High performance control for trajectory tracking of symmetric VTOL vehicles with significant aerodynamic effects

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Abstract—In this paper, we propose a high performance, Lyapunov based control design for symmetric VTOL vehicles to track trajectories at which aerodynamic effects are significant. We consider the case where an accurate model of aerodynamic forces may be obtained either analytically or empirically. A Lyapunov based control strategy is proposed. To avoid lengthy and noisy control expressions resulting from twice differentiating this aerodynamic model (as required by the standard backstepping algorithm), we propose a backstepping technique designed alongside two coupled filters. The first filter enables a cascade approach to be used, while the second provides a smoothed version of the higher order derivatives of the aerodynamic term for use in the dynamic inversion of the model. The resulting controller is described by equations significantly less complex and lengthy than if a standard linearization technique were used. Simulation results are presented for the specific case of a simple cylindrical VTOL vehicle. These results demonstrate the controller’s ability to reject an initial condition mismatch, and invert the system dynamics sufficiently despite the approximations made to the backstepping technique used.

I. INTRODUCTION

Nonlinear stabilisation and control of VTOL vehicles has been a heavily researched area over the last 15 years. Throughout the 1990s, much attention was paid to the PV- 
TOL system model due the interesting non-minimum phase characteristics arising from input coupling [10]. Many techniques have been published addressing methods overcoming these non-minimum phase dynamics [10], [14], [4]. However, the most popular method used involves the application of a decoupling change of co-ordinates (e.g. see [18], [19], [1], [5], [23]). Once decoupled, the resulting triangular, cascade structure leads naturally to backstepping control designs [13]. Most published works exploit this cascade structure, resulting in closed-loop dynamics of a cascade nonlinear system describing vehicle translation with an exponentially stable subsystem describing vehicle rotation [19], [24], [7], [25], [3]. More recently, interest has been focused on global control of 6 degree of freedom (DOF) VTOL vehicle models. Much of this work has been inspired directly by that done earlier for the PVTOL system. However, in general an equivalent decoupling change of co-ordinates can not be found for these vehicles. Published controller designs typically ignore input coupling and use the resulting cascaded structure for control via dynamic linearization [12], high gain designs [16], [8], or full backstepping design using dynamic extension [15]. Robustness arguments are generally used to ensure the control laws are stable despite the neglected input coupling.

A special class of 6 DOF VTOL vehicles are those with an axis of symmetry in the direction of primary thrust. It has been shown that in this case, the decoupling change of co-ordinates used for the PVTOL system can be generalised to decouple the 6 DOF VTOL system. This was demonstrated in [17] by proposing that the weight of a model helicopter be distributed such that it is symmetric. After decoupling, the resulting system dynamics are triangular and can thus be globally stabilised using linearization or backstepping techniques. A large class of UAVs (in particular those utilising ducted fans) exhibit this symmetric structure, such as the Bertin LAAS-CNRS hovereye [20], the Honeywell MAV [2] and Allied Aerospace iSTAR [6], just to name a few.

In this paper, we consider high performance tracking control for symmetric VTOL vehicles at velocities where aerodynamic effects are prevalent. High performance tracking control requires either high gain or an accurate knowledge of the vehicle model. Often, high gain can not be achieved for VTOL vehicles due to physical actuator limitations. The main complexity in a detailed model of the VTOL vehicle lies in the aerodynamic terms. We consider the case where a reasonably accurate model is obtainable. In general, this model will be heavily nonlinear and algebraically complicated. Model based, high performance control must use model knowledge to feed-forward compensate the system’s dynamics. Backstepping control achieves this by propagating derivatives of virtual errors in the backstepping procedure. Applying classical backstepping approaches [13] to the dynamic model of a VTOL vehicle with aerodynamics leads to algebraic formule of high complexity. The problems of generating control laws of such algebraic complexity was largely overcome in the late 90s by symbolic computation software [21], [22]. However, such complicated expressions still cause problems with code verification and management. Furthermore, although a complex model may be a good local representation of a phenomena, the non-linearities are still approximations of real world dynamics. When derivatives are taken of such models, there exists significant potential for the introduction of large errors due to the potential sensitivity of non-linear models. Linear models do not suffer as greatly from this problem, as only the first order trend of the phenomenon is modelled and its derivative will thus be a reasonable approximation of the first order trend of the
II. SYSTEM DYNAMICS.

A. Non-Minimum Phase Representation of Vehicle Dynamics

The class of vehicles considered are symmetric VTOL vehicles, such as that pictured in figure 1. This class of vehicles typically has four independent control inputs \( T_x, T_y, T_z \), and \( \tau_z \). Input \( T_z \) is primary thrust, a force intended to produce a translational acceleration. Inputs \( T_x \) and \( T_y \) are independent control forces applied with some aerodynamic terms from the vehicle center of gravity (CG), to control vehicle orientation. These forces are typically generated via thrust vectoring. Most vehicles of this type are also capable of producing a torque input \( \tau_z \) to control vehicle orientation about an axis in the direction of primary thrust \( e_z \). By symmetric, we mean that the vehicle has an axis of symmetry \( e_z \) in the direction of primary thrust \( T_z \). Many VTOL UAVs have this symmetric property, particularly those utilising ducted fans. For simplicity we consider the simple case of a vehicle with a cylindrical body. However, the control methodology proposed could be applied to any symmetric vehicle.

To provide a framework in which to represent vehicle dynamics we define the inertial reference frame \( \mathcal{E} \triangleq \{ E_x, E_y, E_z \} \), and body fixed reference frame \( \mathcal{B} \triangleq \{ e_x, e_y, e_z \} \), located at the vehicle CG. Variables \( E_x, e_x \), etc. are unit vectors that denote the co-ordinate axis of their respective frames. We define the relative orientation between these frames using the rotation matrix \( R : \mathcal{B} \to \mathcal{E} \). This matrix is a function of the orientation angles \( \eta \triangleq (\phi, \theta, \psi) \). Here, we define these angles as being applied in the order 'roll \( \phi \) - pitch \( \theta \) - yaw \( \psi \)', rather than using conventional 'yaw \( \psi \) - pitch \( \theta \) - roll \( \phi \)' Euler angles as it simplifies equations arising in controller design. The corresponding rotation matrix can be shown to be:

\[
R(\eta) = \begin{bmatrix}
    c\theta c\psi & -c\theta s\psi & s\theta \\
    s\theta c\psi + s\phi s\psi & c\phi c\psi - s\phi s\theta s\psi & s\phi c\theta \\
    -c\phi s\psi + s\phi c\psi s\phi c\theta & s\phi c\psi + c\phi s\phi c\theta & c\phi c\theta \\
\end{bmatrix}
\]  

With reference to figure 1, the rigid body dynamics of the vehicle may be written using Newtonian mechanics as:

\[
\dot{x} = v \\
m\ddot{v} = R(\eta) (T_z e_z + T_x e_x + T_y e_y) + A(\eta, v) - mgE_z \\
\dot{\eta} = T(\eta) \Omega \\
\dot{\Omega} = -\Omega \times \Omega + lT_y e_x - lT_x e_y + \tau_z e_z.
\]  

Here \( x \in \mathbb{R}^3 \) and \( v \in \mathbb{R}^3 \) denote the position and velocity of the vehicle’s CG with respect to the inertial reference frame \( \mathcal{E} \). The variable \( \Omega \triangleq (\omega_x, \omega_y, \omega_z) \) denotes the angular velocity of frame \( \mathcal{B} \) with respect to \( \mathcal{E} \). The variable \( m \) is vehicle mass, \( g \) acceleration due to gravity, and \( l \) the eccentricity of \( T_x e_x \) and \( T_y e_y \) with respect to the CG. The function \( A(\eta, v) \) describes the aerodynamic forces on the vehicle. This term is discussed in more detail, later in this section. The matrix \( T(\eta) \) is the velocity Jacobian that relates \( \dot{\eta} \) to \( \Omega \):

\[
T(\eta) = \frac{1}{c\theta} \begin{bmatrix}
    c\psi & -s\psi & 0 \\
    c\theta s\psi & c\theta c\psi & 0 \\
    -s\theta s\psi & s\theta c\psi & c\theta \\
\end{bmatrix}.
\]  

The inertia tensor \( I \) has the form:

\[
I = \begin{bmatrix}
    I_{xx} & 0 & 0 \\
    0 & I_{yy} & 0 \\
    0 & 0 & I_{zz} \\
\end{bmatrix}.
\]  

As the vehicle is symmetric with respect to the plane \((e_x, e_z)\) and \((e_y, e_z)\), it follows that \( I_{xx} = I_{yy} \).

Since it is accepted that a typical VTOL UAV system has actuators of insufficient authority to cancel unmodelled
aerodynamic effects through simple high gain feedback, a key aspect to providing high performance control is obtaining a reasonable model of these effects for the system considered. There has been intense research in the understanding of aerodynamic lift and drag forces over the last century with a recent focus on small scale airframes such as UAVs. The resulting mathematical models provide a good prediction of real world aerodynamic effects within the normal operating envelope of the vehicles considered. These models, however, suffer from significant algebraic complexity, and are still only approximations of the true real world effects. In practice, such models are good at providing a direct estimate of forces applied to the vehicle; their first order derivatives may be used as a noisy estimate of impulsive effects on the vehicle that is reasonably valid at low frequencies; while second and third order derivatives of aerodynamic models tend to be inaccurate and should not be used in the control design. This poses significant problems in dynamic inversion based control design algorithms that attempt to use the aerodynamic model information to obtain high performance control.

B. Minimum Phase Representation of Vehicle Dynamics

As stated earlier, control inputs $T_x$ and $T_y$ are intended to control vehicle orientation as they are applied eccentrically with respect to the vehicle CG, and $T_z$ is intended to control the translational acceleration of the vehicle $\dot{v}$. However, as shown by equation 2, $T_x$ and $T_y$ also influence vehicle translation. This input coupling makes the system non-minimum phase. Non-minimum phase characteristics of this type of system are discussed in detail in many existing works (e.g. [10], [12], [20]) and are thus not reviewed here. However, it is important to note that this input coupling results in a dynamic structure that is non-triangular, as required for backstepping control design [13]. Before proceeding, it is thus necessary to find a representation of vehicle dynamics that has either the required triangular structure, or is at very least is minimum phase. Fortunately, a non-linear change of co-ordinates exists that can achieve this [17]. This change of co-ordinates is well understood and only included here for completeness.

1) Decoupling of yaw dynamics: Here, we follow the lead of [20], and firstly analyse the yaw dynamics of this symmetric vehicle. Due to the symmetry of the inertia tensor (i.e. $I_{xx} = I_{yy}$), it is straightforward to show that:

$$\Omega \times \Omega = \begin{bmatrix} (I_{zz} - I_{xx}) \omega_y \omega_z \\ (I_{xx} - I_{zz}) \omega_x \omega_z \\ 0 \end{bmatrix}.$$  \hspace{1cm} (5)

Combining this with equation 2, the yaw dynamics of the vehicle may be written as $\dot{\omega}_z = \frac{1}{I_{zz}} \tau_z$. Thus, the yaw dynamics of the vehicle describing $\omega_z$ are decoupled form the rest of the vehicle dynamics. Assuming $\omega_z(0) = 0$, and applying $\tau_z = 0$, we ensure $\omega_z = 0$ for all $t > 0$. Consequently, the inertial cross coupling $\Omega \times \Omega$ will disappear. In practice it is more sensible to chose $\tau_z = -k \omega_z$ such that any non zero initial condition or disturbance in $\omega_z$ will be rejected.

2) Decoupling change of coordinates: Following the work of [19], we use the non-linear change of co-ordinates:

$$\lambda = x + c(e_z - E_z)$$

$$\sigma = v + \Omega \times c e_z$$  \hspace{1cm} (6)

where $c \in \mathbb{R}$ is to be defined. It is insightful to think of $\lambda \triangleq [\lambda_x \ \lambda_y \ \lambda_z]$ and $\sigma \triangleq [\sigma_x \ \sigma_y \ \sigma_z]$ as the position and velocity of a point $e(e_z - E_z)$ from the vehicle’s CG, which we will call the ‘control point’. Combining this with $\omega_z = 0$, the system’s dynamics may be re-written as:

$$\dot{\lambda} = \sigma$$

$$m\dot{\sigma} = (T_z - c(e_z^2 + \omega_e^2)) e_z + \left(1 - \frac{mlc}{I_{xx}}\right) T_y e_y +$$

$$+ \left(1 - \frac{mlc}{I_{yy}}\right) T_x e_x + A(\eta, \sigma - \Omega \times c e_z) - mg E_z$$

$$\ddot{\eta} = T(\eta) \Omega$$

$$I\dot{\Omega} = I_T e_x - IT_y e_y + \tau_z e_z.$$  \hspace{1cm} (7)

Due to the symmetry of the vehicle (i.e. $I_{xx} = I_{yy}$) we may choose $c = \frac{\lambda_z}{\omega_z}$ such that the input coupling disappears. However, the translational acceleration $\dot{\sigma}$ is now coupled to the angular velocity through the vehicle aerodynamics (i.e. $A(\eta, \sigma - \Omega \times c e_z)$). To overcome this, we make the approximation: $A(\eta, \sigma - \Omega \times c e_z) \approx A(\eta, \sigma)$. This may be justified by the observation that under normal operating conditions, at velocities where aerodynamic effects are prevalent, $\sigma >> \Omega \times c e_z$ such that $\sigma \approx v$. Combining all of this with the linearising input augmentation $\bar{T}_z = T_z - c(e_z^2 + \omega_e^2)$, the dynamics become:

$$\dot{\lambda} = \sigma$$

$$m\ddot{\sigma} = \bar{T}_z e_z + A(\eta, \sigma) - mg E_z$$

$$\ddot{\eta} = T(\eta) \Omega$$

$$I\dot{\Omega} = I_T e_x - IT_y e_y + \tau_z e_z.$$  \hspace{1cm} (8)

Input coupling has thus been removed, and the system exhibits the triangular structure required for backstepping design.

III. CONTROLLER DESIGN

A. Control Objective

The control design objective is to force the system outputs $\lambda(t)$ to track a demand trajectory $\lambda_d(t)$, given that a sufficient number of derivatives of the demand trajectory are available to the controller. Furthermore, we wish to avoid expressions involving second derivatives of $A(\eta, \sigma)$, as they are likely to be both noisy and unmanageably lengthy. The approach taken uses the full aerodynamic model $A(\eta, \sigma)$ in the backstepping procedure, however, we introduce two coupled filters that provides a smoothed version of the higher order derivatives of this term for use in the dynamic inversion of the model. These filters are driven by the first order derivatives of $A(\eta, \sigma)$, and consequently, ensures good feedforward tracking of the predicted aerodynamic effects,
while at the same time filtering the high frequency error out of the derivative of $A(\varphi, \sigma)$. The filter gains may be tuned to trade off between high trust in the aerodynamic model and noise in the higher order derivatives of $A(\varphi, \sigma)$. By using the filtered estimates of $A(\varphi, \sigma)$ derivatives in the backstepping procedure we ensure that the correct feedforward terms are used in the control to invert the actual aerodynamic effects modelled by $A(\varphi, \sigma)$ without corrupting the high frequency response of the control signal.

B. Control Design Procedure

Define the first error variable $\delta_1 \triangleq \lambda - \lambda_d$ and the control Lyapunov function (CLF):

$$V_1 \triangleq \frac{1}{2} \delta_1^T \delta_1,$$

with the derivative:

$$\dot{V}_1 = \delta_1^T (\sigma - \dot{\lambda}_d).$$

Here, $\sigma$ appears as our first virtual input. We define its desired value as $\sigma_d \triangleq \lambda_d - k_1 \delta_1$, along with the corresponding error variable $\delta_2 \triangleq \sigma - \sigma_d$, such that equation 9 may be written as:

$$\dot{V}_1 = -k_1 \|\delta_1\|^2 + \delta_1^T \delta_2.$$

Backstepping an integrator, we define the second CLF as:

$$V_2 \triangleq V_1 + \frac{1}{2} \delta_2^T \delta_2.$$

Differentiating this with respect to time we arrive at:

$$\dot{V}_2 = -k_1 \|\delta_1\|^2 + \delta_1^T \delta_2 + \frac{1}{2} \delta_2^T (m R(\eta) \dot{T}_z e_z + A(\sigma, \eta) - g E_z - \dot{\sigma}_d).$$

It should be noted that aerodynamic forces $A(\sigma, \eta)$ are not a function of yaw ($\psi$), due to vehicle symmetry and our chosen angle set. Thus, $\phi$, $\theta$, and $\dot{T}_z$ appear as our second virtual inputs\(^2\) and we define their desired values such that:

$$\frac{1}{m} R(\eta_d) \dot{T}_z e_z + \frac{1}{m} A(\sigma, \eta_d) - g E_z \triangleq \delta_d - \delta_1 - k_2 \delta_2 \triangleq a_d \quad (14)$$

where $\eta_d = [\phi_d \quad \theta_d \quad \psi^*_d]$ and $\dot{T}_z e_z$ are the desired values of virtual inputs, and $\psi^*_d$ may be chosen arbitrarily\(^3\). Variable $a_d \triangleq [a_{d_x} \quad a_{d_y} \quad a_{dz}]^T$ is introduced to shorthand notation and may be considered as a desired translational acceleration. It is important to note that $\frac{1}{m} R(\eta)$ can easily be expressed as an analytic function of state variables. We now define the third error variable:

$$\delta_3 \triangleq \frac{1}{m} R(\eta) \dot{T}_z e_z + \frac{1}{m} A(\sigma, \eta) - g E_z - a_d \quad (15)$$

such that:

$$\dot{V}_2 = -k_1 \|\delta_1\|^2 - k_2 \|\delta_2\|^2 + \delta_2^T \delta_3.$$  

At this stage we depart from the conventional backstepping technique that would attempt to force $\delta_3$ to zero directly by including it in the next CLF. If we were to continue with the conventional backstepping technique by dynamically extending the $\dot{T}_z$ input, the derivative of the final CLF would include the unmanageably large terms; $\frac{\partial^2 A(\sigma, \eta)}{\partial \sigma^2}$, $\frac{\partial^2 A(\sigma, \eta)}{\partial \eta^2}$, and $\frac{\partial^2 A(\sigma, \eta)}{\partial \eta \partial \sigma}$. Consequently, these terms would be included in the control law. We avoid this as follows; Here we assume that it is not possible to algebraically determine $\eta_d$ or $\dot{T}_z$, resulting in a closed-loop cascade structure. However, as we can not directly calculate $\eta_d$ or $\dot{T}_z$, our approach is to design a filter to estimate these values denoted $\tilde{\eta}_d$ and $\tilde{T}_z$ respectively. To achieve this, we split $\delta_3$ into two components such that:

$$\delta_3 \triangleq \frac{1}{m} R(\eta_d) \dot{T}_z e_z + \frac{1}{m} A(\sigma, \eta_d) - g E_z - a_d$$


$$\dot{\tilde{\eta}}_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} (\eta - \tilde{\eta}_d).$$

Backstepping another integrator, we defining the new CLF and filter Lyapunov function as:

$$V_3 \triangleq \frac{1}{2} \delta_3^T \delta_3,$$

$$\dot{L}_3 \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \left( \frac{1}{m} R \dot{T}_z \tilde{\eta}_d + \frac{1}{m} R(\tilde{\eta}_d) \dot{T}_z e_z \right)$$

$$+ \frac{1}{m} A(\sigma, \tilde{\eta}_d) - g E_z - a_d$$

respectively, with the derivatives:

$$\dot{V}_3 = \delta_3^T \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (T(\eta) \Omega - \tilde{\eta}_d)$$

$$\dot{L}_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{\partial R(\tilde{\eta}_d)}{\partial \tilde{\eta}_d} \dot{T}_z \tilde{\eta}_d + \frac{1}{m} R(\tilde{\eta}_d) \dot{T}_z e_z$$

$$+ \frac{1}{m} \frac{\partial A(\sigma, \tilde{\eta}_d)}{\partial \sigma} \dot{\sigma} + \frac{1}{m} \frac{\partial A(\sigma, \tilde{\eta}_d)}{\partial \eta} \dot{\eta}_d - \frac{da_d}{dt}$$

\(^2\)It is straightforward to show that $R(\eta) e_z = [s \theta - \dot{\phi} \cdot \cos \theta \quad \phi \cdot \sin \theta]^T$, and thus $\dot{\psi}$ will not appear as a virtual input through $\frac{1}{m} \dot{T}_z R(\eta) e_z$. This is a consequence of the chosen angle set (i.e. ‘roll $\phi$·pitch $\theta$·yaw $\psi$), and agrees with intuition that only two independent angles are required to define the orientation of the thrust component $T_z e_z$.

\(^3\)In reality we need not decide on a value for $\psi^*_d$. Due to the decoupling of the yaw dynamics and the chosen angle set, it will be shown that the controller does not require knowledge of $\psi^*_d$. 


We wish to define filter equations for $\hat{\delta}_d$ and $\hat{T}_{zd}$ such that:

$$\frac{1}{m} \frac{\partial R(\hat{\eta}_d)}{\partial \hat{\eta}_d} e_z \hat{T}_{zd} \hat{\delta}_d + \frac{1}{m} R(\hat{\eta}_d) e_z \hat{T}_{zd} + \frac{1}{m} \frac{\partial A(\sigma, \hat{\eta}_d)}{\partial \sigma} \dot{\sigma} + \frac{1}{m} \frac{\partial A(\sigma, \hat{\eta}_d)}{\partial \sigma} \dot{\hat{\eta}_d} - \frac{d a_d}{d t} = -k_3 \delta_3$$

(23)

It is possible that our filter may converge to values of $\hat{T}_{zd} < 0$. Clearly this is impractical, as the majority of VTOL vehicles cannot generate negative thrust. To avoid this, we define $\hat{T}_{zd} \triangleq \sqrt{T_{zd}^2} = ||T||$, and estimate $T$ rather than $\hat{T}_{zd}$. Substituting this into equation 23 and rearranging we arrive at:

$$\frac{1}{m} \left[ \frac{\partial R(\hat{\eta}_d)}{\partial \hat{\eta}_d} \right] e_z \hat{T}_{zd} + \frac{1}{m} \frac{\partial A(\sigma, \hat{\eta}_d)}{\partial \sigma} \dot{\sigma} + \frac{d a_d}{d t}$$

$$= -k_3 \delta_3 - \frac{1}{m} \frac{\partial A(\sigma, \hat{\eta}_d)}{\partial \sigma} \dot{\hat{\eta}_d} + \frac{d a_d}{d t}$$

(24)

Matrix $M \in \mathbb{R}^{3 \times 4}$ is not square, and thus multiple solutions exist for $[\hat{\eta}_d \ \hat{T}_{zd}]^T$. However, noting that $\frac{\partial R(\hat{\eta}_d)}{\partial \hat{\eta}_d} = 0$ (i.e., the third column of $M$ is 0)\(^4\), we may rewrite 24 as:

$$\begin{bmatrix}
M(\sigma, \hat{\eta}_d, T) = \begin{bmatrix}
100 \\
010 \\
000 \\
001
\end{bmatrix}
\end{bmatrix}^{-1} \begin{bmatrix}
100 \\
010 \\
001
\end{bmatrix} \hat{T}
$$

(37)

$$\hat{T} = -k_3 \delta_3 - \frac{1}{m} \frac{\partial A(\sigma, \hat{\eta}_d)}{\partial \sigma} \dot{\hat{\eta}_d} + \frac{d a_d}{d t}$$

(25)

such that:

$$\begin{bmatrix}
100 \\
010 \\
001
\end{bmatrix} \hat{T} = M(\sigma, \hat{\eta}_d, T)^{-1} \begin{bmatrix}
100 \\
010 \\
001
\end{bmatrix} \hat{T}$$

(38)

and equation 22 becomes:

$$\dot{L}_1 = -k_3 \| \delta_3 \|^2$$

(27)

To continue backstepping, $\Omega$ appears as our virtual input and we define its desired value such that:

$$\begin{bmatrix}
100 \\
010 \\
001
\end{bmatrix} \begin{bmatrix}
100 \\
010 \\
001
\end{bmatrix} \dot{\hat{T}} \triangleq \dot{\hat{T}}$$

(28)

It is important to note that $\hat{T}$ is available directly from the filter shown in equation 26. The desired value of the angular velocity virtual input is given by $\Omega_d \triangleq \begin{bmatrix} \omega_{xd} & \omega_{yd} & \omega_{zd} \end{bmatrix}^T$. It will be shown that $\omega_{zd}$ may be chosen arbitrarily (recall that $\omega_z$ has been decoupled from the system). The definition of the next error variable follows as:

$$\delta_4 \triangleq \begin{bmatrix}
100 \\
010 \\
001
\end{bmatrix} (\hat{T}(\eta) \Omega_d - \hat{T}(\eta) \Omega_d)$$

(29)

\(^4\)This arises due to the chosen angle set and explains why $\omega_{zd}$ may be chosen arbitrarily.

We do not wish to use the conventional backstepping method and try to force $\delta_4$ to zero by including it in the next CLF, as this will require knowledge of $\hat{\Omega}_d$ and thus $\hat{\delta}_d$, obtainable only by explicit differentiation of equation 26. Explicitly differentiating equation 26 will result in expressions involving second derivatives of $A(\sigma, \eta)$, which we wish to avoid. To overcome this, we design a filter that estimates $\Omega_d$, and force $\Omega$ to converge to this estimate. Thus we split $\delta_4$ into two components:

$$\delta_4 = \begin{bmatrix}
\tilde{\delta}_4 \\
\tilde{\delta}_4
\end{bmatrix}$$

(31)

such that equation 21 may be written as:

$$\dot{V}_3 = -k_3 \| \delta_3 \|^2 + \delta_4^T \delta_4$$

(32)

$$\dot{L}_2 = -k_3 \| \delta_3 \|^2$$

(33)

with the derivatives:

$$\dot{V}_4 = -k_3 \| \delta_3 \|^2 + \frac{1}{2} \delta_4^T \delta_4 + \frac{1}{2} \tilde{\delta}_4^T \tilde{\delta}_4$$

(34)

$$\dot{L}_2 = -k_3 \| \delta_3 \|^2$$

(35)

We now design the filter law:

$$\dot{\hat{\Omega}}_d = -k_3 \Omega + \hat{\Omega}_d - \hat{\Omega}_d$$

(36)

such that

$$\dot{\hat{\Omega}}_d = -k_3 \Omega + \hat{\Omega}_d - \hat{\Omega}_d$$

(37)
Note that we need not compute $\omega_d^e$, as $\hat{\omega}_e$ and $\hat{\omega}_d$ will not depend on it (thanks to the form of $T^{-1}(\eta)$ resulting from our choice of rotation angles). We have not cancelled $\bar{\eta}_d$, as for reasons stated earlier, the purpose of the filter is to avoid it. As a penalty, the derivative of the second filter’s Lyapunov function has the perturbation term $\tilde{\delta}_4^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{\bar{\eta}}_d$ (see equation 37) which must be dominated using a high filter gain.

As our physical control inputs $T_x$ and $T_y$ have appeared in the derivative of our final CLF (see equation 34), we define then as:

$$\begin{bmatrix} T_y \\
-T_x \end{bmatrix} = \frac{I_2}{T} \hat{\bar{\Omega}}_d - \begin{bmatrix} 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} T^{-1}(\eta) \times \left( T(\eta)(\Omega - \bar{\Omega}_d) + \begin{bmatrix} 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left( \hat{\delta}_3 + \bar{k}_3 \hat{\delta}_4 \right) \right),$$

such that:

$$\dot{V}_4 = -\bar{k}_3 \left\| \hat{\delta}_3 \right\|^2 - \bar{k}_4 \left\| \hat{\delta}_4 \right\|^2 + \delta_4^T \hat{\delta}_4.$$  \hspace{1cm} (39)

As with $\psi_d^e$, we need not compute $\omega_e^d$, as $T_x$ and $T_y$ will not depend on it (thanks to the form of $T(\eta)$, once again resulting from our choice of rotation angles).

C. Control Equations

The resulting controller takes the form of a static feedback control law given by equations 38 and $T_z = \|T\|$, with two filters given by equations 26 and 36. All terms in the filters are easily computed as functions of state variables. The two filters serve different purposes. The purpose of the filter defined by equation 26 is to estimate the desired values of the virtual input $\eta_d$, and the true input $T_z = T_{zd} = \|T\|$. The purpose of the filter defined by equation 36 is to provide an estimate of the desired value of the virtual input $\bar{\eta}_d$ and its derivative $\bar{\Omega}_d$ denoted $\bar{\Omega}_d$ and $\bar{\Omega}_d$ respectively. It is easily demonstrated that the equations defining this controller are far simpler than those defining a full backstepping controller for the system.

D. Filter Initial Conditions

As we have used dynamic filters in our controller design, we must choose appropriate initial conditions for $\bar{\eta}_d$, $\bar{\Omega}_d$, $T_0$ and $\bar{\Omega}_d$, and $T_0$ and $\bar{\Omega}_d$, respectively. Appropriate solutions for $\bar{\eta}_d$ and $T_0$ are those that set $\bar{\delta}_3 = 0$ (see equation 17). However, this is not practical, as we can not easily determine these analytically. If the initial conditions are such that the aerodynamic force on the vehicle is far less than that produced by the vehicle thrust (as is likely to be the case at vehicle launch), an appropriate choice for initial conditions are those that set $\bar{\delta}_3 = 0$, assuming $A(\bar{\sigma}, \bar{\eta}_d) = 0$. From equation 14, these are thus the solutions to:

$$\frac{1}{m} R(\bar{\eta}_d) \|T_0\| e_z - g E_z = a_d.$$  \hspace{1cm} (40)

which can be shown to be:

$$T_0 = \pm m \sqrt{a_{dx}^2 + a_{dy}^2 + (a_d + g)^2} \quad \bar{\phi}_{d0} = \text{atan}\{-a_{dy}, (a_d + g)\} \quad \bar{\theta}_{d0} = \text{atan}\left(\frac{a_{dx}}{\sqrt{a_{dx}^2 + (a_d + g)^2}}\right).$$  \hspace{1cm} (41)

With regard to $\bar{\Omega}_{d0}$, an appropriate choice is that which satisfies $\delta_4 = 0$, subject to the above values of $\bar{\eta}_d$ and $T_{zd}$.

From equation 31, the initial condition $\bar{\Omega}_{d0}$ will thus satisfy:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} T(\eta) \bar{\Omega}_{d0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left( \bar{\eta}_d - \bar{k}_3 (\eta - \bar{\eta}_d) \right),$$

which can be shown to be:

$$\bar{\Omega}_{d0} = \begin{bmatrix} c\theta v \psi \\ -c\theta \psi \bar{v} \psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left( \bar{\eta}_d - \bar{k}_3 (\eta - \bar{\eta}_d) \right),$$

where $\bar{\eta}_d$ is easily calculated by one iteration of the filter law given in equation 26.

E. Stability Analysis

Criterion 1: The variable $\bar{\eta}_d$ is bounded along the closed-loop trajectory such that $\|\bar{\eta}_d\| \leq B$.

Criterion 2: The closed-loop cascaded sub-systems are non-peaking.

Lemma 1: If Criteria 1 and 2 are satisfied, then controller gains $\bar{k}_3$ and $\bar{k}_4$ may be chosen sufficiently large such that the the control law given by equations 38, $T_z = \|T\|$, 26 and 36 applied to the system 8 renders the tracking error $\delta_1$ bounded.

Proof: First, recall the Lyapunov function from equation 12, and define the new Lyapunov function $V_T = V_4 + L_1 + L_2$:

$$V_T = \frac{1}{2} \delta_1^T \delta_1 + \frac{1}{2} \delta_2^T \delta_2.$$  \hspace{1cm} (44)

$$V_T = \frac{1}{2} \delta_3^T \delta_3 + \frac{1}{2} \delta_4^T \delta_4 + \frac{1}{2} \delta_5^T \delta_5.$$  \hspace{1cm} (45)

Here, $V_T$ describes the dynamics of tracking errors related to the translational dynamics of the system, while $V_4$ describes dynamics of tracking errors associated with the rotational behaviour and associated filters. Differentiating these Lyapunov functions with respect to time we arrive at:

$$\dot{V}_T = -k_1 \left\| \delta_1 \right\|^2 - k_2 \left\| \delta_2 \right\|^2 + \delta_2^T \left( \hat{\delta}_3 + \hat{\delta}_4 \right).$$  \hspace{1cm} (46)

$$\dot{V}_T = -\bar{k}_3 \left\| \delta_3 \right\|^2 - \bar{k}_3 \left\| \delta_4 \right\|^2 - k_4 \left\| \delta_4 \right\|^2 - \delta_4^T \delta_4 + \delta_4^T \delta_4 - \bar{k}_3 \left\| \delta_3 \right\|^2 - \bar{k}_3 \left\| \delta_4 \right\|^2 + \bar{k}_3 \left\| \delta_4 \right\|^2 \left[ \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} \right] \bar{\eta}_d.$$  \hspace{1cm} (47)
where \( \delta_3 \to 0 \) as \( \dot{\delta}_3 \to 0 \). Provided \( \dot{k}_3, \dot{k}_4 \geq \frac{1}{2} \) the cross term \( \dot{\delta}_3^T \dot{\delta}_4 \) between the controller and second filter will be dominated such that:

\[
\dot{V}_T \leq -\dot{k}_3 \left\| \dot{\delta}_3 \right\|^2 - \dot{k}_4 \left\| \dot{\delta}_4 \right\|^2 - \ddot{\delta}_3 \left\| \dot{\delta}_3 \right\|^2
\]

and thus:

\[
\dot{V}_T \leq -\dot{k}_3 \left\| \dot{\delta}_3 \right\|^2 - \dot{k}_4 \left\| \dot{\delta}_4 \right\|^2 - \ddot{\delta}_3 \left\| \dot{\delta}_3 \right\|^2
\]

We may rewrite this as:

\[
\dot{V}_T \leq -\dot{k}_3 \left\| \dot{\delta}_3 \right\|^2 - \dot{k}_4 \left\| \dot{\delta}_4 \right\|^2 - \ddot{\delta}_3 \left\| \dot{\delta}_3 \right\|^2
\]

As a consequence of Lyapunov stability theory, \( V_T \) converges asymptotically to a set such that \( V_T \geq 0 \) [11]. Within this set one has:

\[
\dot{k}_3 \left\| \dot{\delta}_3 \right\|^2 + \dot{k}_4 \left\| \dot{\delta}_4 \right\|^2 + \ddot{\delta}_3 \left\| \dot{\delta}_3 \right\|^2 \leq \frac{B}{4 \dot{k}_4}
\]

As we may chose \( \dot{k}_4 \) arbitrarily large, we may make the set in which \( \delta_3, \dot{\delta}_4, \ddot{\delta}_3 \) converge to arbitrarily small. As \( \dot{k}_4 \) is a filter gain, setting it arbitrarily high will not result in large actuator demands. However, in practice \( \dot{k}_4 \) should be limited due to noise considerations. As \( \delta_3 \) and \( \dot{\delta}_3 \) may be forced to converge to an arbitrarily small set and \( \delta_3 = \dot{\delta}_3 + \ddot{\delta}_3 \), where \( \delta_3 \to 0 \) as \( \dot{\delta}_3 \to 0 \), it follows from equation 46 and criterion 2 that the tracking error \( \dot{\delta}_1 \) will be bounded.

IV. Simulation Results

A. Aerodynamic Model

For demonstration purposes, we use a simple aerodynamic model of a cylinder. The aerodynamic drag \( F_D \) on a body in a moving fluid is often expressed as [9]:

\[
F_D = C_D (U) \frac{1}{2} \rho U^2 S
\]

where \( C_D (U) \) is a coefficient of drag, \( \rho \) the fluid density, \( U \) free stream velocity and \( S \) a reference area. A reasonable first order approximation of the aerodynamic forces on a cylinder may be derived using equation 52, where \( U \) is replaced by the component of vehicle velocity in the normal and axial vehicle directions. Such an aerodynamic model may be written as:

\[
A(\sigma, \eta) = -C_A (V_A) S_A \frac{1}{2} \rho V_A \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R^T \sigma
\]

\[
C_N (V_N) S_N \frac{1}{2} \rho V_N \begin{bmatrix} 100 \\ 010 \\ 000 \end{bmatrix} R^T \sigma
\]

where \( C_A (V_A) \) and \( C_N (V_N) \) are coefficients of axial and normal force respectively, \( S_A \) and \( S_N \) are axial and normal reference areas. \( V_A \) and \( V_N \) are axial and normal velocity components given by:

\[
V_A = \left\| \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} R^T \sigma \right\|,
\]

\[
V_N = \left\| \begin{bmatrix} 100 \\ 010 \\ 000 \end{bmatrix} R^T \sigma \right\|
\]

Due to the symmetry of the vehicle and the defined angle set, \( A(\sigma, \eta) \) does not depend on \( \psi \). Partial Derivatives of \( A(\sigma, \eta) \) used in controller design are relatively straightforward to obtain and omitted for brevity.

B. Trajectory Demand

We propose the continuously differentiable trajectory for \( t \in (0, T_f) \) given by:

\[
\lambda_d (t) = \begin{bmatrix} r_x \cos \dot{\omega} (t) \\ r_y \sin \dot{\omega} (t) \\ r_z \sin \dot{\omega} (t) \end{bmatrix}
\]

\[
\dot{\omega} (t) = \pi \left( 1 - \cos \left( \frac{\pi t}{T_f} \right) \right).
\]

Values of \( r_x = 250 \text{m}, r_y = 500 \text{m}, r_z = 250 \text{m} \) and \( T_f = 60 \text{s} \) were chosen.

C. System Response

Numerical simulations were carried out for a cylindrical vehicle 1 m in height and 0.15 m in diameter, of uniform density with \( m = 25 \text{kg} \). For simplicity, aerodynamic coefficients were set constant as \( C_A = C_N = 0.8 \). Controller gains \( k_1 = k_2 = \dot{k}_3 = \dot{k}_4 = 1 \) and \( \dot{k}_4 = 10 \) were chosen. Figures 2 and 3 show the system response with initial conditions \( \sigma (0) = \eta (0) = \Omega (0) = 0 \) and \( \lambda (0) = \begin{bmatrix} r_x & 0 & 0 \end{bmatrix} \). Clearly the controller rejects the initial condition mismatch in \( \sigma (0) \), \( \eta (0) \) and \( \Omega (0) \), and inverts the system dynamics sufficiently despite the approximations made to the backstepping technique used for controller design.

V. Conclusion

In this paper, we propose a high performance control design for the trajectory tracking control of symmetric VTOL vehicles at velocities where aerodynamic forces are significant. We consider the specific case where an accurate model of vehicle aerodynamics is obtainable. To avoid lengthy and noisy expressions resulting from twice differentiating the aerodynamic model (as required for a standard backstepping/linearization technique), we introduce two coupled filters.
These filters both enable the system to be cast into a cascade structure and provide a smoothed version of the higher order derivatives of the aerodynamic term for use in the dynamic inversion of the model. The equations defining the resulting control law were far simpler than those derived if a full backstepping design had been used. Simulation results showing favorable performance were presented for the trajectory tracking of a simple cylindrical VTOL vehicle.

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