A mixed $\mathcal{H}_2/\mathcal{H}_\infty$ scheduling control scheme for a two degree-of-freedom aeroelastic system under varying airspeed and gust conditions

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This article investigates the control of a two degree-of-freedom aeroelastic system with a torsional stiffness nonlinearity. The dynamics are transformed into a Linear Fractional Representation (LFR) such that the nonlinear effects of airspeed on the dynamics act as a gain feedback to the nominal system. A controller in LFR, which allows it to schedule with airspeed, is then synthesised using Linear Matrix Inequalities (LMIs). The performance objectives of the controller are the minimisation of the $\mathcal{H}_2$ norm from a gust input to the pitch and plunge outputs, and the minimisation of an $\mathcal{H}_\infty$ norm that corresponds to the systematic method of $\mathcal{H}_\infty$ loop-shaping. This method has a rigorous mathematical background that allows upper limits on these criteria to be established. The nonlinear system and controller are simulated under a variety of varying airspeed and gust conditions and is shown to exhibit both good robustness and rejection of gust disturbances.

Nomenclature

- $\nu$ Upper $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance bounds
- $w_i, z_i$ Performance channels inputs and outputs
- $c, b, S$ Aerofoil chord, half-chord and span
- $\rho$ Air density
- $U$ Free-stream air-speed
- $y, u$ Measured plant output and control input
- $I, \alpha, m, W$ Rotational moment of inertia, wing mass and total system mass
- $G, G_s$ Nominal and scaled plant
- $a$ Distance from the elastic axis to the midchord, normalised against $b$
- $x_a$ Distance from the elastic axis to the centre of gravity of the aerofoil, normalised against $b$
- $\alpha, h, \beta$ Pitch, plunge and trailing-edge surface deflection states
- $C_{l_\alpha}, C_{m_\alpha}$ Aerofoil coefficient of lift and moment
- $c_\alpha, c_h$ Pitch and plunge damping coefficients
- $k_\alpha, k_h$ Pitch and plunge stiffness coefficients
- $x, x_K$ Plant state vector and the controller state vector
- $C_{l_\beta}, C_{m_\beta}$ Trailing-edge actuator coefficient of lift and moment
- $w_G$ Vertical gust velocity
I. Introduction

Aeroelasticity is the study of complex mechanical and aerodynamic interactions. Many different phenomena are present in these systems, including, but not limited to, limit cycle oscillations, flutter, gust load and divergence.\(^1\) These phenomena have been well understood for many years, and traditional design approaches to eliminate the effects of some of these destructive phenomena, for example flutter and divergence, involve designing the aerodynamic member such that the onset of these phenomena occur well outside the flight envelope. However through the use of active-control techniques it is possible to suppress and even utilise some of these aeroelastic effects to reduce weight and/or increase performance.\(^1\)

The two degree-of-freedom wing section which is allowed to pitch and plunge through springs has long been a testbed for novel aeroelastic control methodologies, with several different experimental apparatus, including the Benchmark Active Control Technology (BACT)\(^1–4\) and the Nonlinear Aeroelastic Test Apparatus (NATA)\(^5,6\) to validate models and evaluate control methods. In the past there have been many different control schemes that address different aeroelastic phenomena for this wing section. This work includes several flutter suppression control schemes, such as the use of pole-placement,\(^7\) mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance criteria,\(^8\) Linear Fractional Transformation (LFT) gain scheduling,\(^9,10\) Linear Parameter Varying (LPV) and Robust Multivariable techniques,\(^4\) robust control through simultaneously solving several Linear Matrix Inequalities (LMIs)\(^11\) and feedback compensation of transport lags,\(^12\) to name but a few.

There has also been interest in the models which feature a torsional stiffness nonlinearity, namely increasing stiffness with increasing pitch angle, which leads to limit cycle oscillations.\(^5,13–15\) Ref. 15 illustrate how a dynamic model using nonlinear torsional stiffness and assuming quasi-steady aerodynamics matches experimental data very well. This model, as well as variants thereof, have been used with several different active control schemes to suppress the limit cycle oscillations, including a Linear Quadratic Gaussian (LQG) controller,\(^14\) adaptive feedback linearisation schemes,\(^13,15\) an adaptive backstepping scheme with state estimation filters,\(^16,17\) and several adaptive backstepping schemes for model variations with unsteady aerodynamics\(^18\) and models with a leading-edge actuator.\(^6,19\)

This article investigates limit cycle oscillation suppression of a two degree-of-freedom aeroelastic system with a torsional stiffness nonlinearity. The control scheme utilised is a LFT/LPV gain-scheduling controller similar to those in Refs. 9 and 10, but with the contribution of utilising mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance criteria. This mixed objective design, despite being considerably more involved than a single criteria case, allows the controller to be directly designed to suppress limit-cycle oscillations, minimise the response to a gust disturbance (the $\mathcal{H}_2$ criteria) and maximise the closed-loop robustness using a loop-shaping $\mathcal{H}_\infty$ design across a chosen airspeed range. This is in opposition to many previous control schemes which have typically designed for a single criteria or at a single operating airspeed.

The LFT/LPV gain-scheduling approach used in this paper is based on representing and solving the control problem as a set of LMIs. LMIs are a set of matrix inequalities that are affine in a set of variables. This affine nature renders the inequalities convex, thus large problems can be solved relatively quickly. The emergence of several efficient solvers, such as the excellent SeDuMi\(^20\) (and the equally excellent LMI frontend for MATLAB; YALMIP\(^21\)), and the continuing exponential increase in computing power has led to the popularity of LMIs.

An LPV system is a nonlinear system whose properties vary with some set of parameters, and the plant itself can be considered linear at each point of the set of parameters. When using LMIs to synthesise control laws for LPV systems, it is possible to apply the rigorous and mature linear control methodologies to determine such criteria as stability, robustness and performance across the parameter set. These control laws can also be gain-scheduled while providing the same guarantees of performance, robustness and stability. This is in contrast to ad hoc gain-scheduling methods that have been popular in the past. More detailed discussions of gain-scheduling are given in Refs. 22–24. Some applications of
LPV control of the nonlinear aeroelastic system are given in the already mentioned Ref. 3, as well as Ref. 25.

The LFT/LPV approach used in this paper has successfully been used in the construction of controllers for an arbitrary uncertain system, and auto-pilot control of a missile over a wide range of operating conditions. However it is worth noting that the performance criteria in these articles and that used in this paper differ greatly. Ref. 26 minimise the $\mathcal{H}_\infty$ gain across an uncertain variable and the $\mathcal{H}_2$ performance across two scheduled variables, while Ref. 27 minimise the $\mathcal{H}_\infty$ gain across an uncertain channel and a weighted $\mathcal{S}_K$ channel, and the $\mathcal{H}_2$ performance across the sensitivity function. The $\mathcal{H}_2$ performance channel used in this paper is the gust response, and the $\mathcal{H}_\infty$ channel the four plants that correspond to the systematic method of loop-shaping.

This paper is organised into the following sections, §II contains a derivation of the nonlinear aeroelastic dynamic model, §III details the controller performance criteria and formulation of the LMI problem, §IV presents simulations and a discussion of the results, and finally some concluding remarks are made in §V.

II. Dynamic model

The aeroelastic model used in this study is based on the typical-section aeroelastic model, and used in several previously published aeroelasticity studies. The model consists of a rigid two degree-of-freedom aerofoil section, as shown in Figure 1. The model is connected to a rigid mount by a rotational spring, with nonlinear stiffness $k_\alpha(\alpha)$, and a linear translational spring, of stiffness $k_h$, which allows the model to pitch ($\alpha$) and plunge ($h$). The aerofoil section also has a trailing-edge actuator, the deflection of which relative to the aerofoil is denoted $\beta$.

![Two degree-of-freedom aeroelastic model, showing dimensional parameters and dynamic states.](image)

The mechanical dynamics, which neglect the inertial terms of the trailing-edge surface, of this system are represented in matrix form as a second-order system, given by

\[
\begin{bmatrix}
    m_T & m_W x_a b \\
    m_W x_a b & I_a
\end{bmatrix}
\begin{bmatrix}
    \dot{h} \\
    \dot{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
    c_h & 0 \\
    0 & c_\alpha
\end{bmatrix}
\begin{bmatrix}
    \dot{h} \\
    \dot{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
    k_h & 0 \\
    0 & k_\alpha(\alpha)
\end{bmatrix}
\begin{bmatrix}
    h \\
    \alpha
\end{bmatrix}
= \begin{bmatrix}
    -L \\
    \mathcal{M}_y
\end{bmatrix}
\] (1)

where $m_T$ is the total mass of the translating system (including any mounting apparatus), $m_W$ is the mass of the aerofoil, $I_a$ is the rotational moment of inertia of the aerofoil, and $c_\alpha$ and $c_h$ are the damping coefficients for pitch and plunge motion respectively. The off-diagonal terms in the inertia matrix, $m_W x_a b$, represent the line moment of inertia of the model, and effectively couple the pitch and plunge states.

The aerodynamic model used for this study is a quasi-steady model, which has the aerodynamic lift force and moment as

\[
L = \rho U^2 b SC_{\alpha} \left( \frac{\dot{h}}{U} + \left( \frac{1}{2} - a \right) \frac{\dot{\alpha}}{U} + \frac{w_G}{U} \right) + \rho U^2 b SC_{\beta} \beta
\] (2a)

\[
\mathcal{M}_y = \rho U^2 b^2 SC_{\alpha\alpha} \left( \frac{\dot{h}}{U} + \left( \frac{1}{2} - a \right) \frac{\dot{\alpha}}{U} + \frac{w_G}{U} \right) + \rho U^2 b^2 SC_{\alpha\beta} \beta.
\] (2b)
the effective angle of attack due to vertical gusts (\(w_G\)). When substituting the aerodynamic Equation (2) into the mechanical dynamics Equation (1), the aerodynamic terms that depend upon the aerofoil motion can be rearranged to make the dynamics appear in the form of a second order system as

\[
\begin{bmatrix}
  m_T & m_W \dot{x} \\
  m_W \dot{x} & I_a
\end{bmatrix}
\begin{bmatrix}
  \dot{h} \\
  \dot{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
  c_h + \rho U b SC_{1a} & \rho U b^2 SC_{1a} (1 - a) \\
  -\rho U b^2 SC_{ma} & c_a - \rho U b^2 SC_{ma} (1 - a)
\end{bmatrix}
\begin{bmatrix}
  h \\
  \alpha
\end{bmatrix}
+ \begin{bmatrix}
  k_h & \rho U b SC_{1a} \\
  0 & k_a(\alpha) - \rho U b^2 SC_{ma}
\end{bmatrix}
\begin{bmatrix}
  \dot{h} \\
  \dot{\alpha}
\end{bmatrix}
= \begin{bmatrix}
  -\rho U^2 b SC_{13} & -\rho U b SC_{1a} \\
  \rho U^2 b^2 SC_{ma,3} & \rho U b^2 SC_{ma}
\end{bmatrix}
\begin{bmatrix}
  \beta \\
  \dot{w}_G
\end{bmatrix}.
\]

Choosing a system state vector as \(x = [\dot{h} \ \dot{\alpha} \ h \ \alpha]^T\), with control input \(u = \beta\) and measurable output \(y = \alpha\), a standard state-space representation, with parameter dependence shown, can be written as

\[
\begin{bmatrix}
  \dot{x} \\
  y
\end{bmatrix} = \begin{bmatrix}
  A(U, U^2, k_a(\alpha)) & B(U^2) \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  x \\
  u
\end{bmatrix}.
\]

To visualise how this parameter dependence on airspeed and torsional stiffness affects the system dynamics a family of airspeed root loci are shown in Figure 2. The system parameters used to define the system and consequently generate this figure, and used throughout the rest of this article are taken from Ref. 15, and are repeated in Appendix A. The increasing stiffness corresponds to an increasing \(\alpha\), ranging from \(\alpha = 0\) rad to \(\alpha = 0.2\) rad. As can be seen, the increasing stiffness tends to push the poles into the left-half plane, thus increasing stability as would be expected. Additionally due to the high degree of ‘evenness’ in the spring stiffness, and given that this stiffness is at its minimum at the equilibrium point, the aeroelastic system will always be at its least stable at the equilibrium point. While the torsional stiffness presented in Appendix A may appear to be dominated by the \(\alpha^4\) term, which is in fact negative, the operational range for the spring is quite low, ranging from approximately \(-0.2\) to \(0.2\) radians. Thus the stiffness is dominated by the constant and \(\alpha^2\) terms. For these reasons, the torsional spring stiffness nonlinearity will be neglected for the controller design, instead the spring stiffness at the equilibrium point will be used. The effect of airspeed upon the system stability is far more complicated, thus it will be this particular nonlinearity that will form the focus of this investigation.

III. Controller Synthesis

A. Performance Criteria

The control method proposed in this paper is a mixed \(\mathcal{H}_2/\mathcal{H}_\infty\) performance problem. The dynamics are thus augmented to include both of these performance channels as

\[
\begin{bmatrix}
  \dot{x} \\
  z_1 \\
  z_2 \\
  y
\end{bmatrix} = \begin{bmatrix}
  A(\delta) & B_1(\delta) & B_2(\delta) & B(\delta) \\
  C_1 & D_{11} & D_{12} & E_1 \\
  C_2 & D_{21} & D_{22} & E_2 \\
  C & F_1 & F_2 & D
\end{bmatrix}
\begin{bmatrix}
  x \\
  w_1 \\
  w_2 \\
  u
\end{bmatrix}
\]

where the transfer functions from \(w_j\) to \(z_j\) are the channels on which the control objectives are to be imposed, with \(j = 1\) corresponding to the \(\mathcal{H}_2\) performance channel and \(j = 2\) corresponding to the \(\mathcal{H}_\infty\) performance channel.

For this investigation the first control objective is to minimise the \(\mathcal{H}_2\) norm from the vertical gust...
velocity disturbance input, \( w_1 = w_G \), to the pitch and plunge outputs, \( z_1 = [h \quad \alpha]^T \). This corresponds to a minimisation of the RMS gain of the frequency response, as used in optimal control techniques.

The \( \mathcal{H}_\infty \) performance channel was chosen to correspond to a \( \mathcal{H}_\infty \) loop-shaping design, which provides a systematic and simple method for constructing a controller robust to plant uncertainties without knowledge of the uncertainty.\textsuperscript{29,30} A theoretical justification for the loop-shaping process is given in Ref. 29. \( \mathcal{H}_\infty \) loop-shaping is a two-step process, where the open-loop plant singular values are first shaped using pre- and post-compensators, then the shaped plant is robustly stabilised. Standard rules for desired open-loop singular values are that high gain is required at low frequencies, low slope around the 0dB crossover (which also corresponds to the closed-loop bandwidth), and low gain at high frequencies.\textsuperscript{30}

For the aeroelastic system being investigated the pre-compensator consists of a gain of 10, an integrator to increase the low-frequency gain, a pair of complex zeros at \( s = -5 \pm 5i \text{rad/s} \) to reduce the slope at the 0dB crossover, and two poles at \( s = -50 \text{rad/s} \) to reduce the high-frequency gain, demanded actuator bandwidth and yield the pre-compensator strictly-proper. Thus the pre-compensator transfer function is:

\[
W_i = \frac{0.2s^2 + 2s + 10}{0.0004s^3 + 0.04s^2 + s}.
\] (6)

Both the plant singular values, and the compensated plant singular values are shown at various airspeeds in Figure 3.

The robustly stabilising controller, \( K_\infty \), is the controller which minimises the \( \mathcal{H}_\infty \) norm of the set of plants:\textsuperscript{31}

\[
\inf_{K_\infty \text{ stabilising}} \left\| \begin{bmatrix} K_\infty & I \end{bmatrix} (I - G_s K_\infty)^{-1} \begin{bmatrix} I & G_s \end{bmatrix} \right\|_\infty \leq \gamma
\] (7)

where \( G_s = GW_i \) is the scaled plant. It is shown in Ref. 29 that this corresponds to minimising the \( \mathcal{H}_\infty \) norm of the signal relationship from both the scaled plant input and output to the scaled plant output and input, that is minimising the transfer function from \( w_2 \) to \( z_2 \), where:

\[
w_2 = \begin{bmatrix} \hat{u} \\ \hat{y} \end{bmatrix}, \quad z_2 = \begin{bmatrix} \hat{y} \\ \hat{u} \end{bmatrix}
\] (8)
where \(\tilde{u}\) is the scaled plant input, and \(\tilde{y}\) is the plant output, noting that there is no post-compensator in this instance, thus the scaled plant output is equal to the plant output.

### B. Linear Fractional Representation

The nonlinear dependence of the system dynamics on airspeed can be represented in Linear Fractional Representation (LFR) by applying a Linear Fraction Transformation (LFT). LFR was chosen for this system primarily due to the rational dependence of the system dynamics on the airspeed. LFR allows this rational dependence to be resolved to a repeated feedback of linear airspeed terms. During controller synthesis, this allows the LMIs to be solved at only two vertices, the maximum and minimum values of the LFR feedback, which corresponds to the maximum and minimum airspeed being considered. This is opposed to direct LPV schemes where the airspeed and airspeed squared terms are treated as separate variables, and the LMIs solved at a series of grid points constructed across the airspeed and airspeed squared terms, as performed in Ref. 25. The main disadvantage of constructing a grid across these parameters is that overly coarse grids introduce conservatism, whilst fine grids significantly increase the problem complexity.

A typical LFR requires that the uncertainty feedback be normalised to less than unity, thus the airspeed and airspeed squared terms are decomposed into

\[
U = U_0 + \delta U_d
\]

\[
U^2 = U_0^2 + 2\delta U_0 U_d + \delta^2 U_d^2
\]

where \(U_0\) is the median airspeed about which the controller is designed, \(U_d\) is the deviation magnitude of the airspeed and \(\delta \in [-1, 1]\) is the uncertain term. The form of this LFR is illustrated in Figure 4 where the corresponding state representation is
where the uncertainty input, \( w_u \), is the result of passing the uncertainty output, \( z_u \), through the uncertainty gain, \( \Delta \), such that

\[
\dot{w}_u = \Delta z_u
\]

The \( U \) and \( U^2 \) subscripts on the state and input matrices denote the terms in these matrices that are independent of airspeed, dependent on airspeed and dependent on the square of airspeed respectively. The uncertainty output vector and uncertainty matrix are respectively given by

\[
z_u = \begin{bmatrix} A_{U,11} \ddot{h} + A_{U,12} \dot{\alpha} + B_{1U} \end{bmatrix}
\]

where \( A_{U,11} \) denotes the \( (1,1) \) element of the component of the state matrix that depends on \( U \). For this particular LFR, the linear relationship between the airspeed dependent terms of the first two rows (the \( \ddot{h} \) and \( \dot{\alpha} \) output rows) is exploited to reduce the number of LFR elements.

It is important to ensure that this LFR is well-posed, that is, that the uncertainty feedback loop is stable. This corresponds to the test that \( I - D_{uu} \Delta(\delta) \), is invertible for all \( \delta \in \delta \), where \( \delta := [-1, 1] \) is the uncertainty set. Given the sparseness of the \( D_{uu} \) matrix (only two 1’s corresponding to the points where the output uncertainty vector directly outputs the uncertainty input vector) it is easily shown that \( I - D_{uu} \Delta(\delta) \) is always invertible, hence this LFR is well-posed for all \( \delta \in \delta \).
C. Linear Matrix Inequality Computation

The mixed $H_2/H_\infty$ LFT gain-scheduling is performed using the LMIs derived in Ref. 26, and the procedure more explicitly outlined in Ref. 27. This procedure involves an LMI search for a discrete controller in LFR,\textsuperscript{26} of the form

$$
\begin{bmatrix}
x_K(k+1) \\
u(k) \\
z_K(k)
\end{bmatrix} =
\begin{bmatrix}
A_K & B_{K1} & B_{K\Delta} \\
C_{K1} & D_{K11} & D_{K1\Delta} \\
C_{K\Delta} & D_{K\Delta1} & D_{K\Delta\Delta}
\end{bmatrix}
\begin{bmatrix}
x_K(k) \\
y(k) \\
w_K(k)
\end{bmatrix}
$$

(14)

where $x_K$ is the controller state vector, $y(k)$ and $u(k)$ are the controller input and output respectively, which correspond to the plant measurable output and control input respectively, and $w_K(k)$ and $z_K(k)$ are the controller LFR input and output respectively, which feed through the scheduling block $\Delta_K$. The details of this configuration are more clearly shown in Figure 5.

**Figure 5. Closed-loop LFR plant with a gain-scheduling LFR controller**

The LMI search for the controller equation (14) aims to guarantee upper limits $\sqrt{\nu}$ and $\gamma$, on the $H_2$ (from $w_1$ to $z_1$) and $H_\infty$ (from $w_2$ to $z_2$) performance channels respectively.

The controller must be synthesised in the discrete domain due to there being no continuous domain counterpart to this LFR gain-scheduling approach.\textsuperscript{27} This requires that the system model be converted to the discrete domain, which can be accomplished using a bilinear (Tustin) approximation. Once the controller is computed it can either be converted back to the continuous domain using a discrete to continuous bilinear (Tustin) approximation, or directly implemented in the discrete domain. If converting the controller back to the continuous domain, then the sampling frequency chosen for the transformation is unimportant so long as it yields a numerically tractable discrete model. A sampling frequency that is too fast will tend to force the discrete poles to +1, while a sampling frequency that is too slow will tend to force the discrete poles to −1, both of which can lead to numerical problems and should be avoided. The dynamics as shown in Equation (12) were initially converted to the discrete domain using a bilinear transformation. The sampling frequency used in the transformation was chosen to be 20Hz, or approximately two and a half times the speed of fastest poles in the scaled plant, for reasons outlined above. While this is less than the typical rule of thumb, it yielded numerically reliable results, and since the controller is transformed back to the continuous domain after completion numerical reliability is the only concern. Once the model is converted to the discrete domain LMIs were constructed in MATLAB using YALMIP\textsuperscript{21} and solved using SeDuMi.\textsuperscript{20} The performance indicators, $\sqrt{\nu}$ and $\gamma$, were firstly weighted with a variable $\eta$, and the optimisation objective was set to minimise $(\eta\nu + (1 - \eta)\gamma)$. This was repeated
at various points over the range $\eta \in [0, 1]$, and repeated again for several different values of $U_\delta$ about a median airspeed of $U_0 = 18$ m/s. These curves, which represent the minimal achievable performance values are shown in Figure 6.

Figure 6. Optimal performance values with median airspeed, $U_0 = 18$ m/s, and changing airspeed deviation, $U_\delta$. The trade-off between robustness (the $H_\infty$ value) performance (the $H_2$ value) and airspeed design range (the airspeed deviation $U_\delta$) is evident. The point used in the controller synthesis when $U_\delta = 6$ m/s is also highlighted. This point is sub-optimal, so it lies inside the $U_\delta = 6$ m/s curve, not on the $U_\delta = 12$ m/s curve. Note that the curves have been filtered to remove numerical noise.

It is well known that $H_\infty$ control problems typically become numerically intractable at their optimum values,\textsuperscript{30} thus a suboptimal point was chosen for the final controller synthesis. Setting the airspeed deviation $U_\delta = 6$ m/s, the suboptimal point which achieves good numerical tractability was chosen as $\gamma = 3.12$ and $\sqrt{\nu} = 0.073$, which is highlighted in Figure 6. After obtaining a numerically reliable solution, the controller is recovered by reversing the linearising change of variables in Ref. 27.

The resulting closed-loop plant frequency responses for the $H_\infty$ loop-shaping criteria, equation (7), are shown in Figure 7, and the closed-loop frequency responses from the vertical gust input to the pitch and plunge outputs are shown in Figure 8. Note that these frequency responses are for the aeroelastic system with the torsional stiffness linearised to $k_\alpha = 6.833$ Nm/rad.

IV. Results and Discussion

The nonlinear aeroelastic system of §II was simulated with the controller generated in §III. Initially the plant was perturbed to $h = 0.01$m and $\alpha = 0.05$rad, and simulated at $U = 18$m/s without the controller active during which time limit-cycle oscillations developed. These limit-cycle oscillations are a result of the torsional stiffness nonlinearity used in the simulations. The increasing stiffness with deflection angle prevents the system going unstable at airspeeds where the linearised model would be unstable, instead resulting in limit-cycle oscillations.

The controller was activated at $T = 5$ s, after which the controller successfully manages to suppress these limit cycle oscillations, settling the pitching motion in less than a second. This is shown in Figure 9. Notice that when suppressing the limit-cycle oscillations, the commanded trailing-edge surface deflections are quite modest in frequency and the amplitude did not reach the saturation point of $|\beta|_{\text{max}} = 12^\circ$, unlike many previous studies which greatly exceed any reasonable deflection or saturate.\textsuperscript{16,18,19}

For the sake of comparison, the simulation results from the controller in Ref. 17, which is based on nonlinear adaptive control theory, are shown in Figure 10. While the response is very similar to that
Figure 7. Linearised closed-loop Sensitivity (top left), Complementary Sensitivity (top right) and the two other transfer functions which form the $H_\infty$ loop-shaping criteria, Equation (7). The dashed line indicates the guaranteed maximum $H_\infty$ norm, $\gamma$.

Figure 8. Linearised closed-loop frequency response from the vertical gust input, $w_G$, to the pitching angle, $\alpha$ (solid line), and the plunge displacement, $h$ (dashed line). The $H_2$ performance is the integral of the maximum of all curves.
shown in Figure 9, the actuator demands from the controller in this study are less aggressive and more likely to be realised in practice.

![Figure 9. Controller simulation at $U = 18 \text{ m/s}$. The limit cycle oscillations develop, and at $T = 5 \text{s}$ the LFR gain-scheduled controller is activated.](image)

Next, the closed-loop model was simulated while accelerating at $\dot{U} = 2 \text{ m/s}^2$ from an initial speed of $U = 10 \text{ m/s}$. In addition a square-edged (or sharp-edged) gust starting at $x = 24 \text{ m}$ and lasting 75 metres (from $T = 2 \text{s}$ to $T = 6.1 \text{s}$) was applied. Ref. 28 state that a typical gust velocity is $w_G = 50 \text{ft/s}$ ($15.2 \text{ m/s}$), however when the aerofoil is operating at such a low velocity, this gust velocity would cause the linearisation assumption implicit within the quasi-steady aerodynamic model to break down. Thus a much more modest gust velocity of $w_G = 3 \text{ m/s}$ was used. The results of this simulation are shown in Figure 11. The LFR gain-scheduled controller quickly returns the pitching angle back to its equilibrium, again with only moderate trailing-edge surface demands. Notice that the plunge displacement does not return to the origin, but is instead biased by an amount that could be calculated using static deflection.

![Figure 10. Simulation results from the controller in Ref. 17 at $U = 18 \text{ m/s}$. Again the limit-cycle oscillations are allowed to develop, and the controller is activated at $T = 5 \text{s}$. Note the more aggressive actuator demands than those from the controller synthesised in this study.](image)
formula of the plunge spring due to the lift force. The controller is unable to suppress this plunge displacement bias due to the inability of the single actuator to regulate both pitch and plunge at the same time, and the choice of pitching angle as the measurable output. This behaviour can be explained by performing a singular value decomposition of the system with all state outputs, then examining the output vector directions for the zero singular values. Finally note that positive plunge was defined in the opposite direction to the lift force, which is why a positive gust causes a negative plunge deflection.

![Graph showing LFR gain-scheduled controller](image)

Figure 11. Closed loop aerofoil response to a square-edged gust of 3 m/s over 40 m, while the aerofoil is accelerating.

The above simulation was repeated using a sinusoidal gust, representative of a single vortex or turbulent eddy. The maximum gust strength was again \( w_G = 3 \text{ m/s} \), and the gust duration was 50 metres (from \( T = 3 \text{ s} \) to \( T = 5.8 \text{ s} \)). The simulation results are shown in Figure 12. Again the plunge displacement bias closely follows that which would be expected from a static analysis of the lift force for reasons discussed above. While the pitching angle does not return to zero during the gust, its amplitude is kept sufficiently small (note the change in the pitch scale from the previous simulation results). Again the deflection of the trailing-edge surface is quite moderate.

This LFR gain-scheduled controller has been shown to effectively suppress limit-cycle oscillations in the nonlinear aeroelastic system, whilst being able to schedule with airspeed and alleviate the gust response. By combining the systematic method of \( \mathcal{H}_\infty \) loop shaping to generate a robust controller with the \( \mathcal{H}_2 \) performance criteria on the gust response, the controller is able guarantee good robustness and performance close to their optimum values. As previously stated, the LFR control scheme has the advantage of being able to represent the rational dependence on airspeed as a single feedback parameter. However this scheme also does not allow for bounds of the rate of parameter variation to be taken into account, which makes the control scheme conservative. A viable alternative would be to directly generate the controller using LPV methods, which can take into account the maximum rate of parameter variation, however the drawback is the inability to treat the rational dependence on airspeed directly, requiring the
parameter space of airspeed and airspeed squared to be closely gridded to reduce conservatism.

An advantage of designing this controller using this LMI method is the rigorous mathematical foundation which guarantees of stability and performance that are not provided by other scheduling control schemes. Additionally, since the aeroelastic model considered features only minor nonlinearities, it lends itself well to these LMI control schemes. While the torsional stiffness nonlinearity was not directly considered by the control scheme presented here, it could be incorporated into a velocity-based LPV scheme. However as discussed before, the dominating polynomial stiffness terms are useful for stability, and the benefit from cancelling the odd stiffness terms would likely be negligible.

V. Conclusion

A two degree-of-freedom aeroelastic system control method based on LFR gain scheduling techniques has been presented. It was shown that the rational dependence of the aeroelastic system dynamics on airspeed can be transformed into a linear system with airspeed dependence using a Linear Fractional Transformation.

By augmenting the plant with a \( \mathcal{H}_2 \) performance channel from a gust input to pitch and plunge outputs, and a \( \mathcal{H}_\infty \) performance channel that corresponds to the systematic method of loop-shaping, a controller that is robust and rejects gust disturbances was synthesised.

The neglection of the torsional stiffness nonlinearity had negligible effects on the performance of the controlled nonlinear aeroelastic system. This was validated by nonlinear simulations, as the controller was shown to robustly stabilise the aeroelastic system and reject gust disturbances.

Although the controller performs well, it is conservative in that it does not take into account the finite rate of airspeed variation. Thus alternative schemes that directly use LPV techniques may improve
the closed-loop system performance.

A. System Parameters

The system parameters are taken from Ref. 15. The nonlinear torsional spring is modelled as a fifth order polynomial as

\[ k_\alpha(\alpha) = 6.833 + 9.967\alpha + 667.685\alpha^2 + 26.56\alpha^3 - 5087.931\alpha^4 \text{Nm/rad} \] (15)

and the remaining system parameters are given in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_T)</td>
<td>12.387 kg</td>
</tr>
<tr>
<td>(m_W)</td>
<td>2.049 kg</td>
</tr>
<tr>
<td>(I_\alpha)</td>
<td>(m_W x_\alpha^2 b^2 + 0.0517 \text{ kg} \cdot \text{m}^2)</td>
</tr>
<tr>
<td>(S)</td>
<td>0.6 m</td>
</tr>
<tr>
<td>(b)</td>
<td>0.135 m</td>
</tr>
<tr>
<td>(x_\alpha)</td>
<td>([0.0873 - (b + ab)]/b) m</td>
</tr>
<tr>
<td>(\rho)</td>
<td>1.225 kg/m³</td>
</tr>
<tr>
<td>(C_{l\alpha})</td>
<td>6.28</td>
</tr>
<tr>
<td>(C_{l\beta})</td>
<td>3.358</td>
</tr>
<tr>
<td>(C_{m\alpha})</td>
<td>((0.5 + a)C_{l\alpha})</td>
</tr>
<tr>
<td>(C_{m\beta})</td>
<td>-1.94</td>
</tr>
<tr>
<td>(k_h)</td>
<td>2844.4 N/m</td>
</tr>
<tr>
<td>(k_\alpha(\alpha))</td>
<td>see Equation (15)</td>
</tr>
<tr>
<td>(c_h)</td>
<td>27.43 kg/s</td>
</tr>
<tr>
<td>(c_\alpha)</td>
<td>0.036 kg·m²/s</td>
</tr>
<tr>
<td>(a)</td>
<td>-0.6847</td>
</tr>
</tbody>
</table>

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