Modal analysis of a submerged spherical point absorber with asymmetric mass distribution

Fantai Meng\textsuperscript{a,*}, Benjamin Cazzolato\textsuperscript{a}, Boyin Ding\textsuperscript{a}, Maziar Arjomandi\textsuperscript{a}

\textsuperscript{a}The University of Adelaide, School of Mechanical Engineering, Adelaide, Australia

Abstract

Of all the Wave Energy Converter (WEC) categories, the single-tethered point absorber (PA) is one of the most widely used in the ocean renewable energy industry. In most published research, only the heave motion of the buoy is considered in the motion equation for the analysis. This is because the heave motion of the buoy strongly couples to the power take-off device (PTO), whereas the surge and pitch motions barely couple to the PTO. As a result, only the power arising from heave motion of the buoy can be efficiently absorbed when single-tether PTO is used, leading to deficiency of the design in absorbing the power arising from its surge and pitch motion. In this paper, the deficiencies of single-tethered PAs are addressed by simply shifting the center of gravity of the buoy away from the buoy’s geometric centre. A spherical buoy with asymmetric mass is used in this paper for its simplicity. The asymmetric mass distribution of the buoy causes motion coupling across surge, heave and pitch motions, which enables strong coupling between the buoy’s surge motion and the PTO movement. The operation principle and power generation of the spherical point absorber with asymmetric mass distribution (SPAMD) are investigated via a modal analysis conducted on a validated frequency-domain model. The results show that for regular waves the SPAMD can be three times more efficient than the generic PAs.

Keywords:
Wave energy converter (WEC), spherical point absorber with asymmetric mass, hybrid frequency domain model, modal analysis.

*Corresponding author

Email address: fantai.mengo@adelaide.edu.au (Fantai Meng)
1. Introduction

Since the Oil Crisis of the 1970s, ocean wave energy has been regarded as a potential alternative renewable power source. Compared with solar and wind, the power carried by ocean waves is more continuous and predictable. However, it is difficult to extract the energy from the reciprocating ocean wave motion efficiently by using conventional electricity generators. Consequently, commercial-scale wave energy conversion still does not exist.

The single-tethered point absorber (PA) is the wave energy converter (WEC) that has commercial potential and has received significant attention from research areas. For example, Barbarit et al. [1] assessed the competitiveness of three different single-tethered PA designs in their technical review report. In most published work, single-tethered PAs are typically modelled as single degree-of-freedom (DOF) heaving devices, even though in reality the devices move in multiple DOFs (e.g. surge, heave and pitch). This is because, for single-tethered PAs, the heave motion of the buoy strongly couples to the power take-off device (PTO) and therefore this motion can be fully converted to the PTO extension. In contrast, the surge and pitch motions barely couple to the PTO and only a tiny fraction of these motions are converted to useful energy. Figure 1 illustrates the contribution of the PTO extension from the heave and surge motions respectively for a single-tether PA. It is clear that the heave displacement of the buoy completely converts to an equivalent PTO extension, whereas the surge displacement leads to negligible PTO extension. Therefore, for single-tethered PAs, only the heave motion can result in an effective power absorption.

Considering the theoretical capture width of a 3DOF (i.e. surge, heave and pitch) PA can be three times greater than a heave-only PA [2], several conceptual designs have been proposed to maximize the absorption efficiency of the PA by harvesting the energy arising from its surge and pitch motions. One typical solution is to attach multiple PTO tethers to the buoy, to allow the motions of the buoy in multiple degrees of freedom to strongly couple to the PTO elongations; for example, a PA with a three-cable PTO [3]. A three-cable PTO has the capability of absorbing three times more power than a single-tethered heaving PA over a broad frequency range [4], although by having two additional PTOs and mooring points this design results in a higher capital cost. A similar solution is to use two decoupled PTOs in alignment with the heave and pitch directions to capture more wave energy [5]. The theoretical capture width of this approach is equivalent to that of
the PA with a three-cable PTO. However, the PA with two decoupled PTOs is sensitive to wave direction, since the PTO must be aligned to the incoming wavefront to be efficient.

In this paper, a more cost-effective solution that allows a single-tethered PA to harvest energy arising from surge motion of a submerged spherical buoy is proposed. The approach is based on simply offsetting the mass from the centre of the buoy, such that when the buoy is excited in surge, heave motion is also enhanced. A submerged 3DOF (i.e. surge, heave and pitch) PA is employed because it can more efficiently use the surge motion to capture wave energy than an equivalent floating device [6]. In Section 2 the system of spherical point absorber with asymmetric mass distribution (SPAMD) is described, with the settings of operating environment, the asymmetric mass buoy and the PTO clarified. In Section 3 the static stability condition of the SPAMD is investigated. Furthermore, the motion equations are derived in the frequency domain for the subsequent modal analysis. In Section 4 a modal analysis is presented, with the aim of understanding the operation principles of the SPAMD and evaluating its power generation capability.
2. System description

For simplicity, a submerged spherical asymmetric mass buoy with a positive buoyancy is considered in this work. The buoy is tethered by a linear spring-damper PTO to be immersed below the free water surface. The PTO is anchored to the sea bottom via a ball-joint which allows the PTO to align with the mooring tether under tension when the buoy is excited by incident waves. The incident waves are set to be linear monochromatic waves aligned with the vertical $XZ$-plane of the Cartesian space, propagating along the positive $X$ axis, as shown in Figure 2. As the SPAMD is designed to be symmetric about the $XZ$-plane, it is assumed that the buoy only moves in the plane with surge, heave and pitch motion when excited by the plane waves.

The detailed descriptions of the operating environment, the asymmetric mass buoy, and the PTO are presented below.

Operating environment

The SPAMD operates in a finite depth water column. In this paper, the water depth is assumed to be 30 m, which is inspired by Carnegie’s CETO 6 design [7]. The submergence depth of the buoy is 2 m from top of the buoy to the sea surface, to be positioned to gain more incident wave power near the water surface. The frequency of incident monochromatic waves ranges from 0.34 rad/sec to 1.4 rad/sec, covering the major wave frequencies off the Australian coasts [8]. The wave height is set to 0.5 m, which is sufficiently small to meet the linear wave assumption used in the modelling.
Asymmetric mass buoy configuration

Figure 3 shows the free-body diagram of SPAMD buoy in the vertical XZ-plane in the Cartesian space. The buoy coordinates are defined in a body-fixed frame with an origin located at the centre of the buoy. The SPAMD buoy consists of a solid spherical body with a smooth surface and a point mass offset from the centre of the buoy on the XZ-plane.

The solid spherical body has a uniform mass distribution, with a radius of \( r \). The mass of the solid spherical body can be simplified as a point mass, \( m_1 \), located at the geometric centre of the body, with a moment of inertia about the centre of the buoy, \( I_1 = \frac{2}{5} m_1 r^2 \).

The offset point mass, \( m_2 \), is offset from the centre of the buoy by an offset distance, \( r_{gy} \), and an offset angle, \( \varphi \). The angle \( \varphi \) is measured from the positive X axis to the point mass, as shown in Figure 3. The coordinates of the offset point mass are denoted as \((x_2, z_2)\). The offset point mass introduces an additional moment of inertia about the centre of the buoy, \( I_2 = m_2 r_{gy}^2 \).

In this work, the mass of the solid spherical body, \( m_1 \) and the offset point mass, \( m_2 \), are determined by the weight to buoyancy ratio, \( \delta \), and the mass ratio, \( m_1/m_2 \). Table 1 lists the buoy's parameters used in the modal analysis. The weight to buoyancy ratio is set to 0.5, and the mass ratio to 1, both of which are technically achievable in practice. The radius of the

---

**Figure 3**: Free body diagram of the asymmetric mass buoy in the vertical XZ-plane. The origin is at the centre of the buoy.
buoy is 5m, to be consistent with previous research on point absorbers [6]. In order to explore the maximum efficacy of the SPAMD, the mass offset distance is defined as 5m, and the offset angle is set as −30 deg, which are the optimal configuration for most wave conditions [9]. Design optimisation of the SPAMD will also be discussed in the following paper [9].

Table 1: Parameters of the SPAMD buoy

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter</th>
<th>Value/Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight-buoyancy ratio</td>
<td>δ</td>
<td>0.5</td>
</tr>
<tr>
<td>Mass ratio</td>
<td>m₁/m₂</td>
<td>1</td>
</tr>
<tr>
<td>Radius of buoy</td>
<td>r</td>
<td>5m</td>
</tr>
<tr>
<td>Mass offset angle</td>
<td>ϕ</td>
<td>-30 deg</td>
</tr>
<tr>
<td>Mass offset distance</td>
<td>r₉g</td>
<td>5m</td>
</tr>
</tbody>
</table>

Since the offset mass, m₂, causes an additional moment at the centre of the buoy, the tether attachment point on the buoy needs to be adjusted to the opposite side of the buoy to maintain the static stability of the system. The coordinate of the mooring attachment point is denoted as (x_m, z_m). The mooring point angle, which is measured from the Z axis to the tether attachment point, is denoted as α, as shown in Figure 3.

PTO

A linear PTO is connected to the buoy via a mooring tether with nominal length of L. It consists of a linear spring with stiffness of K_{pto}, a linear damper with damping coefficient of B_{pto} and a pretension force of F_{pre} to balance the net buoyancy of the buoy, as shown in Figure 4.

The PTO does not contain any hard-stop motion constraints. With applied PTO force, the asymmetric mass buoy can harmonically oscillate under wave excitation. The total power output of the system is equal to the power consumed by the linear PTO damper, given by

\[ P_{\text{total}} = \frac{1}{2} B_{\text{pto}} |\Delta \ddot{L}|^2 \]  

(1)

K_{pto}, B_{pto} and L are the key parameters affecting the efficiency of the
SPAMD. These coefficients are optimized to achieve maximum power output in the modal analysis in Section 4.

3. Equation of motion

In this section, the system stability condition is investigated and a frequency-domain model is developed. The system stability condition determines the acceptable mass-offset that prevents the tether from reeling on the planar constrained buoy during system operation. The frequency-domain model of the SPAMD is formulated as a basis for the subsequent modal analysis. The model is built upon linear wave theory with the consideration of viscous drag forces.

In practice, the viscous drag forces are the dominant forces that dissipate kinetic energy from the buoy’s motion. Therefore, it is necessary to model the viscous drag forces in the motion equation of 3DOF single-tethered PAs, otherwise the PA will experience excessive motion, in particular at resonance, where power absorption efficiency of the SPAMD will be overestimated. As commonly expressed in a quadratic form, the viscous drag forces need to be linearised to be used in the frequency domain model, which leads to a hybrid frequency-domain model. This hybrid frequency-domain model is validated against an equivalent time-domain model, which is presented in [11]. The results show that the response of the hybrid frequency-domain model matches well with the response of the time-domain model, even at
3.1. System stability condition

With reference to the geometric centre of the buoy in Figure 3, the torque generated by the offset point mass $m_2$ should be balanced by the torque generated by the PTO pretension force $F_{pre}$ when the buoy is at rest, hence

$$F_{pre} r \sin(\alpha) = m_2 g r_{gy} \cos(\varphi). \quad (2)$$

When the buoy rotates a positive pitch angle of $\theta_y$, which is less than $\pi/2$, the net moment $M_{net}$ can be calculated by

$$M_{net} = m_2 g r_{gy} \cos(\varphi - \theta_y) - F_{pre} r \sin(\alpha + \theta_y). \quad (3)$$

Eqn (3) can be simplified to

$$M_{net} = -\sin(\theta_y) [F_{pre} r \sin(\alpha) - m_2 g r_{gy} \sin(\varphi)] \quad (4)$$

For the system to be stable in the pitch direction, the derivative of the net moment with respect to the pitch angle must be negative, hence

$$\frac{\partial M_{net}}{\partial \theta_y} = -\cos(\theta_y) [F_{pre} r \sin(\alpha) - m_2 g r_{gy} \sin(\varphi)] < 0. \quad (5)$$

Eqn (5) can be simplified as

$$F_{pre} r > m_2 g r_{gy}. \quad (6)$$

As defined in Section 2, the pretension force $F_{pre}$ is given by

$$F_{pre} = \rho V g - m g = (m_w - m) g. \quad (7)$$

in which $\rho$ is water density, $V = \frac{4}{3} \pi r^3$ is the volume of spherical buoy, $m_w = \rho V$ is the mass of water displaced by the submerged spherical buoy. Substituting Eqn (7) into Eqn (6) gives the stability condition

$$\frac{r_{gy}}{r} < \left( \frac{m_w - m}{m_2} \right). \quad (8)$$

Therefore, the SPAMD is stable only if the inequality presented in Eqn (8) is met. In this work, the defined SPAMD is stable, i.e., $\frac{r_{gy}}{r} = 1$ and $rac{m_w - m}{m_2} = 2$. 

---

resonance.

---
3.2. Frequency domain modelling

The motion of the SPAMD can be described by a frequency-domain motion equation formulated at the centre of the buoy, given by

\[
(M + A(\omega)) \ddot{x} + (B(\omega)) \dot{x} = \hat{F}_{\text{exc}} + \hat{F}_{\text{re}} + \hat{F}_{\text{pto}} + \hat{F}_{\text{vis}},
\]

in which \(M\) is the mass matrix of the asymmetric mass buoy, \(A(\omega)\) is the hydrodynamic added mass matrix, \(B(\omega)\) is the hydrodynamic damping coefficient matrix, which are calculated by using the method described in [12], \(\dot{x}\) is the buoy velocity vector \((\dot{x}, \dot{z}, \dot{\theta}_y)^T\), where \(x\) is the surge displacement, \(z\) is the heave displacement and \(\theta_y\) is the pitch angular displacement, \(\hat{F}_{\text{exc}}\) is the wave excitation force vector, which is also calculated by using the method described in [12], \(\hat{F}_{\text{re}}\) is the net restoring force vector, \(\hat{F}_{\text{pto}}\) is the PTO control force vector and \(\hat{F}_{\text{vis}}\) is the viscous force vector.

The mass matrix of spherical buoy

According to Lee [13], for a 3DOF (i.e. surge, heave and pitch) spherical buoy with an additional offset point mass \(m_2\), the mass matrix with respect
to the geometric centre of the buoy is given by

\[
M = \begin{pmatrix}
  m_1 + m_2 & 0 & m_2 z_2 \\
  0 & m_1 + m_2 & -m_2 x_2 \\
  m_2 z_2 & -m_2 x_2 & I_y
\end{pmatrix},
\]

(10)
in which the total moment of inertia \( I_y = I_1 + I_2 = \frac{2}{5} m_1 r^2 + m_2 r_{gy}^2 \). From

Figure 3 as \( \varphi \gg \theta_y \), the coordinate of the offset mass is given by \((x_2, z_2) = (r_{gy} \cos(\varphi + \theta_y), r_{gy} \sin(\varphi + \theta_y)) \approx (r_{gy} \cos(\varphi), r_{gy} \sin(\varphi))\).

**The net hydrostatic restoring force vector**

The net hydrostatic restoring force vector consists of hydrostatic restoring forces in the surge and heave directions, and restoring toque generated by the gravity of the offset mass \( m_2 \) in the pitch direction, given by

\[
\dot{F}_{re} = \begin{pmatrix}
  0 \\
  \rho V g - (m_1 + m_2) g \\
  m_2 r_{gy} \cos(\varphi + \theta_y)
\end{pmatrix},
\]

(11)

**The excitation force vector**

The excitation force vector applied to the fully submerged spherical buoy is given by

\[
\dot{F}_{exc} = \begin{pmatrix}
  F_{exc,x} \\
  F_{exc,z} \\
  0
\end{pmatrix},
\]

(12)
in which $F_{ex,x}$ and $F_{ex,z}$ are calculated by using the equations given by [12], the pitch excitation moment is negligible because the spherical buoy is axisymmetric about the pitch axis.

The PTO control force vector

According to Section 2, the PTO control force along the tether is the sum of the PTO spring force, the PTO damping force and the pretension force. By mapping the PTO control force to the Cartesian axes, the PTO control force vector is given by

$$\hat{\mathbf{F}}_{pto} = (-F_{pre} - B_{pto} \Delta \dot{L} - K_{pto} \Delta L) \mathbf{T},$$

(13)

in which the transformation vector $\mathbf{T}$ converts the PTO control force to the Cartesian axes.

As illustrated in Figure 5, when the buoy moves to an arbitrary position, the motion of the tether attachment point can be decomposed into a translation from $(x_m, z_m)$ to $(x_m, z_m)$ and a rotation from $(x_m, z_m)$ to $(x, z)$. Consequently, the change in PTO length $\Delta L$ is given by

$$\Delta L = \sqrt{(L + z + r \sin(\alpha) \theta_y)^2 + (x - r \cos(\alpha) \theta_y)^2 - L}.$$  

(14)

As $L$ is greater than $x$ and $z$, therefore $L + z + r \sin(\alpha) \theta_y \gg x - r \cos(\alpha) \theta_y$, and Eqn (14) can be simplified as

$$\Delta L \approx z + r \sin(\alpha) \theta_y = \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} x \\ z \\ \theta_y \end{pmatrix},$$

(15)

in which the inverse Jacobian matrix that converts the buoy’s motion in the Cartesian space to the PTO elongation is given by

$$\mathbf{J}^{-1} = \begin{pmatrix} 0 & 1 & r \sin(\alpha) \end{pmatrix}.$$  

(16)

Noting that from Eqn (16), it can be seen that only heave and pitch motion directly couple to the PTO.
The transformation vector $T$ is given by

$$T = \begin{pmatrix} \sin(\beta) \\ \cos(\beta) \\ r_{\text{dis}} \end{pmatrix},$$

where $\beta$ is tether angle, $r_{\text{dis}}$ is instantaneous lever arm between the geometric centre of the buoy to the PTO control force vector, as shown in Figure 5.

The tether angle $\beta$ is assumed small as $L$ is significantly greater than $x$, therefore $\cos(\beta) \approx 1$ and

$$\sin(\beta) \approx \tan(\beta) = \frac{x - r \cos(\alpha) \theta_y}{L + z + r \sin(\alpha) \theta_y}.$$  

As $L \gg z + r \sin(\alpha) \theta_y$, Eqn (18) can be further simplified as

$$\sin(\beta) \approx \frac{x - r \cos(\alpha) \theta_y}{L}.$$  

From Figure 5, the instantaneous lever arm from the centre of the buoy to the tether projection is given by

$$r_{\text{dis}} = r \sin(\alpha + \theta_y - \beta) \approx r \left( \sin(\alpha) + \cos(\alpha) \theta_y - \cos(\alpha) \frac{x - r \cos(\alpha) \theta_y}{L} \right).$$

The viscous drag force vector

According to the Morison equation [14], the viscous drag force is expressed as a quadratic function of the body’s velocity. For a smooth spherical buoy, the viscous drag force vector is given by

$$\mathbf{F}_{\text{vis}} = -0.5 \rho C_D S \begin{pmatrix} |\dot{x}| \dot{x} \\ |\dot{z}| \dot{z} \\ 0 \end{pmatrix},$$

in which $C_D$ is the viscous drag coefficient ($C_D = 0.18$ for the $1 \times 10^6 < Re < 5 \times 10^6$ [15]) and $S$ is the nominal cross-section area of the spherical buoy.
(S = \pi r^2). It should be noted that for a smooth spherical buoy, the viscous drag coefficient in the pitch direction is negligible.

Eqn (21) represents a nonlinear time-domain expression. In order to consider the viscous drag forces in the frequency-domain model, the quadratic viscous drag forces must be linearised. The essence of the linearisation is to use a linear term |\dot{x}_{iter}|\dot{x} (or |\dot{z}_{iter}|\dot{z}) to approach to the quadratic term |\dot{x}|\dot{x} (or |\dot{z}|\dot{z}) iteratively, whereby \dot{x}_{iter} and \dot{z}_{iter} are iterative variables. Figure 6 shows the flow chart of the linearisation process in the surge direction, whereby \dot{x}_{iter,k} is the value of the iterative variable in the k-th iteration, and \dot{x}_k is the system response in the surge direction. The initial value of \dot{x}_{iter,0} is initialized as 1, which is a fair estimation based on existing numerical and experimental data. The viscous drag force is linearised as \hat{F}_{vis,surge} = -0.5\rho C_D S |\dot{x}_{iter,k-1}|\dot{x}. Then the motion equation is solved to obtain the surge response of the system \dot{x}_1. If the convergence criterion |\frac{\dot{x}_k - \dot{x}_{iter,k-1}}{|\dot{x}_k|}| < 5\% is met, \dot{x}_k and \dot{x}_{iter,k-1} are regarded to be equivalent and the iteration stops. Otherwise, \dot{x}_{iter} will be updated to \dot{x}_{iter,k-1} = \frac{\dot{x}_{iter,k-2} + \dot{x}_k}{2} which will be used in next iteration. The same linearisation process can be applied to linearise the viscous drag force in the heave direction.

4. Modal analysis

In this section, a modal analysis is conducted to understand the operating principles of the single-tethered SPAMD and to evaluate its power generation, in combination with the dynamic response of the PA. There are three aspects to be investigated through the modal analysis, namely the system natural frequencies, mode shapes and power output. Background information and governing equations that will be used in modal analysis are presented first.

Natural frequency of the decoupled system

The motion of the 3DOF single-tethered generic PA can be decomposed into three independent vibration modes, namely the surge mode, heave mode and pitch mode. The heave mode is induced by the up and down motion of the buoy along the vertical axis. Therefore, the natural frequency of the buoy’s heave mode \omega_z is governed by the PTO stiffness \(K_{pto}\) and is independent of the nominal tether length \(L\). It is given by

\[
\omega_z = \sqrt{\frac{K_{pto}}{(m + A_z(\omega))}},
\] (22)
$k = 1$, $\dot{x}_{iter,k-1} = 1$

$F_{vis,surge} = -0.5 \rho C_D S |\dot{x}_{iter,k-1}| \ddot{x}$

Solve motion equation

$\dot{x}_k$

$\frac{\dot{x}_k - \dot{x}_{iter,k-1}}{\dot{x}_k} < 5\%$

$k = k + 1$

$\dot{x}_{iter,k-1} = \frac{\dot{x}_{iter,k-2} + \dot{x}_k}{2}$

Figure 6: Flow chart illustrating the process of linearising the surge viscous drag force in the frequency domain.
in which $A_z(\omega)$ is the hydrodynamic added mass in the heave direction.

The surge mode is induced by an inverted pendulum motion arising from the buoy-tether arrangement, as shown in Figure 7. The surge mode is contained in an oscillation about the anchor when the buoy is driven by the surge excitation force. Therefore, the natural frequency of the surge mode is inversely proportional to the square root of the nominal tether length $L$, and is given by

$$\omega_s = \sqrt{\frac{g(\rho V - m)}{(L + r)(m + A_x(\omega))}}$$

(23)

in which $A_x(\omega)$ is the hydrodynamic added mass in the surge direction.

Figure 7: Schematic of the oscillation of the single-tethered PA oscillating about the anchor when driven by the surge excitation force.

The natural frequency of the pitch mode has little impact on the power absorption of the system. This is because the natural frequency of the buoy’s pitch mode tends to be significantly higher than the wave frequencies of interest, as illustrated in Section 4.2. Therefore, natural frequency of the pitch mode is not discussed in this work.

Eqn (22) and Eqn (23) can be used to estimate the modal behaviour of the generic single-tethered PA, as well as to provide initial estimates for the search of the optimal PTO stiffness and nominal tether length that maximises the efficacy of the PAs in Section 4.1.
Mode shape and natural frequency of the fully coupled system

The eigenvectors (mode shapes) and corresponding eigenvalues (natural frequencies) are calculated by solving the characteristic equation of the system

\[(M + A(\omega)) - \lambda K = 0, \quad (24)\]

in which \(\lambda\) are the eigenvalues, \(M\) is the mass matrix, \(A(\omega)\) is the hydrodynamic added mass matrix, given by Eq (10), \(K\) is the stiffness matrix, derived by re-arranging the Eqn (9) and given by

\[
K = \begin{pmatrix}
\frac{F_{pre}}{L} & 0 & -\frac{F_{pre} \cos \alpha}{L} \\
0 & K_{pto} & K_{pto} \sin \alpha \\
-\frac{F_{pre} \cos \alpha}{L} & K_{pto} \sin \alpha & -m_2gr_g \sin \phi + \frac{F_{pre} (r \cos \alpha)^2}{L} + K_{pto} (r \sin \alpha)^2
\end{pmatrix}. \quad (25)
\]

As the asymmetric mass buoy has 3DOFs, the resulting eigenvector matrix \(V\) is 3-by-3, given by

\[
V = \begin{pmatrix}
v_{1,x} & v_{2,x} & v_{3,x} \\
v_{1,z} & v_{2,z} & v_{3,z} \\
v_{1,\theta_y} & v_{2,\theta_y} & v_{3,\theta_y}
\end{pmatrix}, \quad (26)
\]

in which the columns correspond to the oscillation Mode 1, Mode 2 and Mode 3. The rows of the matrix represent the projections of the oscillation modes along the surge, heave and pitch axis of the buoy in Cartesian space.

Unlike Eqn (22) and Eqn (23) that provide an estimation of the natural frequencies, Eqn (24) computes the natural frequencies of the oscillation modes of the fully coupled system.

Relative capture width and power output

In this work, relative capture width (RCW) is used to evaluate the efficiency of the single-tethered PAs. According to [2], the relative capture
width is defined as

$$RCW = \frac{P_{total}}{2rJ},$$

(27)
in which $J$ is the power transport per unit width of wave frontage given by $2r$.

The efficiency improvement ratio, $q$, is defined as the ratio of the RCW of the SPAMD to the RCW of the generic PA, given by

$$q = \frac{RCW_{SPAMD}}{RCW_{generic}}.$$  

(28)

As mentioned in Section 2, the total power output of the SPAMD is equal to the power consumed by the PTO damper. The total power output can be decomposed into power components arising from buoy motion in each DOF by substituting Eqn (15) into Eqn (1), leading to

$$P_{total} = \frac{1}{2} B_{pto} \left| \hat{z} + r \sin(\alpha) \hat{\theta}_y \right|^2$$

$$= \frac{1}{2} B_{pto} \hat{z}^* \hat{z}^* + \frac{1}{2} B_{pto} (r \sin \alpha)^2 \hat{\theta}_y^* \hat{\theta}_y^* + \frac{1}{2} B_{pto} r \sin \alpha (\hat{z} \hat{\theta}_y^* + \hat{z}^* \hat{\theta}_y^*)$$

(29)
in which the $\hat{z}^*$ and $\hat{\theta}_y^*$ are the conjugates of $\hat{z}$ and $\hat{\theta}_y$.

In Eqn (29), it can be seen that the total power output $P_{total}$ consists of three components, namely the power arising from the heave motion $P_{heave}$, the power arising from the pitch motion $P_{pitch}$ and a cross term arising from the heave-pitch coupled motion $P_{cross}$, which indicates the energy flow between the two motions.

In Section 4.2, Eqn (29) is used to understand the contributions of buoy’s motion to the power output of the system.

4.1. Modal analysis as a function of nominal tether length

In this section, the modal analysis considers the impact of the nominal tether length on the efficiency of the SPAMD at a typical wave frequency of $\omega = 0.48$ rad/sec ($T = 13$ sec), which is a common peak wave frequency off the coast of Perth [8]. The nominal tether length ranges from 0.5 to 10 times the buoy’s radius ($0.5r \leq L \leq 10r$), over 500 sampled nominal tether lengths.
At each sampled nominal tether length, the optimal PTO stiffness $K_{pto}$ and PTO damping coefficient $B_{pto}$ that maximise the RCW of the system are determined by using the MATLAB optimization function “fmincon”, within the defined range of tether length $L$.

The modal analysis on the nominal tether length is firstly conducted for the generic symmetric mass PA as a benchmark. Figure 8 illustrates the natural frequencies (see Figure 8a), mode shapes (see Figure 8b), PTO extension velocities arising from oscillation modes (see Figure 8c) and RCW (see Figure 8d) respectively, versus the ratio of the nominal tether length to the buoy radius. As shown in Figure 8a, the natural frequency of oscillation Mode 1 declines as the nominal tether length increases, which implies that Mode 1 is surge-dominant. Conversely, the natural frequency of oscillation Mode 2 remains constant and equal to the excitation frequency over various nominal tether lengths, which means Mode 2 is heave-dominant. Furthermore, as the generic PA is a decoupled system, it can be seen that natural frequencies of Mode 1 and 2 perfectly match the natural frequencies given by Eqn (22) and (23). The oscillation Mode 3 is pitch-dominant and its natural frequency is considerably higher than the incident wave frequency. Therefore, Mode 3 is not shown in Figure 8a. From the eigenvector plots shown in Figure 8b, it can be seen that Mode 1 only contains surge motion, which means Mode 1 oscillates purely along the surge. In contrast, Mode 2 oscillates purely along the heave. The mode shapes of the generic buoy in the Cartesian space are illustrated in Figure 9. Figure 8c shows the normalised contribution of the oscillation modes to the PTO extension velocity, which is mapped from the mode shapes via inverse Jacobian matrix. As the surge motion of the generic buoy is poorly coupled to the single-tether PTO, it can be seen that only Mode 2, which oscillates purely along the heave, can contribute to the PTO extension, and thus to power output. Consequently, the resulting RCW remains constant over various nominal tether lengths, as shown in Figure 8d.

The same modal analysis is then undertaken for the SPAMD. Figure 10 illustrates natural frequencies (see Figure 10a), mode shapes (see Figure 10b), PTO extension velocities arising from oscillation modes (see Figure 10c) and RCW (see Figure 10d) of the asymmetric mass buoy respectively, versus the ratio of nominal tether length to buoy radius. In regards to Figure 10a, it should be noted that when calculating the natural frequencies of the decoupled modes using Eqn (22) and (23), the optimal parameters (stiffness, damping coefficient and nominal tether length) of the SPAMD were used.
Figure 8: Modal analysis of the generic spherical PA for various ratios of nominal tether length to the buoy’s radius, for $\omega = 0.48$ rad/sec: (a) natural frequencies of Mode 1 and 2, compared against the natural frequencies of the decoupled modes given by (22) and Eqn (23); (b) eigenvectors of Mode 1 and 2; (c) normalised contributions of Mode 1 and 2 to PTO extension velocity; (d) RCWs of the generic PA.

Figure 9: Mode shapes of the generic PA: (a) surging Mode 1; (b) heaving Mode 2; (c) pitching Mode 3.
This results in a small difference in the natural frequencies compared to those displayed in Figure 8a. In comparison to the generic PA, the oscillation modes of the SPAMD contain motions along multiple DOFs because of the strong motion coupling arising from offsetting the centre of mass. From Figure 10a, it can be seen that the natural frequency of the oscillation Mode 1 changes from heave-dominant to surge-dominant, as the nominal tether length increases. In contrast, the natural frequency of the oscillation Mode 2 changes from surge-dominant to heave-dominant, as the nominal tether length increases. Furthermore, as the surge and heave motions are coupled in the SPAMD, the natural frequencies of Mode 1 and 2 deviate from the natural frequencies given by Eqn (22) and (23) at small tether length to buoy radius ratios. When the SPAMD operates at optimal nominal tether lengths (the shaded range in Figure 10a), the natural frequencies of Mode 1 and Mode 2 are approximately equal and approach the natural frequency given by Eqn (23) and Eqn (22). For the greatest tether length to buoy radius ratios, the eigenvector of the Mode 1 and 2 are similar to the generic case, where the surge and heave motions of the PA are weakly coupled. The oscillation Mode 3 is pitch-dominant and its natural frequency is considerably higher than the incident wave frequency. Therefore, Mode 3 is not shown in the Figure 10a.

From Figure 10b, it can be seen that Mode 1 and 2 contain both surge and heave motions. When the motions of the system becomes the most strongly motion-coupled (the shaded range in Figure 10b), the surge motion component of Mode 2 (Mode 2 (x)) is approximately equal to the heave motion component of Mode 2 (Mode 2 (z)), which implies Mode 2 oscillates at 45 degrees and Mode 1 oscillates at -45 degrees in the Cartesian space. These two modes become spatially orthogonal in the Cartesian space [16], as illustrated in Figure 11. Figure 10c shows the normalised contribution of the oscillation modes to the PTO extension velocity. It can be seen that Mode 1 and Mode 2 result in almost equal contribution to the PTO extension at optimal tether length. Figure 10d shows the change in RCW for the SPAMD versus the ratio of the nominal tether length to the buoy radius. As the surge motion can couple to the PTO via surge-heave motion coupling, the RCW of the SPAMD (see Figure 10d) is significantly higher in comparison to RCW of the generic PA (see Figure 8d). When the SPAMD operates with the optimal nominal tether length, the RCW of the SPAMD is almost 3 times that of the generic PAs. This is because the capture width of the additional surge motion can be theoretically twice that of the heave motion [2].
Figure 10: Modal analysis of the SPAMD for various ratios of nominal tether length to the buoy’s radius, when the wave frequency is 0.48 rad/sec: (a) natural frequencies of Mode 1 and 2, compared against the natural frequencies of the decoupled modes given by Eqn (22) and (23); (b) eigenvectors of Mode 1 and 2; (c) normalised contributions of Mode 1 and 2 to PTO extension velocity; (d) RCWs of the SPAMD.

Figure 11: Mode shapes of the SPAMD under the optimal condition of nominal tether length: (a) Mode 1 oscillating at -45 degrees in the Cartesian space; (b) Mode 2 oscillating at 45 degrees in the Cartesian space; (c) Mode 3 pitching in the Cartesian space.
4.2. Modal analysis as a function of wave frequency

The modal analysis of the SPAMD was extended to understand the operation principles of the PA at optimal working conditions across the wave frequencies of interest. The wave frequencies ranged from 0.34 to 1.4 rad/sec, and the system was analysed by using 30 discrete frequencies within this range. For each sampled wave frequency, the optimal PTO stiffness, the optimal PTO damping coefficient and the optimal nominal tether length were found by using the MATLAB optimization function “fmincon”, within the defined range of tether length $L$.

Figure 12 illustrates the natural frequencies (see Figure 12a), mode shapes (see Figures 12b and 12c), optimal nominal tether length (see Figure 12d) and wave energy harvesting efficiency (see Figure 12e) of the SPAMD respectively, across the wave frequencies of interest. The SPAMD operates under three different regimes (i.e. I, II and III) throughout the frequencies of interest, which are discussed in the following.

Regime I ($0.34 \text{ rad/sec} < \omega < 0.5 \text{ rad/sec}$)

Modes 1 and 2 oscillate orthogonally in the Cartesian space, in resonance with the incident waves. From Figure 12a, it can be seen that the natural frequencies of Modes 1 and 2 match the wave frequencies, which implies that these two modes become resonant. Note that the oscillation Mode 3, which is pitch-dominant, has a significantly higher natural frequency in comparison to the wave frequency range of interest. Figures 12b and 12c show that Mode 1 and 2 contain almost equal surge and heave motions in Regime I, which implies these two modes oscillate orthogonally in the Cartesian space, as shown in Figure 11. In Figure 12d, the optimal nominal tether length declines as the wave frequency increases in order to match the natural frequencies of Mode 1 and Mode 2 to the wave frequency. As the wave energy is captured by surge and heave motions of the buoy, the RCW of the SPAMD is improved almost three times in comparison to the generic PA, as shown in Figure 12e.

Regime II ($0.5 \text{ rad/sec} < \omega < 0.8 \text{ rad/sec}$)

Only Mode 2 (surge-dominant) oscillates in resonance with the incident waves. From Figure 12a, it can be seen that only the natural frequency of Mode 2 matches the wave frequency, whereas the natural frequency of Mode 1 increasingly deviates from the wave frequency. The natural frequency curve
Figure 12: Modal analysis of the SPAMD buoy with optimal PTO configurations and nominal tether lengths for wave frequencies ranging from 0.34 rad/sec to 1.4 rad/sec: (a) the natural frequencies of three modes; (b) eigenvectors of Mode 1; (c) eigenvectors of Mode 2; (d) the optimal nominal tether length; (e) RCWs of the SPAMD and the generic PA and the q factor.
of Mode 1 reaches a notch at the wave frequency of 0.8 rad/sec. The factors
determine the shape of the notch will be presented in a subsequent pa-
paper [9]. Figures 12b and 12c show that Mode 1 tends to be heave-dominant,
whereas Mode 2 tends to be surge-dominant until the wave frequency of 0.8
rad/sec. According to the Figure 12d, when the wave frequency is beyond
0.8 rad/sec, nominal tether length reaches its lower limit ($L_{min} = 2.5$ m).
Therefore, the system can only be tuned to be surge resonant at the wave
frequencies less than 0.8 rad/sec. As the SPAMD is incapable of heaving in
resonance with incident waves, the RCW of the SPAMD declines from almost
3 times to twice that of the generic PA, as shown in Figure 12e.

Regime III (0.8 rad/sec < $\omega$ < 1.4 rad/sec)

Only Mode 2 (heave-dominant) oscillates in resonance with the incident
waves. From Figure 12a, it can be seen that only the natural frequency of
Mode 2 can match the wave frequency. In contrast, the natural frequency of
Mode 1 is much lower than the wave frequency. As aforementioned, at wave
frequencies over 0.8 rad/s, the tether reaches its lower limit, and consequently
the SPAMD can only capture small amount of wave energy via the surge
motion. From Figure 12c, it can be seen that Mode 2 contains multiple
motions (i.e. surge and heave), in order to use the resonant heave motion
to capture the wave energy. As a result, the RCW of the SPAMD starts to
converge to that of the generic PA, as shown in Figure 12e.

4.3. Power analysis

The surge motion of the SPAMD buoy couples to the PTO via motion
coupling (surge-heave or surge-pitch) arising from offsetting the centre of
mass of the buoy. A power analysis has been performed to understand the
contributions of the buoy motions to the total power output of the system.
Figure 13 shows the power output contribution from the buoy’s motions that
is calculated by Eqn (29) (see Figure 13a) and the corresponding velocity
amplitude of the buoy’s heave, surge and pitch movements (see Figure 13b,
13c and 13d) across the three operation regimes. From Figure 13a, in regime
I, the entire power output of the SPAMD directly arises from the buoy’s
heave motion. This is because the heave oscillation of the buoy is enhanced
by the surge motion via surge-heave motion coupling, as shown in Figure
13c. Furthermore, as shown in Figure 13b, the generic buoy has larger surge
motion than the SPAMD. This is because of the strong surge-heave motion
coupling at low frequencies, which weakens the surge motion of SPAMD as
energy is extracted by the PTO. As it is the low frequency surge (and sway) motion that defined the buoy’s mooring watch circle, such surge suppression is an advantage. In contrast, the power arising from the heave-pitch coupled motion is negative, which implies that a part of the power returns to the environment via this coupled motion. This phenomenon will be investigated in future. In regime II, the pitch motion of the buoy is enhanced by the resonant surge motion via surge-pitch motion coupling, as evident in Figure 13d. Consequently, from Figure 13a it can be seen that the power arising from the heave motion declines, whereas the power arising from the pitch and the heave-pitch coupled motion increase. In regime III, the power arising from the heave motion increasingly becomes dominant in the power output, whereas the power from the pitch motion and from the heave-pitch coupled motion declines. This is because the SPAMD mainly uses the resonant heave motion of Mode 2 to capture the wave energy.

![Graphs](image)

Figure 13: Power analysis of the SPAMD and velocity amplitudes in the heave and pitch directions across the wave frequencies of interest: (a) power output analysis of the SPAMD; (b) surge velocity amplitude; (c) heave velocity amplitude; (d) pitch velocity amplitude.
In Figure 14, the optimal PTO configuration (i.e. $K_{pto}$ and $B_{pto}$) of the asymmetric mass and the generic PAs are compared over the three regimes. In Regime I, as the SPAMD mainly harvests the power arising from the buoy’s heave motion, the optimal PTO stiffness is identical to that of the generic PA. In addition, the optimal PTO damping coefficient is three times greater than that of the generic PA because the heave motion of the SPAMD is enhanced by the surge motion via surge-heave motion coupling. In Regime II, as SPAMD mainly harvests the power arising from the buoy’s pitch motion, the optimal PTO stiffness and damping coefficient are distinct from these of the generic PA. In Regime III, as the SPAMD mainly utilizes resonant heave motion to capture wave energy, the optimal PTO configuration of the SPAMD approaches to that of the generic PA. Therefore, it can be seen that optimal PTO configuration of the SPAMD is distinct from that of the generic PA at most predefined wave frequencies.

![Figure 14: The comparison of optimal PTO configuration between the SPAMD and the generic PAs across the three regimes: (a) optimal PTO stiffness; (b) optimal PTO damping coefficient.](image)

5. Conclusion

In this paper, an asymmetric mass buoy is employed to improve the efficacy of fully submerged single-tethered spherical point absorbers by harvesting the wave energy from both surge and heave directions. The results of the modal analysis show that the efficiency of the SPAMD is significantly affected by the nominal tether length. Furthermore, the SPAMD operates under three...
different operation regimes across the wave frequencies of interest. In regime I, Modes 1 and 2 oscillate orthogonally in the Cartesian space, in resonance with the incident waves; in regime II, only mode 2 (surge-dominant) oscillates in resonance with the incident wave; in regime III, only the heave motion of Mode 2 oscillates in resonance with the incident wave. The maximum efficiency of the SPAMD can be 3 times higher than the generic PAs for long waves, which implies that the SPAMD has significant commercial potential. It should be noted that there are two main limitations of this study. The efficiency improvement of the SPAMD is evaluated in monochromatic plane waves and could be less in real sea states. Furthermore, the sway and yaw dynamics of the SPAMD are neglected in the current modelling, but will be investigated in future work.

Acknowledgement

This research is supported by China Scholarship Council (CSC), Australia Research Council (ARC) Linkage Grant (LP13010011F) and the Research Training Program (RTP). We also thank Nataliia Sergiienko, Alison-Jane Hunter and Jonathan David Piper for their comments that greatly improved the manuscript.

Reference


