An Approximate Backstepping Control Law for the Global Stabilisation of Symmetric VTOL Vehicles.

R. Wood 1#, B. Cazzolato 1

1 School of Mechanical Engineering, The University of Adelaide
SA 5005 Australia, # rohin.wood@mecheng.adelaide.edu.au

Abstract

In this paper we propose an approximate backstepping control law for the stabilisation of Six Degree of Freedom symmetric VTOL vehicles. This control law results in closed-loop dynamics with a stable cascade structure. An additional innovation is to design feedback such that the interconnection term between resulting cascaded sub-systems is minimised. The proposed control law is proven to be globally, exponentially stable. Simulation results show aggressive stabilisation and the benefits of minimising the closed-loop interconnection term.

1. INTRODUCTION

Nonlinear stabilisation and control of VTOL vehicles has been a heavily researched area over the last 15 years. Throughout the 1990s, much attention was paid to the PVTOL system model due to its interesting non-minimum phase characteristics. These characteristics arise via input coupling between roll control and vehicle translation [1]. Most published control approaches for the PVTOL system focused on overcoming the problems associated with the system’s non-minimum phase behaviour. These control techniques included the use of minimum phase approximations achieved by neglecting input coupling, robust techniques and digital model based methods [1], [2], [3]. However, the most popular method involved the application of a diffeomorphism to transform the system into an equivalent, minimum phase representation. This idea was first published by the authors of [4], who demonstrated that changing the outputs (or states) from translation at the PVTOL vehicle’s centre of gravity to its center of percussion decouples the system, making it differentially flat and consequently minimum phase. The concept of changing variables to decouple the system has since been used by many researchers [5], [6], [7], [8]. Once decoupled, the resulting triangular, cascade structure leads naturally to backstepping control designs [9]. One popular control technique resulted in closed-loop dynamics of a cascade nonlinear system describing vehicle translation with an exponentially stable linear subsystem describing vehicle rotation. This method was first presented in [10] as applied to a minimum phase approximation of the PVTOL model, and then used along with a decoupling change of coordinates in [5]. Many other authors have since incorporated the ideas of this method into their designs (e.g. [11], [12], [13]).

More recently, interest has been focused on global control of 6 degree of freedom (DOF) VTOL vehicle models, with research being driven by the push to develop globally stable control laws for VTOL UAVs. Much of this work has been inspired directly by that done earlier for the PVTOL system, as 6 DOF VTOL systems display similar non-minimum phase dynamics. A number of works controlling four-rotor mini rotocraft have used approximations that transform the problem into two parallel PVTOL systems (e.g [14]). Much work has also been published addressing the global control of model helicopters. Non-minimum phase characteristics are typically addressed by designing control laws that ignore input coupling, commonly referred to as body forces. The resulting approximate model has a cascade structure that can be controlled via dynamic linearization [15], high gain designs, resulting in a time scale separation between the cascaded systems [16], [17], and full backstepping design using dynamic extension of the thrust input [18]. Robustness arguments are generally used to ensure the control laws are stable despite the neglected input coupling.

A special class of symmetric VTOL vehicles are those where the principal moment of inertia about axis perpendicular to the direction of primary thrust are equal. It has been shown that in this case, the decoupling change of co-ordinates discovered for the PVTOL system in [4] can be generalised to decouple the 6 DOF VTOL system. This was demonstrated in [19] by proposing that the weight of a model helicopter be distributed such that it is symmetric. After decoupling, the resulting representation of system dynamics is differentially flat and can thus be globally stabilised using linearization or backstepping techniques (e.g. [20]). A large class of UAVs (in particular those utilising ducted fans) exhibit this symmetric structure, such as the Bertin LAAS-CNRS, the Honeywell MAV ACTD and Allied Aerospace iSTAR, just to name a few.

In this paper, we propose a new non-linear control law for the stabilisation of symmetric VTOL vehicles based on an extension of the work in [5]. Firstly, the non-linear dynamics of the vehicle are presented. A non-linear change of co-ordinates, motivated by the work in [5] and [20], is then used to decouple the system. The approximate backstepping technique introduced in [10] for the PVTOL system is then extended to control the 6 DOF model. The resulting closed-loop dynamics comprise of an upper nonlinear system describing vehicle translation with an exponentially stable linear subsystem describing vehicle rotation. This cascade structure is then improved by proposing a different feedback law for pri-
mary thrust that minimises the interconnection term between cascaded systems. The essence of this idea was contained in [13] for the control of a 3 DOF system. Here, we extend and clarify the idea to 6 DOF by demonstrating that it may be cast into an approximate backstepping framework. Global stability of the proposed controller is then proven, followed by simulation results demonstrating aggressive stabilisation, and the benefits of the improved control law. The paper concludes with a brief summary.

2. DYNAMIC MODEL OF SYMMETRIC VTOL VEHICLE

A typical ducted fan type VTOL UAV arrangement is shown in figure 1. This class of UAV controls its orientation and displacement via thrust vectoring, typically using a series of vanes at the exhaust exit. A torque about the longitudinal axis may be produced via differentially pitching the exhaust vanes or differentially varying the speed of a vehicle’s counter rotating fans. To provide a framework in which to represent vehicle dynamics we define the inertial reference frame $\mathcal{E} \triangleq \{ E_x, E_y, E_z \}$, and body fixed reference frame $\mathcal{B} \triangleq \{ e_x, e_y, e_z \}$, located at the vehicle center of gravity (CG). Variables $E_x, e_x$ etc. are unit vectors that denote the co-ordinate axis of their respective frames. We define the relative orientation between these frames using the rotation matrix $R : \mathcal{B} \to \mathcal{E}$. This matrix is a function of the orientation angles $\eta \triangleq (\phi, \theta, \psi)$. Here, we define these angles as being roll $\phi$ - pitch $\theta$ - yaw $\psi$, rather than using conventional 'yaw $\psi$ - pitch $\theta$ - roll $\phi$' Euler angles as it simplifies equations arising in controller design. The corresponding rotation matrix can be shown to be $^1$:

$$
R (\eta) = \begin{bmatrix}
c\theta c\psi & -c\theta s\psi & s\theta \\
c\phi s\theta c\psi + c\psi s\theta & c\psi s\theta c\phi + s\psi c\phi & s\psi s\theta \\
-c\phi c\theta s\psi + s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi & c\phi c\theta
\end{bmatrix}
$$

(1)

With reference to figure 1, the rigid body dynamics of the vehicle may be written using Newtonian mechanics as:

$$
\dot{x} = v
$$

$$
m\dot{v} = R (\eta) (T_x e_x + T_y e_y + T_z e_z) - mg E_z
$$

$$
\dot{\eta} = T (\eta) \Omega
$$

$$
\dot{\Omega} = -\Omega \times \Omega + IT_y e_x - IT_x e_y + \tau_z e_z.
$$

(2)

Here $x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the position and velocity of the vehicle’s CG with respect to the inertial reference frame $\mathcal{E}$. The variable $\Omega \triangleq (\omega_x, \omega_y, \omega_z)$ denotes the angular velocity of frame $\mathcal{B}$ with respect to $\mathcal{E}$. The variable $\tau_z$ denotes the magnitude of the control moment applied to the vehicle in the direction $e_z$. Variables $T_x, T_y, T_z$ are the magnitudes of thrust components in the directions $e_x, e_y, e_z$ respectively, $m$ is vehicle mass, $g$ acceleration due to gravity, and $l$ the eccentricity of $T_x e_x$ and $T_y e_y$ with respect to the CG.

$^1$For brevity, we use the notation $c\theta \triangleq \cos \theta$ and $s\theta \triangleq \sin \theta$.

The corresponding rotation matrix can be shown to be $^1$:

$$
T (\eta) = \frac{1}{c\theta} \begin{bmatrix}
c\psi & -s\psi & 0 \\
c\psi c\theta & c\psi s\theta & s\phi c\theta \\
c\phi s\theta c\psi + c\psi s\theta & c\psi s\theta c\phi + s\psi c\phi & s\psi s\theta
\end{bmatrix}
$$

(3)

The inertia tensor $I$ has the form:

$$
I = \begin{bmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{bmatrix}
$$

(4)

As the vehicle is symmetric with respect to planes $(e_x, e_z)$ and $(e_y, e_z)$, it follows that $I_{xx} = I_{yy}$.

With this type of vehicle, $T_x e_x$ and $T_y e_y$ are intended to control vehicle orientation as they are applied eccentrically with respect to the vehicle CG, and $T_z e_z$ is intended to control the translational acceleration of the vehicle $\dot{v}$. However, as shown by equation 2, $T_x e_x$ and $T_y e_y$ also influence vehicle translation. This input coupling makes the system non-minimum phase. Non-minimum phase characteristics of this type of system are discussed in detail in many existing works (e.g. [1], [15], [20]) and are thus not reviewed here. However, it is important to note that this input coupling results in a dynamic structure that is non-triangular, as required for backstepping control design.

3. APPROXIMATE BACKSTEPPING CONTROL DESIGN

A. Decoupling of yaw dynamics

Here, we follow the lead of [20], and before proceeding with controller design, analyse the yaw dynamics of this symmetric vehicle. Due to the symmetry of the inertia tensor (i.e. $I_{xx} = I_{yy}$), it is trivial to show that:

$$
\Omega \times \Omega = \begin{bmatrix}
(I_{zz} - I_{xx}) \omega_x \omega_z \\
(I_{xx} - I_{zz}) \omega_y \omega_z \\
0
\end{bmatrix}
$$

(5)
Combining this with equation 2, it follows that \( \dot{\omega}_z = \frac{1}{I} \tau_z \). Thus, the yaw dynamics of the vehicle describing \( \omega_z \) are decoupled from the rest of the vehicle dynamics. Assuming \( \omega_z(0) = 0 \) and applying \( \tau_z = 0 \) then we ensure \( \omega_z = 0 \) for all \( t > 0 \), and inertial cross coupling \( \Omega \times \Omega \) will disappear. However, in practice it is more sensible to choose \( \tau_z = -k \omega_z \) such that any non zero initial condition or disturbance will be rejected.

### B. Decoupling change of coordinates

If a backstepping controller is to be designed, the system’s dynamics must take the required triangular form. Motivated by the work of [5], we propose the non-linear change of coordinates:

\[
\lambda = x + c (e_z - E_z)
\]

\[
\sigma = \nu + \Omega \times e e_z
\]

where \( c \in \mathbb{R} \) is to be defined. It is insightful to think of this as the dynamics of a point \( c (e_z - E_z) \) from the vehicle’s CG, which we will call the ‘control point’. Combining this with \( \omega_z = 0 \), the system’s dynamics may be re-written as:

\[
\dot{\lambda} = \sigma
\]

\[
m \dot{\sigma} = (T_z - c (\omega_x^2 + \omega_y^2)) e_z
\]

\[
+ \left(1 - \frac{m l c}{I_{xx}}\right) T_y e_y + \left(1 - \frac{m l c}{I_{yy}}\right) T_x e_x - m g E_z
\]

\[
\dot{\eta} = T (\eta) \Omega
\]

\[
\dot{\Omega} = l T_y e_x - l T_x e_y + \tau_z e_z.
\]

Due to the symmetry of the vehicle (i.e. \( I_{xx} = I_{yy} \)) we may choose \( c = \frac{l m}{m} \) such that the input coupling disappears. Combining this with the linearising input augmentation \( T_z = T_z - c (\omega_x^2 + \omega_y^2) \), the dynamics become:

\[
\dot{\lambda} = \sigma
\]

\[
m \dot{\sigma} = T_z e_z - m g E_z
\]

\[
\dot{\eta} = T (\eta) \Omega
\]

\[
\dot{\Omega} = l T_y e_x - l T_x e_y + \tau_z e_z
\]

Input coupling has thus been removed and the system exhibits the triangular structure required for backstepping design.

### C. Control Design

Firstly, we simply extend the ideas of [10] for the control of a 3 DOF VTOL system, to design an approximate backstepping controller for this 6 DOF VTOL model. Defining the first error variable \( \delta_1 \equiv \lambda \), the first control Lyapunov function (CLF) is defined as:

\[
V_1 \equiv \frac{1}{2} \delta_1^T \delta_1
\]

and has the derivative:

\[
\dot{V}_1 = \delta_1^T \sigma.
\]

Here, \( \sigma \) appears as our first virtual input. We define its desired value as \( \sigma_d \equiv -k_1 \delta_1 \), along with the corresponding error variable \( \delta_2 \equiv \sigma - \sigma_d \), such that equation 9 may be written as:

\[
\dot{V}_1 = -k_1 \| \delta_1 \| + \delta_1^T \delta_2.
\]

Differentiating this with respect to time we arrive at:

\[
\dot{V}_2 = -k_1 \| \delta_1 \| + \delta_1^T \delta_2 + \delta_2^T (\frac{1}{m} \dot{\bar{T}}_z R (\eta) e_z - g E_z - \dot{\delta}_d)
\]

Here, \( \phi, \theta \) and \( \bar{T}_z \) appear as our second virtual inputs and we define their desired values such that:

\[
\frac{1}{m} \dot{\bar{T}}_z R (\eta_d) e_z - g E_z \equiv \dot{\delta}_d - \delta_1 - k_2 \delta_2
\]

\[
\dot{\delta}_d = \delta_3 + \frac{1}{m} \dot{\bar{T}}_z R (\eta_d) e_z
\]

\[
\dot{\delta}_3 = \frac{1}{m} \dot{\bar{T}}_z R (\eta) e_z - \frac{1}{m} \dot{\bar{T}}_z R (\eta_d) e_z.
\]

We may rewrite equation 13 as:

\[
\dot{V}_2 = -k_1 \| \delta_1 \|^2 - k_2 \| \delta_2 \|^2 + \delta_2^T \delta_3
\]

Here we depart from the conventional backstepping method of defining the next CLF by augmenting \( V_2 \) a positive definite function of \( \delta_3 \). This would require dynamic extension of the \( \bar{T}_zd \) input and leads to complex control equations. Rather we explicitly set \( \bar{T}_z = \bar{T}_zd \) and define the new error variable \( \delta_3 = \eta - \eta_d \). We then continue the backstepping method by designing a controller that will force \( \delta_3 \) to converge to 0. Although it will not be possible to cancel the cross term \( \delta_1^T \delta_3 \) in equation 20, as \( \delta_3 \to 0 \), \( \delta_3 \to 0 \) thus we will arrive at a closed-loop cascade structure. Defining the CLF:

\[
V_3 \equiv \frac{1}{2} \delta_3^T \delta_3
\]

\[2\text{It is trivial to show that } R (\eta) e_z = [\begin{array}{l} \sin \theta - \sin \phi \theta \cos \phi \theta \\ \cos \theta \end{array}]^T, \text{ and thus yaw } \psi \text{ will not appear as a virtual input through } \frac{1}{m} \dot{\bar{T}}_z R (\eta) e_z. \text{ This is a consequence of the chosen angle set (i.e. roll } \phi \cdot \text{ pitch } \theta \cdot \text{ yaw } \psi \text{), and agrees with intuition that only two angles are required to define the orientation of the thrust component } T_x e_z.\]

\[\text{In order to achieve true global stability, } \phi, \theta \text{ is calculated using a version of the atan2 function which remembers the immediate past value of } \phi, \theta \text{ to avoid the discontinuity at } \frac{\pi}{2}. \text{ Thus, } \phi, \theta \in \mathbb{R} \text{. Physically, this means that the vehicle may execute a } 2\pi \text{ roll, and be stabilised at any } \phi, \theta = k \pi, k \in \mathbb{Z}.\]
with the derivative:
\[
\dot{V}_3 = \delta_3^T (T (\eta) \Omega - \dot{\eta}_d). \tag{22}
\]

Here \( \Omega = \begin{bmatrix} \Omega_x & \Omega_y & 0 \end{bmatrix} \) appears as our virtual input and we define its desired value as:
\[
T (\eta) \Omega_d = T (\eta) \Omega - \dot{\eta}_d \tag{23}
\]

Note that we only have two virtual inputs and three equations, however the third equation simply describes the yaw \( \psi \) angle dynamics: \( \psi = \tan \theta (\omega_x \cos \psi + \omega_y \sin \psi) \) which we do not wish to control. The fourth error variable follows as:
\[
\delta_4 \triangleq T (\eta) (\Omega - \Omega_d) \tag{24}
\]

such that:
\[
\dot{V}_3 = -k_3 \left\| \delta_3 \right\|^2 + \delta_4^T \delta_4. \tag{25}
\]

Backstepping the final integrator, we define the last CLF as:
\[
V_4 \triangleq V_3 + \frac{1}{2} \delta_4^2 \delta_4 \tag{26}
\]

with the derivative:
\[
\dot{V}_4 = -k_3 \left\| \delta_3 \right\|^2 + \delta_3^T \delta_4 + \delta_4^T \left( \dot{T} (\eta) \Omega \right)
+ \left( \frac{T_y e_x - T_x e_y}{I_1} \right) T (\eta) \left( \frac{\partial T (\eta) \Omega_d}{\partial \eta} \right) - \frac{d \dot{T} (\eta) \Omega_d}{d t}. \tag{27}
\]

As our physical control inputs \( T_x \) and \( T_y \) have appeared, we define them as:
\[
\begin{bmatrix} T_y \\ T_x \end{bmatrix} = \frac{I_1}{T} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} T^{-1} (\eta) \times \left( \frac{\partial T (\eta) \Omega_d}{d t} - \ddot{T} (\eta) \Omega + \ddot{\delta}_3 - k_3 \delta_4 \right) \tag{28}
\]

such that:
\[
\dot{V}_4 = -k_3 \left\| \delta_3 \right\|^2 - k_4 \left\| \delta_4 \right\|^2. \tag{29}
\]

It is important to note that \( \frac{\partial T (\eta) \Omega_d}{d t} \) depends on knowledge of \( \dot{\eta}_d \) and \( \dot{\eta}_d \) which are easily shown to be analytic functions of state variables.

D. Closed-loop dynamics

Applying the control laws given in equations 16 and 28, in error co-ordinates the closed loop system dynamics become:
\[
\dot{\delta}_1 = -k_1 \delta_1 + \delta_2 \tag{30}
\]
\[
\dot{\delta}_2 = \delta_1 - k_2 \delta_4 + \delta_4 \left( \delta_3 \right)
\]

and:
\[
\dot{\delta}_3 = -k_3 \delta_3 + \delta_4 \tag{31}
\]
\[
\dot{\delta}_4 = \delta_3 - k_4 \delta_4.
\]

Thus, the controlled system has a cascade structure of an upper sub-system describing translational dynamics and an exponentially stable cascaded sub-system describing orientation. The translational sub-system is linear and exponentially stable, with the exception of the perturbation \( \delta_3 \left( \delta_3 \right) \).

E. Improving the closed-loop dynamics

Clearly, it is desirable that the perturbation term \( \delta_3 \left( \delta_3 \right) \) in equation 30 be small, as this represents a perturbation to an otherwise exponentially stable sub-system. Analogously, it is also desirable that the cross term \( \delta_4^T \delta_3 \) in equation 20 be as small as possible as this represents a perturbation to an otherwise negative definite CLF derivative. With this in mind, we propose an alternative feedback law for \( T_z \) that minimises the \( L_2 \) norm of \( \delta_3 \). Provided the resulting control law ensures \( \delta_3 \rightarrow 0 \) as \( \delta_3 \rightarrow 0 \), the closed-loop cascade structure will be preserved. Denoting this new feedback law as \( q \) and substituting \( T_z = q \) into equation 19, we may write \( \left\| \delta_3 \right\| \) as:
\[
\left\| \delta_3 \right\| = \left\| \frac{1}{m} q R (\eta) e_z - \frac{1}{m} T_z R (\eta_d) e_z \right\| \tag{32}
\]

As we wish to minimise \( \left\| \delta_3 \right\| \) which is nonnegative, we may equivalently minimise \( \left\| \delta_3 \right\|^2 \). As this is quadratic with respect to \( q \), it is convex, and \( q \) is thus the solution to:
\[
\frac{\partial \left\| \delta_3 \right\|^2}{\partial q} = 2 \frac{1}{m} q R (\eta) e_z - \frac{1}{m} T_z R (\eta_d) e_z \cdot R (\eta) e_z = 0 \tag{33}
\]
resulting in:
\[
q = T_{zd} (R (\eta) e_z) \cdot R (\eta_d) e_z = T_{zd} (s d b d e \theta_d + c \theta c \theta_s d \theta \phi_d + c \theta c \theta_d c \theta c \phi_d) \tag{34}
\]
It is interesting to note that \( \left| q \right| \leq \left| T_{zd} \right| \) will hold for all state configurations. This suggests that the improved feedback structure will result in less control action than if the originally designed feedback law \( T_z = T_{zd} \) is used.

F. Stability Analysis

Theorem 1: The feedback control laws defined by equations 34 and 28 globally, exponentially stabilise the system defined by equation 8.

Proof: Firstly recall the quadratic Lyapunov function \( V_4 \) and its negative definite derivative \( \dot{V} \leq 0 \) (see equations 26 and 29). As a consequence of Lyapunov stability theory, the closed-loop sub-system defined by equation 31 is exponentially stable. In particular, \( \dot{\delta}_3 \) will converge exponentially to zero, and thus its \( L_2 \) norm may be shown to be bound over time:
\[
\left\| \delta_3 (t) \right\| \leq \gamma (\left\| \delta_3 (0) \right\|) e^{-\alpha t} \quad \forall t > 0 \tag{35}
\]
for some \( \alpha \in \mathbb{R}^+ \) and \( \gamma \) a non-decreasing function with \( \gamma (0) = 0 \). Substituting equation 34 into 32 we arrive at:
\[
\left\| \delta_3 \right\| = \frac{1}{m} T_{zd} \left\| (R (\eta) e_z) \cdot R (\eta_d) e_z - R (\eta_d) e_z \right\|. \tag{36}
\]
Noting that:
\[
(R (\eta) e_z) \cdot R (\eta_d) e_z - R (\eta_d) e_z = R (\eta) e_z \cdot R (\eta_d) e_z - R (\eta_d) e_z \cdot R (\eta) e_z
= 0 \tag{37}
\]
it is apparent that $\{ R(\eta) e_z \bullet R(\eta_d) e_z \} R(\eta) e_z - R(\eta_d) e_z$ is perpendicular to $R(\eta) e_z$, and thus we may write:

$$\| \delta_3 \| = \frac{1}{m} T_{zd} \| ((R(\eta) e_z \bullet R(\eta_d) e_z) R(\eta) e_z - R(\eta_d) e_z) \times R(\eta) e_z \|$$

$$= \frac{1}{m} T_{zd} \| R(\eta_d) e_z \times R(\eta) e_z \|$$

$$= \frac{1}{m} T_{zd} \left( c^2 \theta^2 c^2 \theta d^2 (\phi - \phi_d) + c^2 \theta^2 c^2 \theta d \right.$$

$$- 2 c \theta d \theta c \theta c (\phi - \phi_d) + \delta^2 \theta^2 c^2 \theta d \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{m} T_{zd} \left( |s(\phi - \phi_d)| + |s(\theta - \theta_d)| \right.$$

$$+ \sqrt{ |2 c \theta d \theta c \theta c (1 - c (\phi - \phi_d)) | }$$

$$\leq \frac{1}{m} T_{zd} \left( |s(\phi - \phi_d)| + |s(\theta - \theta_d)| + \left| s \left( \frac{\phi - \phi_d}{2} \right) \right| \right)$$

$$\leq \frac{1}{m} T_{zd} \left( \frac{3}{2} |s(\phi - \phi_d)| + |s(\theta - \theta_d)| \right)$$

$$\leq \frac{1}{m} T_{zd} \left( |s(\phi - \phi_d)| + |s(\theta - \theta_d)| \right)$$

$$\leq \frac{1}{m} T_{zd} \left( |s(\phi - \phi_d)| + |s(\theta - \theta_d)| \right)$$

$$\leq \frac{1}{m} T_{zd} \left( \frac{3}{2} m \sigma \right)$$

$$\leq m \sigma_{\lambda_{\max}} \| \chi \| + m g.$$

(38)

Recalling equation 20 and using equations 42, 35 and the fact that $\| \mu \| \geq \| \delta_2 \|$ it can be shown that:

$$\dot{V}_2 = - k_1 \| \delta_1 \|^2 - k_2 \| \delta_2 \|^2 + \delta_3^2$$

$$\leq \| \delta_2 \| \| \delta_3 \| \frac{3}{\sqrt{2}} \left( \frac{\sigma_{\lambda_{\max}}}{\sigma_{\lambda_{\min}}} \right) \| \mu \| + g$$

$$\leq \| \mu \|^2 \left( \left( \| \delta_3 (0) \| \right)^e^{-at} \left( \frac{\sigma_{\lambda_{\max}}}{\sigma_{\lambda_{\min}}} \right) \right)$$

for $\| \mu \| \geq g \pi \sigma_{\max} \sigma_{\min}$

(43)

Noting that $\| \mu \|^2 = 2 V_2$, we may write:

$$\dot{V}_2 \leq 3 \sqrt{2} V_2 \gamma \left( \| \delta_3 (0) \| \right)^e^{-at} \left( \frac{\tau_{\lambda_{\max}}}{\tau_{\lambda_{\min}}} \right)$$

(44)

and thus:

$$V_2 (\mu (t)) \leq V_2 (\mu (0)) e^{-\gamma (\| \delta_3 (0) \|) t}$$

(45)

As $V_2 (\mu)$ is radially unbounded, boundedness of $V_2 (\mu (t))$ implies boundedness of $\| \mu (t) \|$. Global stability follows from proposition 4.1 of [10].

4. Simulation Results

Simulations were performed for a $m = 5$ kg vehicle, with principle moments of inertia $I_{xx} = I_{yy} = 0.25$ kg-m$^2$, $I_{zz} = 0.2$ kg-m$^2$ and thrust eccentricity $l = 0.3$ m. Control gains $k_1, k_2, k_3$ and $k_4$ were selected such that when $\delta_3 = 0$, the eigenvalues of the closed loop sub-system describing vehicle translation (see equation 30) were $-2, -2$, and the eigenvalues of the closed loop sub-system describing vehicle orientation (see equation 31) were $-4, -4$.

Figure 2 shows the response of the controlled system, initially stationary and at $\lambda_0 = 0$ but with a nonzero rotation of $\phi_0 = -\frac{\pi}{2}$. Results for both the original controller using $T_z = T_{zd}$, and the improved controller using $T_z = Q = \emptyset$ are displayed. Figure 2 (a) demonstrates that both controllers aggressively reject the non-zero initial condition and return to equilibrium at $\lambda = 0$. However, it can be seen that the improved controller undergoes less translational displacement. This is to be expected, as the perturbation term to the translational dynamics $\delta_3$ has been minimised. Furthermore, figure 2 (b) shows that the improved controller uses less total thrust $T = \sqrt{T_z^2 + T_{zd}^2}$.

Figure 3 shows the response of the controlled system, initially stationary but displaced to $\lambda_0 = [5, 5, 5]$ and with a rotation of $\phi_0 = -\frac{\pi}{2}$. Again, figure 3 (a) demonstrates that both controllers aggressively reject the non-zero initial condition and the improved controller undergoes significantly less translational displacement. As expected, figure 3 (b) shows that the improved controller uses less total thrust.

5. Conclusion

An approximate nonlinear control design for a 6 DOF VTOL UAV has been presented, based on an extension of the ideas in [10] and [5], [10] for the non-linear control of the PVTOL.
An improvement to this design has been made by proposing an alternative feedback law that minimises the perturbation term in the resulting closed-loop translational sub-system. Global stability of this improved control law has been proven. Simulation results demonstrate aggressive disturbance rejection using both controllers, however the improved controller was shown to use less control action whilst achieving a smoother, more desirable response.

**REFERENCES**


