
0_{[N \times N_a]} \\
\kappa_1 \Psi_1 & \cdots & \kappa_1 \Psi_{N_a} \\
\vdots & \ddots & \vdots \\
\kappa_N \Psi_1 & \cdots & \kappa_N \Psi_{N_a}
\end{bmatrix}
\]

\[
C_1^{[N_y \times 2N]} = \begin{bmatrix}
\phi_1(r_{y_1}) & \cdots & \phi_N(r_{y_1}) \\
\vdots & \ddots & \vdots \\
\phi_1(r_{y_{N_y}}) & \cdots & \phi_N(r_{y_{N_y}})
\end{bmatrix}
\]

\[
C_2^{[N_s \times 2N]} = \begin{bmatrix}
\Omega_1 \Psi_1 & \cdots & \Omega_N \Psi_{N_1} \\
\vdots & \ddots & \vdots \\
\Omega_{N_s} \Psi_1 & \cdots & \Omega_{N_s} \Psi_{N_{N_s}}
\end{bmatrix}
\]

The variables defined in section 2.1.
C Piezoelectric actuator power amplifier design

The circuit design of the power amplifier used to drive the piezoelectric actuator in the experiment is shown below.