The application of modal analysis to the design of multi-mode point absorber wave energy converters

Boyin Ding*, Nataliia Sergiienko, Fantai Meng, Benjamin Cazzolato, Peter Hardy, and Maziar Arjomandi

Ocean Wave Energy Research Group, School of Mechanical Engineering, The University of Adelaide, Adelaide, Australia, 5005

Abstract

Point absorbers are one of the most common wave energy converters. They are typically designed to extract power primarily from heave motion, which results in a relative capture width limited to a third of the maximum possible. Over the past few decades, an increasing amount of attention has been given to the design of point absorbing wave energy converters operating in multiple oscillation modes, in an attempt to more efficiently extract power from waves. However, it is not a trivial task as wave energy converters operating in multiple modes demonstrate complex coupled vibrational characteristics across several degrees of freedom. This paper addresses this challenge using modal analysis, a modern approach developed for determining, improving and optimising dynamic characteristics of complex engineering systems. Case studies are conducted on three multi-mode submerged point absorber designs, each with distinct modal behaviour to show the generality and efficacy of the approach. Results show that in combination with knowledge of wave power absorption in the frequency-domain, modal analysis can be used as an effective analytical tool to evaluate the vibrational characteristics and the power absorption potential of the multi-mode system, as well as to explore the corresponding working principles and the physical limits of the design.

Keywords: Point absorber, submerged wave energy converter, power absorption by multiple oscillation modes, modal analysis

Corresponding Author email address: boyin.ding@adelaide.edu.au (Boyin Ding)

1. Introduction

Ocean wave energy has been under the public spotlight over the past few decades for its high energy density, predictability, and consistency. It has shown a great potential to address the increasing global energy demand, with recent estimates suggesting a global wave energy capacity exceeding 2TW [1]. However, wave energy converter (WEC) technologies are still in their pre-commercial phase and exhibit great diversity in design, leading to more than one thousand different WEC prototypes in various stage of development [2]. Oscillating point absorbers (PA) are a popular type of WEC design defined by having dimensions much smaller
than a wavelength and account for 53% of the existing WEC prototypes [3]. PAs are usually
designed as axisymmetric heaving buoys which mainly extract power from heave motion
regardless of wave direction [1].

In 1975 and 1976, Budal and Falnes [4], and Evans [5] found that given unconstrained
motion, the maximum capture width of an oscillating buoy does not depend on its size, shape
or submergence depth, and is only governed by the mode of motion. An axisymmetric buoy
needs to oscillate in at least two modes, one radiating symmetric waves such as heave and the
other one radiating antisymmetric waves such as surge or pitch, to absorb the maximum
available power from waves [6]. Since then, a number of PA prototypes have been proposed to
extract power from multiple modes of motion (multi-mode) [7]. A major group of multi-mode
PA prototypes employ multiple tethers and power-take-off (PTO) machineries to enhance
kinematic coupling between the modes of the buoy (where wave power enters the system) and
the PTO machinery (where wave power is either stored or converted to electricity). A typical
example is the submerged three tether PA proposed by Srokosz in 1979 [8]. In contrast, buoys
with an asymmetric shape or mass distribution have been employed to enhance dynamic
coupling between the modes of the buoy, and thus convert power absorbed by multiple modes
into useful PTO work. Typical examples in this category are the Edinburgh duck [9] and the
submerged asymmetric mass distribution PA proposed by Meng, et al. [10]. Despite the
diversity in the design concepts, multi-mode WECs demonstrate coupled vibrational
characteristics in multiple degrees of freedom (DoFs) which makes them much more complex
than their uni-modal counterparts (e.g. heaving buoy). The complexity associated with a
specific converter design increases as the number of motion modes and moving parts increase.

Modal analysis is a tool commonly employed for determining, improving and optimising
dynamic characteristics of complex engineering systems, and there are numerous applications
of modal analysis reported in the literature covering wide areas of engineering, science and
technology [11]. Nevertheless, to the best knowledge of the authors, modal analysis has not
been widely applied to the field of WEC design since multi-mode WECs are still in their early
research phase. Finnegans and Goggins [12] applied modal analysis to a two-body wave energy
converter to transform the coupled equations of motion into uncoupled modal equations,
reducing the complexity of the solution for the response amplitude operator and other
characteristics in the frequency domain. Meng et al. [13] used modal analysis to explore the
modal behaviour of a submerged asymmetric mass distribution PA with optimal PTO
parameters. These are the only studies in the literatures reporting the use of modal analysis in
WEC research, and neither study reported the full functionalities of the modal analysis
approach nor discussed its true value for multi-mode WEC design.

This paper addresses the complex design problem of multi-mode WECs by applying modal
analysis and well-known theory in wave power absorption. Three multi-mode submerged PA
designs are used as examples to illustrate the methodology in order of increasing complexity
in the modal behaviour, in an attempt to develop a guideline for the WEC designers to follow.
The layout of the paper is as follows: Section 2 describes the fundamental of WEC modelling,
wave power absorption in the frequency domain, and modal analysis; Section 3 describes the
application of modal analysis on a generic single tether spherical buoy PA, highlighting the
procedure, and discussing the outcomes; Sections 4 and 5 escalate the applications to two
advanced multi-mode PA designs with more complex modal behaviour and operating
principles; Section 6 provides a summary of the results.
2. Fundamental Knowledge of Wave Energy Converter Properties

2.1 Motion Equation for Submerged Point Absorbing Wave Energy Converters

Excited by a plane incident wave traveling along the x-axis, an axisymmetric oscillating WEC can be modelled in 3DoFs along buoy Cartesian coordinates as shown in Fig. 1: with a surge mode along the x-axis, heave mode along the z-axis, and pitch mode about the y-axis. This 3DoF motion equation in the time domain can be written based on the well-known Cummins equation [14]:

$$ (\mathbf{M} + \mathbf{A}_\infty) \ddot{\mathbf{x}} + \int_0^t \mathbf{K}_{\text{rad}}(t - \tau) \dot{\mathbf{x}}(\tau) d\tau = \mathbf{F}_{\text{h/stat}} + \mathbf{F}_d + \mathbf{F}_{\text{exc}} + \mathbf{F}_{\text{pto}}, $$

(1)

where the vector \( \mathbf{x} \) contains the surge displacement \( x \), heave displacement \( z \) and pitch angle \( \theta \) of the body at its geometrical centre; \( \mathbf{M} \) is the mass matrix of the oscillating body; \( \mathbf{A}_\infty \) is the hydrodynamic added mass existing at infinite frequency; \( \mathbf{K}_{\text{rad}}(t) \) is the radiation impulse response function; and \( \mathbf{F}_{\text{h/stat}}, \mathbf{F}_d, \mathbf{F}_{\text{exc}}, \) and \( \mathbf{F}_{\text{pto}} \) are the hydrostatic, viscous drag, wave excitation and PTO forces exerted on the body in Cartesian space, respectively.

Sergiienko et al. [15] found that compared with floating converters, fully submerged buoys can more effectively utilise multiple modes of motion to extract power from waves. Therefore, only fully submerged PAs anchored to the sea floor by taut tethers are considered in this study. For a fully submerged buoy, the generalised hydrostatic force is

$$ \mathbf{F}_{\text{h/stat}} = [0 \ (\rho V - m)g \ 0]^T, $$

(2)

where \( \rho \) is the density of water, \( m \) and \( V \) are the mass and volume of the buoy, respectively, and \( g \) is the gravitational acceleration.

The viscous drag forces experienced by the buoy can be written as quadratic functions of the buoy velocity, based on the Morison equation [16]:

$$ \mathbf{F}_d = \begin{pmatrix} -0.5 \rho C_{Dx} A_x |\dot{x} - \dot{x}_f| (\dot{x} - \dot{x}_f) \\ -0.5 \rho C_{Dz} A_z |\dot{z} - \dot{z}_f| (\dot{z} - \dot{z}_f) \\ -0.5 \rho C_{D\theta} D^4 \dot{D} |\dot{\theta}| \dot{\theta} \end{pmatrix}, $$

(3)

where \( C_{Dx}, C_{Dz}, \) and \( C_{D\theta} \) are the viscous drag coefficients of the buoy along surge, heave and pitch axes respectively, whose values for various buoy shapes can be found in [17]; \( A_x \) and \( A_z \) are the cross sectional areas of the buoy along surge and heave axes, respectively; \( D \) is the diameter of the buoy; \( \dot{x}_f \) and \( \dot{z}_f \) are the fluid particle velocities at the position of the geometric centre of the buoy, which are usually assumed to be negligible compared to the buoy velocity;
velocity. Spherical shape buoys are considered in this study for simplicity. Thus, $C_{Dx}=C_{Dz}$, $C_{Dh} = 0$ and $A_x = A_z = \pi r^2$, where $r$ is the radius of the buoy.

The PTO force exerted on the buoy along buoy Cartesian coordinates is given by

$$F_{pto} = J(x)^{-T}F_t,$$

where $F_t$ are the forces generated by the PTO machinery installed along the tethers; $J(x)^{-T}$ is the transpose inverse Jacobian matrix that maps the PTO forces along the tethers to the PTO forces in buoy Cartesian coordinates. The forms and dimensions of $J(x)^{-T}$ and $F_t$ depend on the specific arrangement of the tethers and the PTO machinery that will be illustrated for each of the three submerged PA designs in later sections.

In order to exclude uncertainties associated with a specific PTO machinery design, it is presumed that the PTO force has linear spring and damping effects proportional to the tether elongation and the rate of change of the tether elongation, respectively. Therefore, the PTO force along each tether consists of a static term for keeping the buoy submerged at its rest position, and the spring and damping terms:

$$F_{t,i} = C_{pto,i} - K_{pto,i}\Delta l_t - B_{pto,i}\dot{\Delta l}_t,$$

where $C_{pto}$ is a constant pretension force defined based on the buoyancy force experienced by the buoy and the arrangement of tethers, $K_{pto}$ and $B_{pto}$ are the stiffness and damping coefficients of the PTO machinery along the tether, respectively, $\Delta l$ is the elongation of the tether, and the subscript $i$ denotes the tether number.

$A_\infty$, $K_{rad}(\omega)$ and $F_{exc}$ are basic hydrodynamics arising from wave-buoy interaction and are usually solved by linear wave theory that assumes the wave height relative to the wave length and the resulting buoy motion are both small [18]. Therefore, terms in Equation (1) associated with basic hydrodynamics are linear. In contrast, $F_d$ and $F_{pto}$ are non-linear terms which need to be linearised before the time-domain motion equation can be transformed to the frequency domain form for modal analysis. By applying Lorentz linearisation [19], the quadratic drag terms in $F_d$ can be numerically converted to linear viscous damping terms. $F_{pto}$ contains trigonometric functions of buoy position and can be linearised at the nominal/rest position of the buoy $x_0 = [0 \ 0 \ 0]^T$, assuming small displacements of the buoy in Cartesian space with respect to the tether length which is normally the case in deep water.

After linearisation, Equation (1) can be transformed to the following frequency-domain form [20]:

$$(M + A(\omega))\ddot{x} + (B_{rad}(\omega) + B_d(\omega) + B_{pto})\dot{x} + K_{pto}x = \hat{F}_{exc}(\omega),$$

where the superscript $\hat{x}$ denotes the complex amplitudes covering the magnitude and phase information of the variables. For a spherical shape buoy,

$$A(\omega) = \begin{pmatrix} a_{11}(\omega) & 0 & 0 \\ 0 & a_{33}(\omega) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
where \( \omega \) is the wave frequency; \( a_{ii} \) and \( b_{ii} \) denote the frequency-dependent hydrodynamic coefficients (added mass and radiation damping respectively) of the buoy at its rest position; \( b_s \) and \( b_h \) are the linearised frequency-dependent viscous damping coefficients along surge and heave axes, respectively; \( \mathbf{F}_s^\ast \) and \( \mathbf{F}_h^\ast \) represent the wave excitation force along surge and heave axes of the buoy. The hydrodynamic coefficients and the wave excitation forces can be calculated using boundary element solvers such as WAMIT, AQWA and NEMOH for any buoy shapes. For submerged spherical buoys, \( a_{ii}, b_{ii}, \mathbf{F}_s^\ast \) and \( \mathbf{F}_h^\ast \) can be solved using an analytical method described in [21]. \( \mathbf{M}, \mathbf{B}_{pto}, \text{and } \mathbf{K}_{pto} \) in Equation (6) are dependent on the specific PA design and thus will be discussed for each case in later sections. The hydrostatic force disappears in Equation (6) as it is cancelled by the constant PTO pretension force.

After the motion response of the buoy in Cartesian space (e.g. buoy velocity) is solved by Equation (6), the corresponding elongation response of the tethers (e.g. tether elongation velocity) can be obtained:

\[
\hat{\Delta l} = \mathbf{J}(\mathbf{x}_o)^{-1}\hat{\mathbf{x}},
\]

where \( \mathbf{J}(\mathbf{x}_o)^{-1} \) is the inverse Jacobian matrix at the buoy nominal/rest position. The time-averaged power absorbed by the PA, more specifically by the PTO machinery installed along the tethers, is then equal to the mechanical power dissipated by the PTO dampers:

\[
P_a = \sum_{i=1}^{n} \frac{1}{2} B_{pto,i} |\Delta \hat{l}_i|^2,
\]

where \( n \) denotes the total number of tethers. The time-averaged absorbed power can also be calculated in the buoy Cartesian coordinates as the difference between the wave power input into the system and the power radiated and dissipated in water [6]:

\[
P_a = P_{in} - P_{out} = \frac{1}{4} (\mathbf{F}_s^{\ast T} \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \mathbf{F}_s^{\ast}) - \frac{1}{2} \hat{\mathbf{x}}^T (\mathbf{B}_{rad} + \mathbf{B}_d) \hat{\mathbf{x}},
\]

where the superscript \( T \ast \) denotes the conjugate transpose. Furthermore, the total absorbed power can be decomposed into power absorption by the surge, heave, and pitch modes if only 3DoF motion are considered:

\[
P_{a,s} = \frac{1}{4} (\hat{\mathbf{F}}_s^{\ast T} \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \hat{\mathbf{F}}_s^{\ast}) - \frac{1}{2} \hat{\mathbf{x}}^T (b_{11} + b_s) \hat{\mathbf{x}},
\]

\[
P_{a,h} = \frac{1}{4} (\hat{\mathbf{F}}_h^{\ast T} \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \hat{\mathbf{F}}_h^{\ast}) - \frac{1}{2} \hat{\mathbf{x}}^T (b_{33} + b_h) \hat{\mathbf{x}},
\]

\[
P_{a,p} = \frac{1}{4} (\hat{\mathbf{F}}_p^{\ast T} \hat{\theta} + \hat{\theta}^T \hat{\mathbf{F}}_p^{\ast}) - \frac{1}{2} \hat{\theta}^T (b_{55} + b_p) \hat{\theta}.
\]
For a spherical buoy, there is neither excitation torque, $\vec{F}_p$, nor hydrodynamic damping, $b_{55}$ and $b_p$, on its rotational axis. Therefore, the power absorbed by the pitch mode of the buoy, $P_{\alpha p}$, is always zero.

2.2 Theory of Wave Power Absorption

2.2.1 Power Limits for Regular Waves

A body placed in water captures wave energy when it moves in an oscillatory manner and radiates waves in order to counteract the incident wave front. Thus, the maximum amount of power that can be removed from waves is defined by the radiating ability of the body. This limit has been derived in [4] and [5] for various motion modes. A well known equation characterising the maximum absorbed power by an axisymmetric body in monochromatic waves is [6]:

$$ P_{\text{rad}}^{\text{max}} = \alpha \frac{J}{2k}, $$

where $J = \rho g^2 D(kh)A^2/(4\omega)$ is the wave-energy transport per unit frontage of the incident wave, $\alpha$ is a coefficient that depends on the motion oscillation mode ($\alpha = 1$ for heave, $\alpha = 2$ for surge or pitch, and $\alpha = 3$ when the body oscillates in heave, surge and/or pitch simultaneously), $k$ is the wavenumber and is given by $k = \omega^2/g$ for deep water, $A$ is the wave amplitude, and $D(kh)$ is the depth function which is equal to 1 for deep water. Therefore, $P_{\text{rad}}^{\text{max}}$ in Equation (12) depends only on the mode of motion and decreases cubically as the wave frequency, $\omega$, increases.

The maximum power indicated by Equation (12) is obtained when the buoy velocities are [6]:

$$ \tilde{x}_{\text{opt}}(\omega) = \frac{1}{2} \vec{F}_s(\omega)/(b_{11}(\omega) + b_s(\omega)), $$

$$ \tilde{z}_{\text{opt}}(\omega) = \frac{1}{2} \vec{F}_h(\omega)/(b_{33}(\omega) + b_h(\omega)), $$

$$ \tilde{\theta}_{\text{opt}}(\omega) = \frac{1}{2} \vec{F}_p(\omega)/(b_{55}(\omega) + b_p(\omega)), $$

for surge, heave, and pitch modes, respectively. In waves with long period (low frequency), hydrodynamic damping coefficients are significantly small compared to the excitation force, and thus according to Equation (13) the body should move with extremely high velocities and displacements to absorb the absolute maximum power, $P_{\text{rad}}^{\text{max}}$. This is not possible in practice due to the design constraints associated with the buoy and the PTO machinery. Large PTO damping is usually used in this scenario to constrain the buoy velocity/displacement within the operating limit of the PTO machinery (e.g. PTO maximum stroke defined to protect the machinery from damage), which consequently reduces the absorbed power. Budal [22] then showed that the power extraction at low frequencies is limited by the swept volume of the body, which is a collective term for the body physical volume and the maximum motion amplitude of each mode subject to the specific constraints in WEC design. The expression of this low-frequency power limit is strongly dependent on the shape of the buoy, and for a submerged spherical buoy is given by [15]:
\[ P_{\text{swept}}^{\text{max}} = pe^{-kh}d_{\text{max}}V\omega^3/(2r) \]  

(14)

for both the surge and heave modes. In Equation (14), \( h_s \) denotes the submergence depth of the buoy (from buoy geometric centre to water surface) and \( d_{\text{max}} \) denotes the maximum buoy displacement of the mode. The \( P_{\text{rad}}^{\text{max}} \) and the \( P_{\text{swept}}^{\text{max}} \) graphs together form the well known Budal diagram [22].

2.2.2 Optimal Reactive Control

In Equation (13), the optimal condition is defined by the velocity of the buoy in each mode being in phase (resonance) with the excitation force and of optimal amplitude, given by the ratio of the relative excitation force to the relative hydrodynamic damping coefficient. This can be achieved by applying optimal reactive control to the PTO machinery. To illustrate the main concept underlying this control strategy, a submerged buoy constrained to move in heave only and subject to reactive PTO control is used as an example. The concept is equally applicable to the other modes such as surge and pitch. The motion equation for a submerged heaving buoy under reactive spring-damper control in the frequency domain can be extracted from Equation (6):

\[ (m + a_{33}(\omega))\ddot{z} + (b_{33}(\omega) + b_h(\omega) + B_{\text{pto}})\dot{z} + K_{\text{pto}}\dot{z} = \hat{F}_h(\omega). \]  

(15)

Substituting \( \dot{z} = j\omega z \) and \( \ddot{z} = -j\omega \dot{z} / \omega \) into Equation (15) results in:

\[ (b_{33}(\omega) + b_h(\omega) + B_{\text{pto}} + j\omega m + j\omega a_{33}(\omega) - j\frac{K_{\text{pto}}}{\omega})\ddot{z} = \hat{F}_h(\omega). \]  

(16)

Comparing Equation (16) with the optimal heave velocity equation in Equation (13b), the following optimal values of the PTO control parameters can be obtained:

\[ K_{\text{pto, opt}} = \omega^2(m + a_{33}(\omega)), \]  

(17a)

\[ B_{\text{pto, opt}} = b_{33}(\omega) + b_h(\omega), \]  

(17b)

where the optimal spring stiffness \( K_{\text{pto, opt}} \) forces the heave velocity to be in phase (resonance) with the excitation force, while the optimal damping \( B_{\text{pto, opt}} \) makes the amplitude of the radiated wave half of the amplitude of the incident wave [6]. Equation (17) is usually called optimal reactive control as the PTO spring provides a reactive force, and thus reactive power to the system. It is also often called complex conjugate, or impedance matching, control, as the optimal PTO load impedance is defined as the complex conjugate of the intrinsic mechanical impedance of the buoy [23]:

\[ \hat{z}_{\text{pto, opt}}(\omega) = \hat{z}_h^*(\omega), \]  

(18)

where \( \hat{z}_{\text{pto, opt}}(\omega) = B_{\text{pto, opt}} - j\frac{K_{\text{pto, opt}}}{\omega} \), and \( \hat{z}_h(\omega) = b_{33}(\omega) + b_h(\omega) + j\omega(m + a_{33}(\omega)) \).
2.3 Modal Analysis

2.3.1 Matrix Eigenvalue Problem for an Undamped Multiple DoF System

The eigenvalue problem is a commonly encountered problem in engineering and is the basis of modal analysis. The solution of an eigenvalue problem provides important physical meaning to a dynamic system. The general motion equation for the free vibration of an undamped \( n \) DoF system is given by:

\[
\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0,
\]

(19)

where \( \mathbf{M} \) and \( \mathbf{K} \) are the mass and stiffness matrices respectively. Assuming harmonic sinusoidal motion and substituting \( \mathbf{x} = \Phi\sin(\omega t + \phi) \) and \( \ddot{x} = -\omega^2\Phi\sin(\omega t + \phi) \) into Equation (19) results in:

\[
(\mathbf{K} - \omega^2\mathbf{M})\Phi = 0.
\]

(20)

The non-trivial solution of Equation (20) is given by:

\[
det(\mathbf{K} - \omega^2\mathbf{M}) = 0,
\]

(21)

where \( det \) is the determinant of the matrix, which can be expanded, forming an \( n \)th order polynomial for \( \omega^2 \). The roots of this polynomial are the eigenvalues of \( \mathbf{K}^{-1}\mathbf{M} \), whose square roots are the system natural frequencies \( [\omega_1, \omega_2, \ldots, \omega_n] \). Substituting each eigenvalue or natural frequency into Equation (20), a corresponding eigenvector \( \Phi \) can be derived. Therefore, the system has \( n \) eigenvectors \( [\Phi_1, \Phi_2, \ldots, \Phi_n] \), denoting the mode shapes of the system, also referred to as normal modes.

2.3.2 Orthogonality of Normal Modes

The orthogonality properties of an undamped \( n \) DoF system are manifested in the relationship between its spatial and modal models. Mode shapes are orthogonal to each other with respect to system mass and stiffness matrices [11]:

\[
\Phi_i^T\mathbf{M}\Phi_j = 0, \text{ for } i \neq j,
\]

\[
\Phi_i^T\mathbf{K}\Phi_j = 0, \text{ for } i \neq j,
\]

\[
\Phi_i^T\mathbf{M}\Phi_i = m_i,
\]

\[
\Phi_i^T\mathbf{K}\Phi_i = k_i,
\]

(22)

where \( \Phi_i \) and \( \Phi_j \) are the \( i \)th and \( j \)th eigenvectors/modes, respectively, \( m_i \) and \( k_i \) are the modal mass and modal stiffness of the \( i \)th mode, respectively. This is known as the principle of orthogonality. This principle can be utilised to transform the (usually coupled) spatial model to the uncoupled modal model for an undamped system [11]. For a system of light damping (\( 0 < \xi_i, \) damping ratio < 0.2 for all modes), the assumption is usually made that the damping matrix satisfies the same modal orthogonality properties as the mass and stiffness matrices [24]:

\[
\Phi_i^T\mathbf{B}\Phi_j = 0, \text{ for } i \neq j,
\]

\[
\Phi_i^T\mathbf{B}\Phi_i = b_i,
\]

(23)
where $\mathbf{B}$ is the damping matrix, and $b_i$ is the modal damping of the $i$th mode. Under this assumption, a lightly damped $n$ DoF system can be transformed from the original spatial form to the following modal form:

$$
\Psi^T \mathbf{M} \Psi \ddot{\mathbf{q}} + \Psi^T \mathbf{B} \Psi \dot{\mathbf{q}} + \Psi^T \mathbf{K} \Psi \mathbf{q} = \mathbf{N}_{exc},
$$

(24)

where $\Psi = [\Phi_1 \ \Phi_2 \ldots \ \Phi_n]$; $\ddot{\mathbf{q}}$ denotes the modal displacement defined by $\Psi^{-1} \dot{\mathbf{x}}$; $\mathbf{N}_{exc}$ denotes the modal excitation force defined by $\Psi^T \mathbf{F}_{exc}$. Substituting Equations (22) and (23) into Equation (24), the following decoupled modal model in matrix form is obtained:

$$
\begin{pmatrix}
    m_1 & 0 & \ldots & 0 \\
    m_2 & b_1 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & m_n
\end{pmatrix}
\ddot{\mathbf{q}} +
\begin{pmatrix}
    0 & b_1 & \ldots & 0 \\
    0 & b_2 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & b_n
\end{pmatrix}
\dot{\mathbf{q}} +
\begin{pmatrix}
    k_1 & 0 & \ldots & 0 \\
    0 & k_2 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & k_n
\end{pmatrix}
\mathbf{q} = \mathbf{N}_{exc}.
$$

(25)

2.3.3 Application in multi-mode WEC Design

To absorb the maximum energy from waves, a multi-mode PA WEC needs to achieve both the phase (resonance) and amplitude optimal conditions for at least two spatial modes: one radiates symmetric waves such as heave and the other radiates antisymmetric waves such as surge or pitch. The optimal phase condition can be achieved when the natural frequency of the mode is equal to the wave frequency, obtained by re-arranging Equation (17a). The amplitude optimal condition can occur when the PTO damping is equal to the hydrodynamic damping of the mode, as shown in Equation (17b).

The matrix eigenvalue problem defined by Equations (20) and (21) outputs the natural frequencies of a multi-mode PA WEC, $\omega_i$, as well as the corresponding mode shapes represented in the buoy Cartesian coordinates (e.g. $\Phi_i = [\Phi_{ix} \ \Phi_{iz} \ \Phi_{i\theta}]^T$ for a 3DoF WEC), which can be used to evaluate the phase optimality of the system modes. A PA WEC is usually lightly damped because it needs to oscillate with waves to extract energy. Thus, the damped spatial model of the PA in buoy Cartesian coordinates as shown in Equation (6) can be transformed to the decoupled modal form as shown in Equation (25) for the ease of evaluating the amplitude optimality of the modes and the controllability of the PA system. Modes that are not utilised or cannot be properly controlled for wave energy extraction should be designed off-resonant with waves to improve the overall durability of the system. Case studies are conducted on three multi-mode WEC designs having representative modal behaviour in the following three sections.

3. Modal Analysis on a Generic Single Tether Spherical Buoy PA

3.1 System Description

Fig. 2 shows a generic single tether spherical buoy (to be referred to as G1TSB) PA at its nominal/rest position, where the only tether is vertical and under tension which maintains the positively buoyant buoy fully submerged. The tension is generated by the PTO unit as part of the PTO force shown in Equation (5), and is given by:

$$
c_{pto} = -(\rho V - m)g.
$$

(26)
3.2 Frequency Domain Model

A time-domain model of the G1TSB PA system is discussed in detail in [20] and [25]. Applying linearisation to the system at its nominal/rest position (for more details about the linearisation procedure, please refer to [20]), the following mass and PTO damping and stiffness matrices associated with Equation (6) are obtained:

\[
\mathbf{M} = \begin{pmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & I_{yy}
\end{pmatrix},
\]

\[
\mathbf{B}_{pto} = \begin{pmatrix}
0 & 0 & 0 \\
0 & B_{pto} & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
\mathbf{K}_{pto} = \begin{pmatrix}
-C_{pto}/l_0 & 0 & C_{pto}r/l_0 \\
0 & K_{pto} & 0 \\
C_{pto}r/l_0 & 0 & -C_{pto}(l_0 + r)/l_0
\end{pmatrix}.
\] (27)

where there are three unknowns: PTO stiffness \(K_{pto}\), PTO damping \(B_{pto}\), and nominal tether length \(l_0\). In this study, it is assumed that any desired \(K_{pto}\) and \(B_{pto}\) can be achieved in the PTO design and \(l_0\) can be varied by changing the height of the mooring base above the ocean floor. The corresponding design challenges and economic viability, however, are out of the scope of the paper. At the nominal position of the buoy, the inverse Jacobian matrix of the G1TSB PA is given by:

\[
\mathbf{J}(\mathbf{x}_0)^{-1} = \begin{pmatrix}
0 & 1 & 0
\end{pmatrix}.
\] (28)
3.3 Natural Frequencies and Mode Shapes

Substituting the mass and stiffness matrices $\mathbf{M}$, $\mathbf{A}(\omega)$ and $\mathbf{K}_{\text{pto}}$ into Equations (20) and (21), the eigenvalue problem for the undamped G1TSB PA system can be solved, for varying nominal tether length $l_0$ and varying PTO stiffness $K_{\text{pto}}$. The resulting natural frequencies and mode shapes of the system are displayed in Fig. 3, as functions of the nominal tether length and the PTO stiffness. Fig. 3(a) shows two convex surfaces, denoting the natural frequencies of Mode 1 and Mode 3. Mode 1 is surge dominant as can be seen from its mode shapes shown in Fig. 3(b). Mode 2 is pitch dominant as evident in Fig. 3(c). Mode 3 is heave dominant as evident in Fig. 3(d). A graphical representation of Modes 1, 2, and 3 in the buoy Cartesian coordinates is shown in Fig. 4. As the pitch mode does not contribute to power absorption for a spherical buoy, Mode 2 is not shown in Fig. 3(a) for better visualisation of the natural frequencies of Mode 1 and Mode 3. As shown in Fig. 3(a), the natural frequency of Mode 1 increases as the nominal tether length decreases, because the G1TSB PA operating in this surge-dominant mode is analogous to an inverted pendulum under small displacements (see Fig. 4). Therefore, ignoring couplings between the surge and pitch modes in $\mathbf{K}_{\text{pto}}$ in Equation (27), the natural frequency of Mode 1 can be approximated as:

$$\omega_1 \approx \frac{(\rho V - m)g}{l_0(m + a_{11}(\omega))}. \quad (29)$$

On the other hand, the natural frequency of Mode 3 increases as the PTO stiffness increases, and is approximately equal to the natural frequency of a 1DoF heaving PA:

$$\omega_3 \approx \sqrt{\frac{K}{m + a_{33}(\omega)}}. \quad (30)$$

It is evident in Fig. 3(a) that by varying the PTO stiffness, the natural frequency of Mode 3 can reach any value between 0.3 and 1.5 rad/s, the typical wave frequency range, whilst the natural frequency of Mode 1 can only reach up to 0.7 rad/s when the nominal tether length decreases to be less than the buoy radius. The line of intersection of the Mode 1 and the Mode 3 surfaces indicates the wave frequency range where both modes can be tuned to resonance (i.e. optimal phase condition), as well as the required combination of nominal tether length and PTO stiffness.

Wave power is absorbed by the oscillation of the buoy, and then eventually absorbed by the PTO machinery or dissipated in the sea. Therefore, it is critical to consider the coupling between the PTO machinery and the normal modes of the buoy in Cartesian coordinates. The inverse Jacobian matrix can be used to transform the mode shapes in Cartesian coordinates to the contribution of the modes on tether elongation:

$$\epsilon_i = \mathbf{J}(x_o)^{-1} \Phi_i. \quad (31)$$

For the G1TSB PA system, the contribution of its three normal modes on the single tether is shown in Fig. 3(e). It is evident that only Mode 3 contributes to tether elongation that drives the PTO unit, and therefore actually contributes to PTO power absorption. As Mode 1 and Mode 2 do not contribute to PTO power generation, they should be designed as off-resonant modes to improve the durability of the whole system. In practice, additional modal coupling will occur due to the real nonlinear dynamics of the PA, however, the power arising from these coupling effects are negligible compared to the power in the primary modes.
Figure 3. Eigenanalysis on the generic single tether spherical buoy PA: (a) Natural frequencies of normal modes, $\omega_i$, plotted from two different visual angles; (b) Mode shapes of Mode 1, $\Phi_1$; (c) Mode shapes of Mode 2, $\Phi_2$; (d) Modes shapes of Mode 3, $\Phi_3$; (e) Contribution of normal modes to tether elongation, vs. nominal tether length $l_0$ (normalised by buoy radius $r$) and PTO stiffness $K_{pto}$ (normalised by the hydrostatic stiffness of a half-submerged buoy $K_{hs} = \frac{\rho \pi r^2}{2}$). In subplots (b), (c), (d) and (e), yellow colour indicates value close to 1; green colour indicates values close to 0; and purple colour indicates value close to -1. For an undamped system, its mode shapes are either in phase or $180^\circ$ out of phase with respect to the spatial modes (e.g. surge, heave, pitch), depending on the sign of the elements of the mode shapes, $\Phi_i = [\Phi_{ix} \quad \Phi_{iz} \quad \Phi_{i\theta}]^T$. 
3.4 Optimality across Wave Frequencies

For the ease of design in the mooring base, the tether is assumed to be anchored to the sea floor level, and thus \( l_0 = d_w - h_s - r = 36.5 \) m. At this nominal tether length, the natural frequencies of the G1TSB PA system as a function of PTO stiffness are shown in Fig. 5. By varying PTO stiffness, Mode 3 can be tuned to resonance or optimal phase condition across the wave frequency range, as also observed from the 3D plot shown in Fig. 3(a), whilst the natural frequencies of Modes 1 and 2 remain constant. The natural frequency of Mode 2 is much higher than the wave frequency upper limit and therefore is always an off-resonant mode as desired. The natural frequency of Mode 1 is at the edge of the wave frequency lower limit. Therefore, there is a probability that Mode 1 can reach resonance, in particular at seas sites where very long waves often occur. The natural frequency of Mode 1 can be further reduced by increasing buoy mass as shown in Equation (29).

As shown in Fig. 3, when the nominal tether length is 36.5 m, the mode shapes of the system form a constant matrix regardless of the variation of PTO stiffness:

\[
\Psi \approx \begin{pmatrix} -1 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\]
The elements of the mode shapes are rounded to one decimal point. Then applying the orthogonality of normal modes described in Section 2.3.2, the motion equation of the G1TSB system given by Equations (6), (7), and (27), can be transformed to the following decoupled modal form:

\[
\begin{pmatrix}
m + a_{11} & 0 & 0 \\
0 & l_{yy} & 0 \\
0 & 0 & m + a_{33}
\end{pmatrix}
\ddot{\tilde{q}} + \begin{pmatrix}
b_{11} + b_s & 0 & 0 \\
0 & (b_{11} + b_s)/100 & 0 \\
0 & 0 & b_{33} + b_h + B_{pto}
\end{pmatrix}
\ddot{\tilde{q}} +
\begin{pmatrix}
-C_{pto}/l_0 & 0 & 0 \\
0 & -C_{pto}r(l_0 + r)/l_0 & 0 \\
0 & 0 & -C_{pto}r/(5l_0) - C_{pto}/(100l_0) - K_{pto}
\end{pmatrix}
\dot{\tilde{q}} = \begin{pmatrix}
-f_s \\
0 \\
0
\end{pmatrix}.
\]

where the scalar numbers arise from the mode shape matrix given by Equation (32). Equation (33) shows that Mode 1 and Mode 2 are not affected by PTO spring, \( K_{pto} \), or damping, \( B_{pto} \), whilst Mode 3 can be reactively controlled by tuning PTO spring and damping to their optimal values as defined in Equation (17).

Since Mode 3 aligns with the heave mode as can be observed from Fig. 3(d), Fig. 4 and Equation (33), the G1TSB PA system can only absorb power from its heave motion, and thus one third of the maximum available wave power. To validate this conclusion made by modal analysis, power absorbed by the surge and heave modes of the reactively controlled system subject to incident waves of 0.1 m amplitude and frequencies between 0.3 and 1.5 rad/s are calculated, with results shown in Fig. 6. Also shown are the low and high frequency power bounds of the heave mode arising due to the swept volume limit (e.g. \( d_{max} = 3 \) m) and the radiation limit of the buoy, respectively. It can be seen from Fig. 6 that the G1TSB PA system only absorbs power from its heave mode. At low frequencies, the power absorbed by the heave mode is much lower than the power bound associated with swept volume limit because the effects of the viscous drag (which dissipates additional power from the system) were not considered which are significant at low wave frequencies but negligible at high wave frequencies.
frequencies [26]. Thus the power absorbed by the heave mode converges to the power bound associated with the radiation limit at high frequencies, meaning the PA can absorb at most one-third of the wave power.

4. Modal Analysis on a Three Tether Spherical Buoy PA

4.1 System Description

As shown in the previous section, the G1TSB PA system has control authority along its heave mode only and therefore is inefficient for power absorption. Thus, a three-tether spherical buoy (to be referred to as 3TSB) PA system was proposed to enhance kinematic coupling between the oscillating modes of the buoy and the PTO machinery installed along the tethers, enabling more control authority of the modes.

Fig. 7 illustrates a schematic of a 3TSB PA system at its rest pose, where the tethers are equally distributed around the buoy, separated by 120° in the horizontal plane. This configuration makes the system insensitive to wave direction. Tether 1 lies on the x-z plane, and thus the wave travelling direction, and tethers 2 and 3 are symmetric about the x-z plane. In this paper, it is assumed that all three tethers point towards the geometric centre of the spherical buoy and the PTO units along the three tethers have identical linear characteristics (e.g. pretension force $C_{pto}$, PTO damping $B_{pto}$ and stiffness $K_{pto}$). The tether attachment points are located on the surface of the buoy hull. The inclination angle of the tethers with respect to the z-axis, $\alpha$, plays an important role in this design as it defines the contribution of the surge and heave modes in the total absorbed power, as well as the effects of PTO control on the surge and heave modes [27]. Due to tether inclination, the pretension force along each tether is given by:

$$C_{pto} = -(\rho V - m)g/(3 \cos(\alpha)).$$  \hfill (34)

The system parameters are kept identical to the ones for the G1TSB PA system. All three tethers are anchored to the sea floor level and thus the nominal tether length $l_0$ is a function of the inclination angle $\alpha$. 

Figure 7. Three tether spherical buoy PA
4.2 Frequency Domain Model

A time-domain model of the 3TSB PA system is discussed in detail in [27]. Applying linearisation to the system at its nominal/rest position (for more details about the linearisation procedure, please refer to [28]), the following mass and PTO damping and stiffness matrices associated with Equation (6) are obtained:

\[
\mathbf{M} = \begin{pmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & l_{yy}
\end{pmatrix},
\]

\[
\mathbf{B}_{pto} = \begin{pmatrix}
\frac{3}{2}B_{pto} \sin^2(\alpha) & 0 & 0 \\
0 & 3B_{pto} \cos^2(\alpha) & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
\mathbf{K}_{pto} = \begin{pmatrix}
\frac{3}{2} \sin^2(\alpha) \left( K_{pto} + \frac{\mathbf{c}_{pto}}{l_0} \right) & 0 & \frac{3\mathbf{c}_{pto} \cos(\alpha)}{l_0} \\
0 & 3 \cos^2(\alpha) \left( K_{pto} + \frac{\mathbf{c}_{pto}}{l_0} \right) & 0 \\
\frac{3\mathbf{c}_{pto} \cos(\alpha)}{l_0} & 0 & \frac{-3\mathbf{c}_{pto} (l_0 + r)(\cos^2(\alpha) + 1)}{2l_0}
\end{pmatrix}. \quad (35)
\]

At the nominal position of the buoy, the inverse Jacobian matrix of the linear 3TSB PA is given by:

\[
\mathbf{J}^{-1}(\mathbf{x}_o) = \begin{pmatrix}
-\sin(\alpha) & \cos(\alpha) & 0 \\
\sin(\alpha)/2 & \cos(\alpha) & 0 \\
\sin(\alpha)/2 & \cos(\alpha) & 0
\end{pmatrix}. \quad (36)
\]

4.3 Natural Frequencies and Mode Shapes

Substituting the mass and stiffness matrices \(\mathbf{M}, \mathbf{A}(\omega)\) and \(\mathbf{K}_{pto}\) into Equations (20) and (21), the eigenvalue problem for the 3TSB PA system can be solved, for varying tether inclination angle \(\alpha\) and varying PTO stiffness \(K_{pto}\). The resulting natural frequencies and mode shapes of the system are displayed in Fig. 8, as functions of the tether inclination angle and the PTO stiffness. Fig. 8(a) shows two convex surfaces, denoting the natural frequencies of Mode 1 and Mode 3 respectively. Mode 1 is surge dominant as can be seen from its mode shapes shown in Fig. 8(b). Mode 2 is pitch dominant as evident in Fig. 8(c). Mode 3 is heave dominant as evident in Fig. 8(d). A graphical representation of Modes 1, 2, and 3 in the buoy Cartesian coordinates is shown in Fig. 9. As the pitch mode does not contribute to power absorption for a spherical buoy, the natural frequency of Mode 2 is not shown in Fig. 8(a). The line of intersection of the natural frequency surfaces of Modes 1 and 3 in Fig. 8(a) indicates that at tether inclination angle of 54.5 degrees, both modes can be simultaneously tuned to resonance (i.e. the optimal phase condition) across the wave frequency range between 0.3 and
1.5 rad/s by varying PTO stiffness. Fig. 10 plots this line of intersection against PTO stiffness. Fig. 8(e) shows the contribution of Mode 1 and Mode 3 to the elongations of the three tethers, mapped from the system mode shapes by the inverse Jacobian matrix. When the tether inclination angle increases from 0 to 90 degrees, the contribution gradually shifts from Mode 3 (heave) dominant (like the G1TSB PA system) to Mode 1 (surge) dominant. At a tether inclination angle of 54.5 degrees, Mode 1 and Mode 3 contribute to approximately equal tether elongations, thus both modes are able to cause considerable PTO power generation.

Figure 8. Eigenanalysis on the three-tether spherical buoy PA: (a) Natural frequencies of normal modes, $\omega_i$, plotted from two different visual angles; (b) Mode shapes of Mode 1; (c) Mode shapes of Mode 2; (d) Modes shapes of Mode 3; (e) Contribution of Mode 1 and Mode 3 to tether elongations, vs. tether inclination angle $\alpha$ in degrees and PTO stiffness $K_{pto}$ (normalised by the hydrostatic stiffness of a half-submerged buoy $K_{hs} = \rho \pi r^2$). In subplots (b), (c), (d) and (e), yellow colour indicates value close to 1; green colour indicates values close to 0; and purple colour indicates value close to -1.
Figure 9. Graphical representation of Modes 1, 2, 3 in the buoy Cartesian coordinates for the three-tether spherical buoy PA regardless of wave frequency. Mode 1 oscillates predominantly along surge; Mode 2 oscillates predominantly along pitch; Mode 3 oscillates predominantly along heave.

Figure 10. Natural frequencies of normal modes, $\omega_i$, vs. PTO stiffness $K_{pto}$ (normalised by the hydrostatic stiffness of a half-submerged buoy $K_{hs} = \rho \pi r^2$), when the tether inclination angle $\alpha$ is 54.5 degrees, for the three tether spherical buoy PA.

### 4.4 Optimality across Wave Frequencies

As discussed in Section 4.3, Mode 1 (surge-dominant) and Mode 3 (heave dominant) can be simultaneously tuned to resonance across the wave frequency range between 0.3 and 1.5 rad/s, for a tether inclination angle of 54.5 degrees and various PTO stiffness. In contrast, the natural frequency of Mode 2 (pitch-dominant) is insensitive to PTO stiffness and is higher than the usual wave frequency range as shown in Fig. 10, which is ideal as Mode 2 (pitch-dominant) is not necessary for absorbing maximum power from waves.

As shown in Fig. 8, the modes shapes of the system are insensitive to the variations of tether inclination angle and PTO stiffness, hence the mode shape matrix can be approximated by:

$$\Psi \approx \begin{pmatrix} -1 & 0.2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (37)

Then applying the orthogonality of normal modes described in Section 2.3.2, the motion equation of the 3TSB PA system can be transformed to the following decoupled modal form:
\[
\begin{pmatrix}
m + a_{11} & 0 & 0 \\
0 & l_{yy} & 0 \\
0 & 0 & m + a_{33}
\end{pmatrix}
\begin{pmatrix}
\dot{q} + \\
\dot{\ddot{q}} + \\
\dot{\ddot{q}}
\end{pmatrix}
\begin{pmatrix}
b_{11} + b_s + \frac{3}{2}B_{pto} \sin^2(\alpha) & 0 & 0 \\
0 & \left(b_{11} + b_s + \frac{3}{2}B_{pto} \sin^2(\alpha)\right)/25 & 0 \\
0 & 0 & b_{33} + b_h + 3B_{pto} \cos^2(\alpha)
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{2}{2} \sin^2(\alpha) \left(K_{pto} + \frac{c_{pto}}{l_0}\right) \\
- \frac{3c_{pto}/l_0}{l_0} \\
- \frac{3}{50} \sin^2(\alpha) \left(K_{pto} + \frac{c_{pto}}{l_0}\right) \\
\frac{3 \cos^2(\alpha) \left(K_{pto} + \frac{c_{pto}}{l_0}\right)}{l_0} \\
- \frac{3c_{pto}/l_0}{l_0} \\
- \frac{3}{50} \sin^2(\alpha) \left(K_{pto} + \frac{c_{pto}}{l_0}\right) \\
\frac{3 \cos^2(\alpha) \left(K_{pto} + \frac{c_{pto}}{l_0}\right)}{l_0}
\end{pmatrix}
\begin{pmatrix}
-F_s \\
F_s/5 \\
F_h
\end{pmatrix}.
\]

Equation (38) shows that the 3TSB PA system has control authority along all three modes, although Mode 2 is much less controllable than the other two modes. In order to absorb the maximum available power from waves, Mode 1 and Mode 3 are also required to satisfy the optimal amplitude conditions, as shown in the decoupled damping matrix in Equation (38):

\[
\frac{3}{2}B_{pto} \sin^2(\alpha) = b_{11} + b_s,
\]

\[
3B_{pto} \cos^2(\alpha) = b_{33} + b_h.
\]

Solving the system of two equations in Equation (39) across the wave frequency range, the frequency-dependent PTO damping and tether inclination angle that result in the optimal amplitude condition for Mode 1 and Mode 3 are obtained, as shown in Fig. 11. The required tether inclination angle varies between 43 and 60 degrees, which conflicts with the previous finding that tether inclination angle should be always fixed at 54.5 degrees to satisfy the optimal phase condition for Mode 1 and Mode 3 across the wave frequency range. Therefore, by having identical PTO characteristics (e.g. spring and damping) on all three tethers, Mode 1 (surge dominant) and Mode 3 (heave dominant) of the 3TSB PA system cannot be simultaneously and optimally controlled except when the system is excited by very low frequency waves. In this case, the determination of the frequency-dependent optimal PTO parameters for the 3TSB PA needs an additional numerical process (e.g. power optimization), which is discussed in detail in [26] and [28].
Figure 11. Tether inclination angle and PTO damping that achieve the optimal amplitude condition for Modes 1 and 3 across the wave frequencies, for the three tether spherical buoy PA.

The power absorbed by the surge and heave modes of the reactively controlled system (with the optimised PTO parameters in [26]) subject to incident waves of 0.1 m amplitude and frequencies between 0.3 and 1.5 rad/s are calculated, with results shown in Fig. 12. Also plotted are the high frequency power bounds of the heave and surge modes due to the radiation limit of the buoy. In addition, the maximum power absorbed by a 1DoF surging buoy and a 1DoF heaving buoy with identical buoy parameters as the 3TSB PA are plotted for a benchmark. It is evident that the surge mode and heave mode of the 3TSB PA absorb slightly less power than the combination of a 1DoF surging buoy and a 1DoF heaving buoy, and thus absorb almost the maximum available power from waves. In general, the surge mode absorbs two times more power than the heave mode, except at low frequencies where the surge mode dissipates more power than the heave mode due to higher viscous losses ($b_s > b_h$). The power absorption curves of the surge and heave modes converge to the corresponding power radiation limits at high frequencies, where viscous losses are negligible compared to radiation loss ($b_{11} \gg b_s$ and $b_{33} \gg b_h$). In order to absorb the absolute maximum available power from waves, the 3TSB PA must have distinct PTO characteristics (e.g. spring and damping) on the three tethers.

Figure 12. Power absorbed by the surge (blue solid line) and heave (red solid line) modes of the 3TSB PA system under reactive control, subject to incident waves of 0.1 m amplitude and 0.3-1.5 rad/s frequencies; power bounds of the heave mode (cyan dashed line) and surge mode (cyan dashdot line) caused by radiation limit, $P_{\text{rad}}^{\text{max}}$ defined by Equation (12); maximum power absorbed by a 1DoF surging buoy (magenta dashed line) and a 1DoF heaving buoy (green dashed line) with identical buoy parameters as the 3TSB PA.
5. Modal Analysis on an Asymmetric Mass Distribution Spherical Buoy PA

5.1 System Description

The 3TSB PA system is almost able to absorb the maximum available power from waves, however its economic viability is governed by the manufacturing costs of the PTO machineries along the tethers. Thus, to reduce the costs associated with PTO machineries, Meng et al. [10] proposed a single tether asymmetric mass distribution spherical buoy (1TAMDSB) PA, as shown in Fig. 13(a). The concept is to simply shift the buoy’s centre of gravity away from its geometric centre to enhance dynamic coupling between the oscillating modes, and thus to provide control authority of all the modes from a single PTO, tether. Fig. 13(b) shows a cross section view of the 1TAMDSB PA system on the x-z plane that the buoy is symmetric about.

The buoy consists of a spherical hull with a radius $r$ and a mass $m_h$, and an additional mass $m_o$ offset from the geometric centre of the buoy, resulting in an eccentric buoy centre of gravity on the x-z plane. The location of $m_o$ on the x-z plane is defined by the offset distance $r_o$ and the offset angle $\varphi$ with respect to the positive x-axis. For the buoy to remain at rest in calm water, the PTO pretension (offset) force is given by

$$C_{pto} = -(\rho V - m_h - m_o)g.$$  \hfill (40)

In addition, in order to balance the torque generated by the offset mass about the geometric centre of the buoy, the tether attachment point is rotated clockwise around the surface of the hull on the x-z plane. The line passing through the geometric centre of buoy and the tether

![PTO Diagram](image)

Figure 13. Single tether asymmetric mass distribution spherical buoy PA (left) and schematic highlighting the buoy variables in the buoy Cartesian coordinates (right).

Table 2. Additional buoy parameters for the asymmetric mass distribution spherical buoy PA

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value/unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_h$</td>
<td>$1.34 \times 10^5$ kg</td>
<td>Hull mass for the buoy with asymmetric mass distribution</td>
</tr>
<tr>
<td>$m_o$</td>
<td>$1.34 \times 10^5$ kg</td>
<td>Offset mass for the buoy with asymmetric mass distribution</td>
</tr>
<tr>
<td>$r_o$</td>
<td>4 m</td>
<td>Offset distance of the offset mass $m_o$</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>$30^\circ$</td>
<td>Offset angle of the offset mass $m_o$</td>
</tr>
<tr>
<td>$I_{yy,o}$</td>
<td>$4.383 \times 10^6$ kg.m$^2$</td>
<td>Moment of inertia of a hollow spherical buoy with offset mass about its pitch axis, given by $I_{yy,o} = \frac{2}{5} m_h r^2 + m_o r_o^2$</td>
</tr>
</tbody>
</table>
attachment point forms an angle $\beta$ with respect to the negative $z$-axis, which can be calculated by solving

$$c_{pto}r \sin(\beta) = m_o g r_o \cos(\varphi).$$  \hspace{1cm} (41)

The system parameters are kept identical to the ones for the GITSB and the 3TSB PA systems, except for the ones specific to the 1TAMDSB PA as listed in Table 2. The settings in Table 2 are optimal for the 1TAMDSB PA system subject to low frequency waves, and are used to demonstrate the application of modal analysis in evaluating a tightly coupled multi-mode WEC system.

5.2 Frequency Domain Model

The linearisation procedure for the 1TAMDSB PA is similar to the one for the GITSB PA, as detailed in [13]. The resulting mass, PTO damping and stiffness matrices associated with Equation (6) are given by:

$$M = \begin{pmatrix}
m_h + m_o & 0 & -m_o r_o \sin(\varphi) \\
0 & m_h + m_o & -m_o r_o \cos(\varphi) \\
-m_o r_o \sin(\varphi) & -m_o r_o \cos(\varphi) & I_{yy,o}
\end{pmatrix},$$

$$B_{pto} = \begin{pmatrix}
0 & 0 & 0 \\
0 & B_{pto} & B_{pto} r \sin(\beta) \\
0 & B_{pto} r \sin(\beta) & B_{pto} r^2 \sin^2(\beta)
\end{pmatrix},$$

$$K_{pto} = \begin{pmatrix}
-C_{pto}/l_0 & 0 & C_{pto} r \cos(\beta)/l_0 \\
0 & K_{pto} & K_{pto} r \sin(\beta) \\
C_{pto} r \cos(\beta)/l_0 & K_{pto} r \sin(\beta) & (-C_{pto} r \cos(\beta) (l_0 + r \cos(\beta))/l_0 \\
-m_o g r_o \sin(\varphi) + K_{pto} r^2 \sin^2(\beta)
\end{pmatrix}.$$  \hspace{1cm} (42)

Compared with the matrices for the GITSB PA in Equation (27), $M$, $B_{pto}$, and $K_{pto}$ for the 1TAMDSB PA in Equation (42) exhibit additional couplings (e.g. heave-pitch and surge-pitch) due to the asymmetric mass distribution.

At the nominal position of the buoy, the inverse Jacobian matrix of the 1TAMDSB PA is given by:

$$J(x_o)^{-1} = \begin{pmatrix}
0 & 1 & r \sin(\beta)
\end{pmatrix}.$$  \hspace{1cm} (43)

5.3 Natural Frequencies and Mode Shapes

Substituting the mass and stiffness matrices $M$, $A(\omega)$ and $K_{pto}$ into Equations (20) and (21), the eigenvalue problem for the 1TAMDSB PA system can be solved for varying nominal tether length $l_0$ and PTO stiffness $K_{pto}$. The resulting natural frequencies and mode shapes of the system are displayed in Fig. 14 as functions of the nominal tether length and the PTO stiffness. Fig. 14(a) shows two convex surfaces, denoting the natural frequencies of Mode 1 and Mode 3 respectively, which do not intersect but become extremely close, as shown in the zoomed-in area. This implies that Mode 1 and Mode 3 cannot be simultaneously tuned to
resonance through varying the nominal tether length or PTO stiffness. Individually, Mode 1 can be tuned to resonance across the typical wave frequency range, whilst Mode 3 can only be tuned to resonance at frequencies between 0.3 and 0.8 rad/s. The natural frequency of Mode 2 is always higher than the typical wave frequency range, thus, Mode 2 is not considered in this

![Figure 14. Eigenanalysis on the single tether asymmetric mass distribution spherical buoy PA: (a) Natural frequencies of normal modes, $\omega_i$, plotted from two different visual angles; (b) Mode shapes of Mode 1; (c) Mode shapes of Mode 3; (d) Contribution of Mode 1 and Mode 3 to tether elongation, vs. nominal tether length $l_0$ (normalised by buoy radius $r$) and PTO stiffness $K_{pto}$ (normalised by the hydrostatic stiffness of a half-submerged buoy $K_{hs} = \rho \pi r^2$). In subplots (b), (c), (d) and (e), yellow colour indicates value close to 1; green colour indicates values close to 0; and purple colour indicates value close to -1.](image)
analysis. The mode shapes of the 1TAMDSB PA system vary as the PTO characteristics change. As shown in Fig. 14(b), Mode 1 gradually switches from a heave dominant mode to a surge dominant mode, when the PTO stiffness and the nominal tether length are such that the natural frequencies of Modes 1 and 3 are almost equal. This region of optimality will be herein referred to as “the natural frequency asymptote”. Mode 3 shows a similar but opposite trend to Mode 1, as shown in Fig. 14(c), transitioning from a surge dominant mode to a heave dominant mode. Fig. 14(d) shows the contribution of Mode 1 and Mode 3 to the elongation of the single tether, mapped from the mode shapes by the inverse Jacobian matrix. It is evident that Mode 1 and Mode 3 have nearly equal contributions to the tether elongation near the natural frequency asymptote. The observations from Fig. 12 imply that the optimal combination of nominal tether length and PTO stiffness must exist near the natural frequency asymptote.

The natural frequencies and mode shapes of the 1TAMDSB system are also plotted in 2D in Figure 15 for a better visualisation. The left subplots are for fixed \( l_0/r = 5 \) and varying \( K_{pto}/K_{hs} \), while the right subplots are for fixed \( K_{pto}/K_{hs} = 6 \) and varying \( l_0/r \). For very low wave frequency waves, Mode 1 and Mode 3 can almost be tuned to resonance simultaneously as evident by the tiny gap between the natural frequency surfaces as shown in the zoomed-in area in Fig. 14(a), which can be more clearly observed from Fig. 15(a). When both Mode 1 and Mode 3 operate near resonance, their mode shapes have almost equal \( \Phi_z \) component and opposite \( \Phi_x \) component as shown in Fig. 15(b), meaning that the two modes oscillate along the \( z \) axis (heave) with equal amplitude and equal phase, and oscillate along the \( x \) axis (surge) with equal amplitude but opposite phase, as illustrated in the buoy Cartesian coordinates in Fig. 16. Consequently, Mode 1 and Mode 3 have almost equal contributions to the tether elongation, as shown in Fig. 15(c), similar to the 3TSB PA. When the wave frequency increases, and thus the required natural frequencies of the modes increase, it becomes difficult to tune both modes into resonance, as shown in Fig. 14(a) and Fig. 15(d). In addition, it can be observed from Fig. 15(d), Fig. 15(e) and Fig. 15(f) that the optimal tether length occurring near the natural frequency asymptote is extremely short, as highlighted in the shaded grey regions where \( l_0/r < 1 \). This generates additional challenges in the practical PA design, and thus a nominal tether length lower limit is introduced for modal analysis (e.g. assuming \( l_0/r \) must be greater than 1). This limit further narrows down the wave frequency range within which Mode 1 can be tuned to resonance to between 0.3-0.7 rad/s. At the nominal tether length lower limit, Modes 1 and 3 start to show surge- and heave- dominant behaviour, as evident in Fig. 15(e), and Mode 3 contributes to more tether elongation than Mode 1, as evident in Fig. 15(f).

### 5.4 Optimality across Wave Frequencies

As the wave frequency increases, it is increasingly difficult to simultaneously tune Modes 1 and 3 into resonance, particularly Mode 1 which can only be tuned to resonance for wave frequencies up to 0.7 rad/s given the lower limit is applied to the nominal tether length \( (l_0/r \geq 1) \). Due to this limit, there is a trend that the mode shapes of Modes 1 and 3 rotate anticlockwise as wave frequency increases, from 45 degrees with respect to the heave and surge spatial modes (as shown in Fig. 16) at very low frequencies to almost aligning with the heave and surge modes at high frequencies:
where the subscripts denote the corresponding wave frequency. Again, applying the

\[ \mathbf{\Psi}_{0.4} \approx \begin{pmatrix} -0.7 & 0.2 & 0.7 \\ 0.7 & 0.7 & 0.7 \\ 0 & 0.7 & 0 \end{pmatrix} \rightarrow \mathbf{\Psi}_{0.7} \approx \begin{pmatrix} -1 & 0 & 0.3 \\ 0.3 & 0.7 & 1 \\ 0 & 0.6 & 0 \end{pmatrix} \rightarrow \mathbf{\Psi}_{1.2} \approx \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0.8 & 1 \\ 0 & 0.5 & 0 \end{pmatrix}, \quad (44) \]

The grey region denotes the impractical case where \( l_0/r < 1 \).

Figure 15. Eigenanalysis on the single tether asymmetric mass distribution spherical buoy PA: (a), (b) and (c) on the left hand side show the natural frequencies, mode shapes, and contribution of modes to tether elongation, respectively, vs. PTO stiffness \( K_{pto}/K_{hs} \), when \( l_0/r = 5 \); (d), (e) and (f) on the right hand side show the natural frequencies, mode shapes, and contribution of modes to tether elongation, respectively, vs. nominal tether length \( l_0 \), when \( K_{pto}/K_{hs} = 6 \).
orthogonality of normal modes, the following decoupled modal forms (without showing Mode 2 for the ease of comparison) are obtained for wave frequencies at 0.4, 0.7 and 1.2 rad/s, respectively:

For $\omega = 0.4$ rad/s,

$$
\begin{pmatrix}
    m + 0.5a_{11} + 0.5a_{33} & 0 \\
    0 & m + 0.5a_{11} + 0.5a_{33}
\end{pmatrix}
\begin{pmatrix}
    \ddot{\mathbf{q}}
\end{pmatrix}

+ 

\begin{pmatrix}
    0.5b_{w1} + 0.5b_{w3} + 0.5B_{pto} \\
    0.5b_{w1} + 0.5b_{w3} + 0.5B_{pto}
\end{pmatrix}
\begin{pmatrix}
    \dot{\mathbf{q}}
\end{pmatrix}

+ 

\begin{pmatrix}
    -0.5C_{pto}/l_0 + 0.5K_{pto} \\
    0
\end{pmatrix}
\begin{pmatrix}
    \mathbf{q}
\end{pmatrix}

= 

\begin{pmatrix}
    -0.7\dot{F}_s + 0.7\dot{F}_h \\
    0.7\dot{F}_s + 0.7\dot{F}_h
\end{pmatrix}.

(45)

For $\omega = 0.7$ rad/s,

$$
\begin{pmatrix}
    1.1m + a_{11} + a_{33} & 0 \\
    0 & 1.1m + a_{11} + a_{33}
\end{pmatrix}
\begin{pmatrix}
    \ddot{\mathbf{q}}
\end{pmatrix}

+ 

\begin{pmatrix}
    b_{w1} + 0.1b_{w3} + 0.1B_{pto} \\
    0.1b_{w1} + b_{w3} + B_{pto}
\end{pmatrix}
\begin{pmatrix}
    \dot{\mathbf{q}}
\end{pmatrix}

+ 

\begin{pmatrix}
    -C_{pto}/l_0 + 0.1K_{pto} \\
    0
\end{pmatrix}
\begin{pmatrix}
    \mathbf{q}
\end{pmatrix}

= 

\begin{pmatrix}
    -\dot{F}_s + 0.3\dot{F}_h \\
    0.3\dot{F}_s + \dot{F}_h
\end{pmatrix}.

(46)

For $\omega = 1.2$ rad/s,

$$
\begin{pmatrix}
    m + a_{11} & 0 \\
    0 & m + a_{33}
\end{pmatrix}
\begin{pmatrix}
    \ddot{\mathbf{q}}
\end{pmatrix}

+ 

\begin{pmatrix}
    b_{w1} & 0 \\
    0 & b_{w3} + B_{pto}
\end{pmatrix}
\begin{pmatrix}
    \dot{\mathbf{q}}
\end{pmatrix}

+ 

\begin{pmatrix}
    -C_{pto}/l_0 & 0 \\
    0 & K_{pto}
\end{pmatrix}
\begin{pmatrix}
    \mathbf{q}
\end{pmatrix}

= 

\begin{pmatrix}
    -\dot{F}_s \\
    \dot{F}_h
\end{pmatrix}.

(47)

where $m = m_h + m_o$, $b_{w1} = b_{11} + b_s$, and $b_{w3} = b_{33} + b_h$. The modal form in Equation (45) shows that at $\omega = 0.4$ rad/s, Mode 1 and Mode 3 have almost identical motion equations and are affected almost equally by the PTO control parameters, $B_{pto}$ and $K_{pto}$, considering round-off errors in the mode shapes used for the transformation. This indicates that Modes 1

Figure 16. Graphical representation of Modes 1, 2 and 3 in the buoy Cartesian coordinates for the single tether asymmetric mass distribution spherical buoy PA at the natural frequency asymptote. Mode 1 oscillates predominantly along surge and heave with equal amplitude but opposite phase. Mode 3 oscillates predominantly along surge and heave with equal amplitude and equal phase. Mode 2 oscillates predominantly along heave and pitch with equal amplitude and equal phase. Please note that the natural frequency of Mode 2 is much higher than the wave frequency range, and thus Mode 2 barely contributes to power absorption.
and 3 can be simultaneously tuned close to their optimal operating conditions for very low frequency waves, and thus can absorb nearly the maximum available wave power. In contrast, the modal form in Equation (46) for $\omega = 0.7 \text{ rad/s}$ shows two distinct motion equations (as evident by the distinct diagonal terms in the matrices) controlled by the same PTO parameters. Thus a compromise exists in tuning both Modes 1 and 3 to their optimal operating conditions for low to medium frequency waves. For example, it is evident in the modal damping matrix that the PTO damping $B_{pto}$ required to tune Modes 1 and 3 to the optimal amplitude condition exhibits a difference in the order of one magnitude. Mode 1 can absorb more power than Mode 3, as Mode 1 couples more with the surge mode, as evident from the modal excitation force in Equation (46). Therefore, Mode 1 has the priority to be tuned to the optimal operating condition whenever possible for low to medium frequency waves. The modal form in Equation (47) for $\omega = 1.2 \text{ rad/s}$ is almost (to the limit of the round-off errors in the mode shapes for transformation) identical to the modal form of the G1TSB PA system, and thus can only absorb about one third of the maximum available wave power for high frequency waves. The exact frequency-dependent optimal PTO parameters for the 1TAMDSB PA were determined by numerical optimization, as discussed in detail in [28].

The power absorbed by the surge and heave modes of the reactively controlled system subject to incident waves of 0.1 m amplitude and frequencies between 0.3 and 1.5 rad/s are calculated, with results shown in Fig. 17. The optimal PTO parameters determined in [28] are adopted. Also plotted are the radiation limit power bounds and the maximum power absorbed by a 1DoF heaving buoy and a 1DoF surging buoy, for a benchmark. It is evident that optimisation results match the conclusions made from modal analysis for the 1TAMDSB PA. At very low frequencies (Region I), the PA absorbs nearly the maximum power from waves similar to the 3TSB PA. Between low and medium frequencies (Region II), the PA can only tune Mode 1 (surge dominant) into the optimal operating condition and therefore can absorb full power from the surge mode and partial power from the heave mode. When the wave frequency further increases (into Region III), the PA can no longer achieve the optimal operating condition for Mode 1 and thus decreases power absorption from Mode 1 while

![Figure 17](https://via.placeholder.com/150)

Figure 17. Power absorbed by the surge (blue solid line) and heave (red solid line) modes of the 1TAMDSB PA system under reactive control, subject to incident waves of 0.1 m amplitude and 0.3-1.5 rad/s frequencies; power bound of the heave mode (cyan dashed line) and surge mode (cyan dashdot line) caused by radiation limit, $P_{rad max}$ defined by Equation (12); maximum power absorbed by a 1DoF surging buoy (magenta dashed line) and a 1DoF heaving buoy (green dashed line) with identical buoy parameters as the 1TAMDSB PA. Black dashed vertical lines divide the frequency-dependent performance of the PA into four regions that have distinct modal behaviour.
increases power absorption from Mode 3. Finally at high frequencies (Region IV), the PA
switches to absorb power from the heave mode dominantly, and thus approaches the
performance of the G1TSB PA.

6. Conclusion

This paper introduced the application of modal analysis in the design of multi-mode WECs
and used three submerged PA systems with distinct modal behaviour as case studies to illustrate
the procedure and the efficacy of the method. The natural frequencies and the mode shapes of
normal modes were used to understand the phase condition of the spatial modes of the systems.
The decoupled modal form was used to understand the amplitude condition of the spatial modes
and the controllability of the systems. Both the phase and the amplitude conditions associated
with modes determine the power absorption potential of the multi-mode WEC design.
Controllability further suggests the compromise faced in multi-mode control and thus the limit
of the proposed design concepts.

Modal analysis provided an in-depth understanding on the three multi-mode submerged PA
systems. The single tether generic spherical buoy PA demonstrates an extremely controllable
mode (Mode 3, heave-dominant) that contributes to 100% of WEC power absorption, as well
as two barely controllable modes (Mode 1, surge-dominant, and Mode 2, pitch-dominant) that
should be designed off-resonance to increase system durability. This PA design can only
efficiently operate in heave, and this is constrained to absorb one third of the maximum power
from the waves. The three tether spherical buoy PA, in contrast, while exhibiting similar modal
behaviour (e.g. mode shapes) to the single tether generic spherical buoy PA, shows almost
equal control authorities along Mode 1 (surge dominant) and Mode 3 (surge dominant) at the
optimal tether inclination angle of 54.5 degrees. Whilst having identical PTO characteristics
on all three tethers does not allow the PA to achieve the absolute optimal phase and amplitude
conditions for both modes, the three tether spherical buoy PA is able to extract nearly the
maximum available power from waves. Finally, the single tether asymmetric mass distribution
spherical buoy PA shows complex modal behaviour that is wave frequency dependent due to
the nature and physical limits of the design. For very low frequency waves, Modes 1 and 3 are
both surge-heave coupled modes, which can be almost simultaneously tuned to optimal,
leading to power absorption capability similar to the three tether spherical buoy PA. For low
to medium frequency waves, Mode 1 becomes surge-dominant and Mode 3 becomes heave-
dominant, which results a trade-off in tuning both modes close to their optimal. Therefore,
Mode 1 (surge-dominant) is optimally controlled as a priority in this wave frequency range as
it absorbs more power from waves. For high frequency waves, Mode 1 (surge-dominant) can
no longer be tuned to the optimal phase condition, and thus Mode 3 (heave-dominant) is
optimally controlled instead, resulting in power absorption performance approaching the single
tether generic spherical buoy PA.

In summary, modal analysis was proven to be an efficient analytical approach that can be
used to assess and understand the optimality and the limit of the multi-mode PA design before
more in-depth studies are conducted using time-consuming numerical simulation (e.g. CFD)
and fine tuning optimisation. Modal analysis can also be used to assist the interpretation of the
optimisation results, which are usually not intuitive for complex multi-mode PA systems
operating in several DoFs, in particular in a later development stage where the system model
gets more complex by involving additional moving parts (e.g. moving components of the PTO transmission system). Future work will focus on investigating the utilisation of modal analysis in the wave-to-wave optimal control of multi-mode PA systems in stochastic seas.

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Reference