ACTIVE CONTROL OF VIBRATION IN STIFFENED STRUCTURES

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Abstract

ACTIVE CONTROL OF VIBRATION
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ABSTRACT

Active control of vibration in structures has been investigated by an increasing number of researchers in recent years. There has been a great deal of theoretical work and some experiment examining the use of point forces for vibration control, and more recently, the use of thin piezoelectric crystals laminated to the surfaces of structures. However, control by point forces is impractical, requiring large reaction masses, and the forces generated by laminated piezoelectric crystals are not sufficient to control vibration in large and heavy structures.

The control of flexural vibrations in stiffened structures using piezoceramic stack actuators placed between stiffener flanges and the structure is examined theoretically and experimentally in this thesis. Used in this way, piezoceramic actuators are capable of developing much higher forces than laminated piezoelectric crystals, and no reaction mass is required. This thesis aims to show the feasibility of active vibration control using piezoceramic actuators and angle stiffeners in a variety of fundamental structures.

The work is divided into three parts. In the first, the simple case of a single actuator used to control vibration in a beam is examined. In the second, vibration in stiffened plates is
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controlled using multiple actuators, and in the third, the control of vibration in a ring-stiffened cylinder is investigated.

In each section, the classical equations of motion are used to develop theoretical models describing the vibration of the structures with and without active vibration control. The effects of the angle stiffener(s) are included in the analysis. The models are used to establish the quantitative effects of variation in frequency, the location of control source(s) and the location of the error sensor(s) on the achievable attenuation and the control forces required for optimal control. Comparison is also made between the results for the cases with multiple control sources driven by the same signal and with multiple independently driven control sources. Both finite and semi-finite structures are examined to enable comparison between the results for travelling waves and standing waves in each of the three structure types.

This thesis attempts to provide physical explanations for all the observed variations in achievable attenuation and control force(s) with varied frequency, control source location and error sensor location. The analysis of the simpler cases aids in interpreting the results for the more complicated cases.

Experimental results are given to demonstrate the accuracy of the theoretical models in each section. Trials are performed on a stiffened beam with a single control source and a single error sensor, a stiffened plate with three control sources and a line of error sensors and a ring-stiffened cylinder with six control sources and a ring of error sensors. The experimental
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results are compared with theory for each structure for the two cases with and without active vibration control.
STATEMENT OF ORIGINALITY

To the best of my knowledge and belief all of the material presented in this thesis, except where otherwise referenced, is my own original work, and has not been presented previously for the award of any other degree or diploma in any University. If accepted for the award of the degree of Doctor of Philosophy, I consent that this thesis be made available for loan and photocopying.

Andrew J. Young
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1.1 INTRODUCTION

In this thesis, the feedforward active control of harmonic flexural vibration in three types of stiffened structures using as control sources piezoceramic actuators placed between the stiffener flange and the structure surface is investigated. The first structure considered is a beam of rectangular cross-section with a mock stiffener mounted across the larger cross-sectional dimension. The analysis of vibration in the beam is treated as a one-dimensional problem. The second structure considered is a rectangular plate with a stiffener mounted across the width of the plate. The transverse vibration of the plate is treated as a two-dimensional problem. Finally, a ring-stiffened cylindrical structure is analysed. Vibration in each of the radial, axial and tangential directions is considered. The thesis is presented in three main chapters, each considering one type of structure, but the study of the more complicated structures makes use of results from the simpler cases.

The control of flexural vibrations in a simple beam is considered in Chapter 2, where the classical one-dimensional equation of motion for flexural vibration is used to develop a theoretical model for the vibration of a beam with a primary point source, an angle stiffener and a control actuator. The effective control signal is a combination of the effects of the point force at the base of the actuator, and the reaction force and moment at the base of the stiffener.
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Figure 1.1 Beam showing primary source, angle stiffener, piezoceramic stack control source and error sensor.

The displacement at a point is the sum of the displacements due to each of the primary source and control source forces and moments. Optimal control is achieved by minimising the total mean square displacement at the location of a single error sensor downstream of the control source.

The theoretical analysis considers four different sets of classical beam supports; infinite, free, simply supported and fixed. The influences of the control source location, the error sensor location and the excitation frequency on the control source amplitude and achievable attenuation are investigated, and the physical reasons for each observation are explained. The effects of introducing a second control source and angle stiffener and a second error sensor are also examined.

Experimental verification of the beam model is undertaken for four different sets of beam terminations; infinite, free, simply supported and vibration isolated. The impedance
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corresponding to each type of termination is first measured from the experimental apparatus. Experimental results are compared with theoretical predictions for the four cases.

The control of flexural vibrations in a plate is considered in Chapter 3, where the classical two-dimensional equation of motion for flexural vibration is used to develop a theoretical model for the vibration of a plate with one or more primary point sources, an angle stiffener and one or more control actuators. A modal analysis of the plate with and without a stiffener attached shows that the stiffener makes a significant difference to the vibration response of the plate, so the theoretical model is modified to include the effects of the stiffener. The effective control signal is a combination of the effects of the point force at the base of each actuator, and the line force and line moment at the base of the stiffener.

Figure 1.2 Plate showing primary sources, angle stiffener, piezoceramic stack control actuators and error sensors.
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The displacement at a point is the sum of the displacements due to each of the primary source and control source forces and moments. Optimal control is achieved by minimising the total mean square displacement at a line of error sensors across the plate downstream of the control sources. Consideration is given to the results achieved with control sources driven by the same signal and with control sources driven independently.

The theoretical analysis considers two different sets of plate supports. In both cases, the edges of the plate are simply supported and the end upstream of the primary sources is modelled as free. In the first case, the end downstream of the error sensors is modelled as infinite, and in the second case the downstream end is modelled as free. The influences of the location of the control sources, the location of the line of error sensors and the excitation frequency on the control source amplitude and achievable attenuation are investigated, and the physical reasons for each observation are explained. The effect of introducing a second angle stiffener and additional control sources is also examined.

Experimental verification of the plate model is performed for the case with the upstream end modelled as free and the downstream end modelled as infinite.

Finally, the more complicated case of control of flexural vibrations in a ring-stiffened cylinder is considered in Chapter 4. The three equations of motion for vibration of a cylinder in the radial, tangential and axial directions are used to develop a theoretical model for the vibration of a cylinder with one or more primary point sources, a ring stiffener and one or
more control actuators. The effective control signal is a combination of the effects of the point force at the base of each actuator, and the line force and line moment around the circumference of the cylinder at the base of the ring stiffener.

![Diagram of cylinder with primary sources, ring stiffener, piezoceramic stack control actuators, and error sensors.](image)

**Figure 1.3** Cylinder showing primary sources, ring stiffener, piezoceramic stack control actuators and error sensors.

The displacement at a point is the sum of the displacements due to each of the primary source and control source forces and moments. Optimal control is achieved by minimising the total mean square displacement at a ring of error sensors around the cylinder circumference downstream of the control sources. Consideration is given to the results achieved with control sources driven by the same signal and with control sources driven independently.
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The theoretical analysis considers two different sets of cylinder supports. In both cases, the end upstream of the primary sources is modelled as simply supported. In the first case, the end downstream of the error sensors is modelled as infinite, and in the second case the downstream end is modelled as simply supported. The influences of the location of the control sources, the location of the ring of error sensors and the excitation frequency on the control source amplitude and achievable attenuation are investigated, and the physical reasons for each observation are explained.

Experimental verification of the cylinder model is performed for the simply supported cylinder.

In Chapter 5 the results of each structural model are reviewed. The similarities between the three cases are described and the trends established from one model to the next are summarised. The difficulties in controlling vibration in the three types of structure are described, and the implications for controlling vibration in real stiffened structures are presented.
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1.2 LITERATURE REVIEW

1.2.1 Analysis of vibration in continuous structures

1.2.1.1 The differential equations of motion

The differential equations of motion for the displacement of simple continuous structures and their basic solutions have been known for a long time. The solutions to the equations of motion for a beam and plate are discussed in various well-known texts (Morse, 1948; Wang, 1953; Timoshenko, 1959; Cremer, Heckl and Ungar, 1973; Graff, 1975, Meirovitch, 1975). Vibration in cylindrical shells is also discussed briefly in the texts by Graff and Cremer, Heckl and Ungar, and in more detail by Timoshenko. Flügge (1960) developed one version of the three-dimensional equations of motion for the vibration of shells and gave solutions for a variety of shell types.

Leissa's famous monographs *Vibration of Plates* (1969) and *Vibration of Shells* (1973a) are comprehensive summaries of the analyses of plate and shell vibration to that time. Leissa's works present results for the free vibration frequencies and modes shapes for plates and shells with a wide array of geometries and boundary conditions.

Noiseux (1970) presented a significant work introducing the concepts of near and far fields of vibrational intensity, using the solutions to the beam and plate equations. The near field exists in the region of a discontinuity, boundary or actuator, and contains reactive and active components. In the far field the reactive component is insignificant. Pavić (1976) determined a method of measuring structural intensity using an array of vibration sensors.
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The concepts of near and far fields of vibrational intensity are important in understanding some of the mechanisms involved in active vibration control, particularly when selecting control source and error sensor locations (Sections 2.3, 3.3 and 4.3).

The vibration of beams is a relatively simple problem, and will not be discussed in any great detail in this review. The analysis of vibration in plates and cylinders has received more attention, and is discussed in more detail in Sections 1.2.1.3 and 1.2.1.4 respectively.

1.2.1.2 Treatment of termination impedances in theoretical analysis

The effect on the vibration response of the types of supports used to mount a beam, plate or shell is significant. In any theoretical analysis, the impedances of the structure supports must be modelled. Traditionally, assumptions corresponding to one of a set of classical impedances have been used to model simply supported structures, free structures, fixed structures and semi-infinite structures.

In 1977, Seybert and Ross developed a method for measuring the acoustic impedance of a duct termination, using two microphones placed in a tube with the system under investigation at one end. The method showed that the incident- and reflected-wave spectra and the phase angle between incident and reflected waves, could be determined from measurement of the auto- and cross-spectra of the two microphone signals. Expressions for the normal specific acoustic impedance and reflection coefficient of the impedance material were then developed. Højbjerg (1991) improved the design of the microphones to be used in the impedance tube.
Until recently, there has been no equivalent method for measuring beam impedances. In 1990, Trolle and Luzzato examined the problem, developing a simple method for identifying the four coefficients in the solution for beam displacement from a minimum of four acceleration measurements. They did not apply their work to the measurement of termination impedances. Fuller et al (1990) used a similar analysis to find the displacement equation coefficients. With the solution for displacement, they calculated the force and moment impedances of a blocking mass on a beam. They ignored the coupling impedances associated with a termination discussed by Cremer, Heckl and Ungar (1973).

Taylor (1990) presented an alternative method using a measurement of structural intensity to identify the impedance of a beam termination, as well as the related reflection coefficients. While reducing the number of acceleration measurements to three, the method for measuring structural intensity and then calculating impedances is significantly more complicated than the method described by Fuller et al. More recent investigations have attempted to improve the accuracy of structural intensity measurements (Halkyard and Mace, 1993; Gibbs and Fuller, 1993; Linjama and Lahti, 1993).

In Section 2.4.1, this thesis examines measurement of the impedance of real beam terminations in order to better compare the experimental and theoretical results for active vibration control. The method used is based on the simple method developed by Fuller et al (1990), but here an attempt is made to include the force and moment coupling impedances in the analysis, rather than ignoring them as did Fuller et al. The termination impedances of
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wire supported, pinned and anechoically terminated beams are measured and compared with the corresponding classical approximations. The number of accelerometers required to give reasonable results using this method is also examined.

When discussing the vibration of plates and cylinders, classical impedance approximations have been used. Development of a method for measuring the impedance of a plate or cylinder support would be far more complicated than that for a beam, and is outside the scope of this work. It is acknowledged that some of the observed differences between theoretical results and experimental data may be due to the inaccuracy of the classical support approximations.

1.2.1.3 Analysis of vibration in rectangular plates

There basics of free vibration in plates are discussed in texts by Timoshenko (1959), Cremer et al. (1973), Graff (1975) and others. Early work was concerned exclusively with determining the natural frequencies of plates with a variety of supports and geometries, as can been seen in Leissa's extensive 1969 summary of works to that time. Leissa (1973b) also gave a broad analytical study of free vibration in rectangular plates, using the Ritz method to examine the natural frequencies of plates with a variety of classical edge conditions. These analyses utilised the two-dimensional displacement equation with various boundary conditions corresponding to each of the edge constraints. Different forms of the general solution to the displacement equation were used to satisfy the boundary conditions in each case, and the characteristic equation solved for the natural frequencies.
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Mukhopadhyay (1978, 1979) developed a new numerical method for determining the natural frequencies of rectangular plates with different degrees of elastic edge restraints, again beginning with the two dimensional displacement equation. The general solution satisfying a given set of boundary conditions was transformed, reduced to an ordinary differential equation, and expressed in finite difference form. The solution of the resulting eigenvalue problem yielded the plate free vibration natural frequencies. Numerical results were presented for some cases, and these results agreed closely with results given in previous work (such as Leissa's).

Other researchers have examined vibration in plates with stiffeners attached using a wide variety of approximate methods. Kirk (1970) used the Ritz method to determine the natural frequencies of a plate with a single stiffener placed on a centre line, and the results compared closely with the exact solution obtained from the plate equation. Wu and Liu (1988) also examined vibration in plates with intermediate stiffening using the Rayleigh-Ritz method. They calculated the first four natural frequencies for some examples, and the results agreed closely with those of Leissa (1973a). Aksu and Ali (1976) used the finite difference method for the free vibration analysis of a rectangular plate with a single stiffener. Cheng and Dade (1990) used bicubic B-splines as coordinate functions to formulate problems based on energy principles, using the technique of piecewise Guass integration collocation. This method could be used with non-uniform thicknesses of plates. Yet another method, utilising Poynting vector formulation, has been used in structural intensity analysis of plates, by Romano et al. (1990).
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The most widely used approximate method is the finite element method, applied to plates with discrete stiffeners by Olson and Hazell (1977) and Gupta et al (1986). Mead, Zhu and Bardell (1988) examined vibration in a flat plate with an orthogonal array of rectangular stiffeners using this method. Koko and Olson (1992) used super plate and beam finite elements which were constructed so that only a single element per bay or span was required, resulting in an economic model of the orthogonally stiffened structure.

These approximate methods have been developed largely because the exact solution of the plate differential equation is cumbersome, particularly when stiffeners are attached (Koko and Olson, 1992). While these approximate methods were useful in analysing the free vibration of plates with or without attached stiffeners, they have not generally been used to develop models for active vibration control. Some researchers in recent years, particularly those interested in developing a theoretical model for the active feedforward vibration control of plates, have returned to the exact solution of the plate equation; however, none have included stiffeners in their analysis (Section 1.2.2.6).


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1.2.1.4 Analysis of vibration in cylindrical shells

There has been much discussion in the literature regarding the equations of motion for cylindrical shells. Unlike the classical equations of motion for the beam and plate, there is no universally accepted set of equations for the cylinder. Because of the complexity of vibrations in cylinders, a large number of assumptions must be made when deriving the equations of motion from the basic strain displacement relationships. Different authors have made distinct assumptions at various points in their derivation, arriving at slightly different equations of motion.

Leissa (1973a) gave an extensive summary of the equations derived by the better known authors. He showed that, with a few exceptions, all the theories were very similar and produced results consistent to within a few percent in most cases. In his discussion, Leissa demonstrated that, of all the theories, Flügge's (1934, 1960) work was the most referred to by other researchers, such as Yu (1955), Hoppmann (1957), Forsberg (1964, 1966), Reismann (1968), Reismann and Pawlik (1968), Smith and Haft (1968) and Warburton (1969). Recent researchers have also used the equations of motion developed by Flügge (e.g. Fuller, 1981; Haung, 1991; Païdoussis et al, 1992).

The main criticism of Flügge's version of the equations of motion for a cylinder has been his omission of inertia terms. Leissa (1973a) showed that the omission of these terms can cause inaccurate results, and most researchers include the inertia terms in their analyses.
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The early work described using the Flügge equations has been directed towards calculation of the natural frequencies of a variety of cylinders. More recent authors include complicating effects in their analyses, such as the effect of wall discontinuities on the propagation of waves (Fuller, 1981) and the coupled vibrations of shells containing liquids (Haung, 1991 and Païdoussis et al, 1992). No author has previously developed and presented quantitative solutions for the acceleration (or velocity or displacement) amplitude distribution over a cylinder in response to a specific form of vibration excitation, from the Flügge equations or any other of the similar equations of motion for cylinders. In Section 4.2 of this thesis, the Flügge equations, with the inertia terms included, are used to develop a model for the vibration of a cylinder in response to the application of a point force, a line force and a line moment.

Wah and Hu (1968) examined the free vibration of ring-stiffened cylinders using a highly simplified set of the equations of motion for shells. They divided the ring-stiffened cylinder into bays and considered the effect of the stiffener on the boundary conditions at the junction of two bays. In this thesis the effects of ring-stiffeners on the vibration response of a cylinder are included in the solution to the full set of Flügge equations, without dividing the cylinder into bays (Section 4.2).

A variety of approximate methods have also been used in the analysis of vibration of stiffened shells. Galletly (1954), Mikulas and MôElman (1965) and Patel and Neubert (1970) used energy methods in their determinations of the natural frequencies of stiffened shells. More
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Finite element analysis has also been used in the analysis of shell vibration (e.g. Bogner et al, 1967; Cantin and Clough, 1968; Henschell, Neale and Warburton, 1971 and Orris and Petyt, 1974). Recently, Mustafa and Ali (1987) applied structural symmetry techniques to the prediction of the natural frequencies of stiffened and unstiffened shells and developed boundary conditions for the analysis of part-shells. Mecitoglu and Dökmeci (1991) used a finite element analysis including smeared stringers and frames. Langley (1992) took stiffeners to be smeared over the surface of an element, with a view to analysing complex aircraft structures using only a few elements. Sinha and Mukhopadhyay (1994) used high precision curved triangular elements in their analysis, allowing greater flexibility in placement of stiffeners in the shell model.

As was the case with the analysis of vibrations in plates, the approximate methods used for the investigation of vibration in cylinders have been developed because the exact solution of the differential equations of motion is unwieldy. While these approximate methods have been useful in analysing the free vibration of cylindrical shells, they have not been used to develop models for active vibration control.
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1.2.2 Active vibration control

1.2.2.1 The origins of active noise and vibration control

The fields of active noise control and active vibration control have much in common. Active noise control can be traced back to Lueg's work in the nineteen thirties, although his ideas were far in advance of the technology required for practical noise control systems (Guicking, 1990). Nearly twenty years later, Olson (1953, 1956) experimented with an "electronic sound absorber". His arrangement consisted of a microphone mounted in close proximity to the face of a loudspeaker cone. The loudspeaker was driven to null the sound pressure at the microphone, creating a quiet area around it. His results were promising, but the electronics technology of his time was still not sufficiently advanced to enable implementation in useful applications. Conover's 1956 application of loudspeakers arranged around a noisy transformer was another early attempt at noise control. However, it was not until the late nineteen sixties and early seventies that electronics technology became advanced enough to make implementation of basic noise control systems practical (Snyder, 1990).

Despite this early investigation into the active control of noise, and discussion of active control of structural damping in the late seventies (Section 1.2.2.2), it wasn't until the eighties that advances in vibration actuator technology made active vibration control a feasible alternative in practical situations. The introduction of the piezoelectric actuator to vibration control by Bailey and Hubbard (1985) marked a new era in active vibration control (Section 1.2.2.3).
1.2.2.2 Development of feedback vibration control methods

Most early active vibration control theory considered modal feedback control of large structures. Balas and Canavin (1977) discussed feedback damping control of large spacecraft structures. Balas (1978) applied theoretical modal control using velocity feedback to a simple beam. Meirovitch and Öz (1980), with later work by Meirovitch and others (Meirovitch, Baruh and Öz, 1983; Meirovitch and Norris, 1984; Zhu and Bardell, 1985; Meirovitch and Bennighof, 1986), expanded the modal control method, with Meirovitch and Bennighof arriving at a method they described as Independent Modal-Space Control (IMSC), where a coordinate transformation was used to decouple a complicated system into a set of independent second order systems in terms of modal coordinates. Baz and Poh (1988) made modifications to the IMSC method to minimise the effect of control spillover into unmodelled modes and also into modelled modes when the number of modelled modes exceeded the number of control sources used.

In 1986 von Flotow presented another feedback control solution which considered modification of the disturbance propagation characteristics of the structure. The discussion indicated that the amount of control achieved using this method would be limited at low frequencies. The results were later compared with the velocity feedback method and with experiments on a beam (von Flotow and Schäfer, 1986).

Feedback control methods like those discussed are suited to the control of vibrations in very large structures with many structural members and in situations where it is difficult to obtain
a suitable reference signal. In feedback systems, design of the control system is dependent to
a large degree on the physical analysis of the structure. In the design of feedforward control
systems, the physical analysis of the structure can be separated from the design of the
electronic controller. In recent years, a significant amount of research effort has been
concerned with feedforward control. In this thesis, the emphasis is on analysis of the physical
mechanisms behind the feedforward active control of harmonic vibration in structures
consisting of relatively few fundamental elements, in structures where reference signals can
be obtained. However, the results relating to the performance of the piezoceramic stack
control actuators are still relevant to feedback control systems.

1.2.2.3 Actuators for active vibration control

Electromagnetic shakers were used as control actuators in much of the early experimental
work on active vibration control (Noiseux, 1970). While electromagnetic shakers are useful
tools in experimental work, their usefulness in practical applications is severely limited by
their size and mass.

Bailey and Hubbard (1985) introduced piezoelectric actuators to active vibration control.
They used the actuators bonded to the surface of a cantilever beam in their feedback vibration
damping design. Crawley and de Luis (1987) presented an analytical and experimental
development of piezoelectric actuators as vibration exciters. Using the models they
developed from stress/strain relationships, Crawley and de Luis were able to predict the
displacement of three real cantilevered beam and piezoelectric actuator arrangements under
steady-state resonance vibration conditions. Their work demonstrated several important results regarding the stress/strain behaviour of piezoelectric actuators, including the effect of stiffer and thinner bonding layers.

Dimitriadis et al (1991) and performed a two-dimensional extension of Crawley and de Luis' work, applying pairs of laminated piezoelectric actuators to a plate. They demonstrated that the location and shape of the actuator dramatically affected the vibration response of the plate. Kim and Jones (1991) performed another study into the use of laminated piezoelectric actuators, following a similar analytical method to that used by Crawley and de Luis (1987) and Dimitriadis et al (1991). They calculated the optimal thickness of piezoelectric actuators, and investigated the influence of the thickness and material properties of the bonding layer and piezoelectric actuators on the effective moment generated and the optimal actuator thickness.

Clark et al (1991) performed tests on a simply supported beam excited by pairs of piezoelectric actuators bonded to either side. They compared test results with theoretical predictions made using the one-dimensional beam equation altered to include the effects of the piezoelectric crystals. They discussed the idea of modelling a piezoelectric element by two pairs of line moments acting along the edges of the element. Their test results agreed to within 25% of the theoretical predictions. The paper concluded that discrepancies between experiment and theory were the result of some of the assumptions they made in modelling the piezoelectric elements and the beam using the one-dimensional beam theory. The
assumptions ignored the increase in beam stiffness due to strain in the normal direction to the vibration and the fact that the piezoelectric elements were not as wide as the beam. The authors suggested that a finite element analysis may be required to predict the beam response more accurately, but for the purposes of choosing optimal actuator locations and relative structural responses, the one-dimensional model was sufficient.

Lester and Lefebvre (1993) extended to a cylinder the theoretical models of Dimitriadis et al (1991) applying piezoelectric actuators to plates. They developed two models; in the first the actuator acted on the cylinder through line moments along the actuator edges, and in the second through in-plane forces along the actuator edges. They used a modal amplitude analysis to show that the in-plane force model predicted better coupling between the actuator and lower order modes than the bending model, and suggested that the in-plane force model may be more suitable in the development of a model for the use of distributed piezoelectric actuators as vibration control actuators. Rivory et al (1994) reviewed and extended previous models of beam - laminated actuator systems and compared theoretical results with experimental data to show that the models did not predict accurately the response of the beam to excitation by laminated actuators at frequencies away from the beam resonances.

The purpose of this thesis is to investigate the use of a piezoceramic stack actuator placed between the flange of a stiffener and the surface of a structure to control vibration in the structure. In this thesis, it is the one-dimensional model that is used to describe the response of the beam to the piezoceramic stack actuators (Section 2.2.3). Since these actuators act
Chapter 1. Introduction

essentially at a point, not over an area like laminated piezoelectric actuators, some of the problems experienced by Clark et al, Rivory et al and others will not be important.

Many other researchers have used piezoelectric actuators in active vibration control experiments. Fansen and Chen (1986) and Baz and Poh (1988, 1990) presented results for active control of beams using piezoelectric actuators, showing again the potential of piezoelectric actuators as control actuators in vibration control where low forces are required. Tzou and Gadre (1989) developed a theoretical model for the active feedback control of a multi-layered shell with distributed piezoelectric control actuators using Love's theory. Their analysis included a detailed examination of the stresses and strains between layers. Other investigators to examine the use of laminated piezoelectric actuators in vibration excitation and control include Wang et al (1991), Liao and Sung (1991), Pan et al (1992), Tzou and Fu (1994a and 1994b) and Clark and Fuller (1994). Clearly there is a perceived need for a vibration actuator that can be used in practical situations, where the large reaction mass of an electromagnetic shaker is unsuitable (Rivory, 1992). The laminated piezoelectric actuator has become the popular alternative, but this type of actuator is fragile and is not capable of generating great amounts of force. This thesis examines the suitability of another type of actuator, the piezoceramic stack actuator, which is capable of producing much higher forces for vibration excitation or vibration control than the thin laminated actuators.

The application of piezoceramic stack actuators to control of vibrations in rotating machinery was considered in a paper by Palazzolo et al in 1989. Their experimental results indicated
Chapter 1. Introduction

that significant reductions in the vibration of rotating machinery could be achieved using two of these actuators in the support structure of the rotating shaft. To the author’s knowledge, no research has been performed considering the use of the stack actuator to control vibrations in any other type of structure, such as the structures considered in this thesis.

1.2.2.4 Error sensors for active vibration control

Traditionally, accelerometers have been used for vibration measurement and as error sensors for active control of vibration. Along with the introduction of piezoelectric materials for vibration excitation, the use of piezoelectrics such as polyvinylidene fluoride (PVDF) for vibration sensing has developed (Bailey and Hubbard, 1985; Burke and Hubbard, 1988; Clark and Fuller, 1992). Cox and Lindner (1991) discussed optical fibre sensors for vibration control. Distributed vibration sensors such as the piezoelectric and optical fibre sensors can be shaped and placed to generate suitable error signals when only certain modes of vibration are to be controlled; for example, when reducing the far field noise emitted from a plate (Clark and Fuller, 1992).

Thomas et al used minimisation of vibrational kinetic energy (1993a) and acoustic potential energy (1993b) as the cost functions for the feedforward active control of harmonic vibration. Clark and Fuller (1994) presented the results of experimental work dealing with the control of harmonic vibrations in an enclosed cylinder using between three and six piezoelectric patches as control sources and microphones or polyvinylidene fluoride vibration sensors as cost
functions. Both investigations showed that simple vibration error sensors generally give poor results for the attenuation of transmitted or radiated sound in comparison to acoustic sensors, because use of vibration sensors does not necessarily lead to minimisation of the modes that contribute most to the radiated or transmitted sound power.

The aim in this thesis is to minimise the total vibratory power transmission downstream of the control sources. Pan and Hansen (1993a,1995a) demonstrated that simple acceleration measurements can be used to optimally reduce total vibration transmission provided the error sensors are not located in the vibratory near field produced by the control sources. Accelerometer measurements are used here as the cost functions for active vibration control.
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1.2.2.5 Feedforward active control of vibration in beams

Much of the research in recent years has focussed on feedforward control of harmonic flexural vibration in simple structures.

Redman-White et al (1987) used feedforward control in their experimental work. They used two closely-spaced point force control actuators to control harmonic flexural vibration in a beam excited by a single point source primary excitation. They showed that minimising vibration at the location of the control source is not sufficient for reducing power transmission downstream; rather, velocity should be minimised at a point far downstream of the control sources. Xia Pan and Hansen (1993a) showed that minimisation of velocity at a point and minimisation of flexural wave power give identical results provided the error sensor is located in the far field of the primary and control sources. Minimisation of power transmission gives better results when the error sensor is in the near field of the control source. In this thesis work, minimisation of velocity (acceleration) at a point is used, with the error sensor generally in the far field of the primary and control sources.

Mace (1987) examined theoretically the active control of harmonic flexural vibration in beams. He discussed the excitation and control of vibration from two types of actuator; point force and point moment. Mace treated the point force and point moment as discontinuities in the shear force and bending moment in the beam with resulting discontinuities in the displacement solution for the beam equation at the point of application of the force or moment. This analysis is followed in this thesis for the treatment of vibration in beams...
(Section 2.2.3) and extended to include line moments and line forces on plates and cylinders (Sections 3.2.3 and 4.2.4).

Gonidou (1988) used a similar analysis in his treatment of beam vibration control. He concluded that good feedforward control can be achieved with a single control source if power transmission is used as the cost function. As already mentioned, Xia Pan and Hansen (1993a) showed that minimising velocity was equivalent to minimising power transmission provided the error sensor was located outside of the control source near field. In the current work, a single control source is considered and acceleration at a point is used as the cost function. The dependence of attenuation on control source - error sensor separation is examined in Section 2.3.3.

In 1990 Gibbs and Fuller controlled harmonic flexural vibration in experiments on a thin beam using laminated piezoelectric actuators and an adaptive feedforward least squares controller, achieving 30 dB or more attenuation for beams with a variety of termination types. Jie Pan and Hansen (1990a) performed some similar experiments on beams, but used a single point force as the control source. Both papers demonstrated that reducing harmonic power transmission using a secondary source had the effect of reducing the power input from the primary source. This idea is consistent with the analogous case for ducts (Snyder, 1990) and is widely accepted.

Xia Pan and Hansen (1993b) solved the one-dimensional beam displacement equation,
treat the point of application of the primary and control point sources as discontinuities in the beam displacement in a similar manner to Mace (1987). Their analysis discussed some of the effects of variations in control source location, error sensor location, beam termination type and frequency on the attenuation of vibration level, but without attempting to explain the physical reasons for the observations. In Section 2.3 this thesis examines the effect of control source location, error sensor location, frequency and type of termination on the magnitude of the control signal and the achievable attenuation, offering explanations for all observations. The aim is to develop a full understanding of the physical behaviour of a beam vibration control system. This work will aid in understanding the behaviour of the more complicated stiffened plate and stiffened cylinder vibration control systems in later chapters.

Recent work by Petersson (1993a) discussed the significance of the moment in a combined moment and force excitation of a beam. Numerical results indicate that moments must be taken into account at all frequencies. When a piezoceramic actuator is placed between the flange of a stiffener and the beam surface, there is a moment reaction as well as a force reaction at the base of the stiffener. Petersson's work indicates that the moment may be a significant part of the effective control signal. In Section 2.2.3.2 of this thesis, the one-dimensional displacement equation is solved for a point moment excitation, treating the moment as a discontinuity in the beam displacement, following the analysis of Xia Pan and Hansen (1993b) for a point force, and Mace (1987). In Section 2.2.5, the effective control signal is analysed in terms of the point forces and the point moment developed by the piezoceramic actuator and angle stiffener combination.
Chapter 1. Introduction

Fuller et al (1990), Jie Pan and Hansen (1990a) and Xia Pan and Hansen (1993a) have made limited comparisons between theoretical results and experimental data for active feedforward control of vibration in beams. In Section 2.5 of this thesis, direct comparison is made between the numerical results and experimental data for vibration of beams both with and without active vibration control for an aluminium beam with four types of termination.

1.2.2.6 Feedforward active control of vibration in plates

Dimitriadis et al (1991) used the two dimensional plate displacement equation in their study of the application of piezoelectric actuators in control of harmonic flexural vibration in unstiffened plates. The displacement equation was solved for the vibration responses to line moments parallel to the x- and y-axes. The results of their work showed that control of vibrations in plates using bonded piezoelectric actuators was possible, and that the location of the actuators and the excitation frequency are important factors in the effectiveness of the control achieved. They did not investigate in any depth the dependency of control effectiveness on actuator location and frequency.

In a two-part presentation, Fuller (1990) and Metcalf et al (1992) investigated analytically and experimentally the effectiveness of active feedforward control of sound transmission and radiation from a plate using one and two point force control sources and acoustic error sensors. Metcalf et al showed that use of acoustic sensors gives greater attenuation of radiated sound than vibration sensors. The purpose of this thesis is to investigate control of vibration, not control of radiated sound, and accordingly vibration sensors are used rather
Recent work by Pan and Hansen (1994) used a classical plate equation solution to compare piezoelectric crystal with point force excitation of beams and plates. An "equivalence ratio" was derived, which could be used to determine the magnitude of a point force which would give the same root mean square displacement amplitude as a piezoelectric actuator at some position downstream of the actuator. Pan and Hansen (1995a) used a similar analysis to investigate active control of power transmission along a semi-infinite plate with a variety of piezoelectric actuator configurations. It was found that the effectiveness of the vibration control depended greatly on location of the control actuator and excitation frequency. In this thesis, the dependence of attenuation and control source amplitude and phase on frequency, control source location and error sensor location are investigated for the control of vibration in plates using three piezoceramic stack control actuators and an angle stiffener (Section 3.3).

In analysing the effectiveness of piezoceramic stack control of a stiffened plate, the plate displacement equation is solved directly to develop models for the plate excited by point forces and line moments parallel to the y-axis, following the work of Dimitriadis et al (1991) and Pan and Hansen (1994). In addition, a new solution is developed for the application of line forces parallel to the y-axis (Section 3.2). The effect of attached stiffeners on the vibration response of the plate is included in the classical model for the first time.

Fuller et al (1989) were the first to demonstrate the potential effectiveness of piezoelectric
Chapter 1. Introduction

actuator control of sound radiated from plates in experimental work. Their results show
global reductions in sound radiated from a panel of the order of 30 dB. To this author's
knowledge, there have been few experimental works dealing with feedforward active control
of vibration in plates and none directly comparing experimental data with theoretical
predictions for the acceleration distribution over a plate with vibration control. In Section 3.5
of this thesis, direct comparison is made between theoretical predictions and experimental
data for the vibration response of a stiffened plate with and without active vibration control.

1.2.2.7 Feedforward active control of vibration in cylinders

The feedforward active control of vibration in cylinders and aircraft fuselages has gained
increasing attention in recent years, but largely in the interests of reducing the sound radiated
from a vibrating shell or transmitted through it rather than reducing the vibration transmitted
along the cylinder.

Fuller and Jones (1987) performed an experimental investigation into the control of acoustic
pressure inside a closed cylinder using an external acoustic monopole primary source, a single
point force vibration control source and microphone sensors mounted on a traverse inside the
cylinder. This work was extended later to include more control sources (Jones and Fuller;
1989). Significant global attenuation was achieved for harmonic excitation. Elliott et al
(1989) have performed successful experiments on the control of aircraft noise in-flight, but
using acoustic rather than vibration control sources. Thomas et al (1993a,1993b) and Clark
and Fuller (1994) investigated the effect of using different acoustic and vibration error
Chapter 1. Introduction

sensors in the active vibration control of cylinders (discussed briefly in Section 1.2.2.4).

The work done by researchers such as Fuller with Jones and Clark, and Thomas et al shows that active vibration control of aircraft noise is an application of vibration control that has great potential for implementation in real systems in the near future. The purpose of this thesis is to investigate the feasibility of using piezoceramic stack actuators as control sources in real active vibration control situations to reduce vibration transmission along large cylinders such as those found in large aircraft and submarines. The aim of this work can be met by demonstrating that piezoceramic stack actuators can be used as control sources to significantly reduce vibration in cylinders (Chapter 4). Investigation of the effects of excitation frequency, number of control sources, control source location and error sensor location on the amount of control achieved is included (Section 4.3) to establish trends that may be significant when taking the next step to practical implementation.

There has been little work done on the feedforward active control of vibrations in cylinders. To the author's knowledge, none has directly compared theoretical results with experimental data. In this thesis, the theoretical model developed to examine the use of the stack actuators in cylinder vibration control is verified experimentally (Section 4.5).
Chapter 1. Introduction

1.3 SUMMARY OF THE MAIN GAPS IN CURRENT KNOWLEDGE ADDRESSED BY THIS THESIS

The primary aim of this thesis is to investigate the potential of the piezoceramic stack actuator in feedforward active control of harmonic vibration. The actuator is suitable specifically for use in stiffened structures (or structures with stiffeners added). This application of piezoceramic stack actuators is new. A theoretical model is developed to describe the response of beams, plates and cylinders to excitation by the stack actuator. The solutions of the classical equations of motion for plate and cylinder structures including the effects of stiffening, and the analysis of the forces and moments applied to all three structure types by the stack actuator and angle stiffener control source are new. The solution of the three dimensional equations of motion for a cylinder to determine the vibration response to force and moment excitations is original.

For each type of structure, the physical reasons for the given results are discussed, particularly in relation to the variation of control effort and attenuation as functions of control location, error sensor location and frequency. Some of this discussion represents an extension of previous work and much of it is new.

Prior to this work, there has been some work published directly comparing theoretical results and experimental data for active vibration control of beams, but almost none for the active vibration control of plates or cylinders. Here direct comparison is made between the theoretical models and experimental data for vibration in beams, plates and cylinders, for a variety of cases with and without active vibration control.
Chapter 2. Control of vibrations in a stiffened beam

CHAPTER 2. FEEDFORWARD ACTIVE CONTROL OF FLEXURAL VIBRATION IN A BEAM USING A PIEZOCERAMIC ACTUATOR AND AN ANGLE STIFFENER

2.1 INTRODUCTION

In this chapter, the active control of flexural vibration in beams using as a control source a piezoceramic actuator placed between a stiffener flange and the beam surface is investigated. The beam is of rectangular cross-section with a mock stiffener mounted across the larger cross-sectional dimension. The analysis of vibration in the beam is treated as a one-dimensional problem. The classical one-dimensional equation of motion for the flexural vibration of a beam is used to develop a theoretical model for the beam with a primary point source and an angle stiffener and control actuator (Section 2.2). The effective control signal is a combination of the effects of the point force at the base of the actuator, and the reaction force and moment at the base of the stiffener (Section 2.2.5).

Figure 2.1 Beam showing primary source, angle stiffener, piezoceramic stack control source and error sensor.
Chapter 2. Control of vibrations in a stiffened beam

The displacement at a point is the sum of the displacements due to each of the primary source and control source forces and moments. Optimal control is achieved by minimising the total mean square displacement at the location of a single error sensor downstream of the control source.

The theoretical analysis considers four different sets of classical beam supports; infinite, free, simply supported and fixed. The influence of the control source location, the error sensor location and the excitation frequency on the control source amplitude and achievable attenuation are investigated, and the physical reasons for each observation are explained (Section 2.3). The effect of introducing a second control source and angle stiffener is also examined (Section 2.3.4).

Experimental verification of the beam model is performed for four different sets of beam terminations; infinite, free, simply supported and vibration isolated (Section 2.4). The impedance corresponding to each type of termination is first measured from the experimental apparatus (Section 2.4.1). Finally, experimental results are compared with theoretical predictions (Section 2.5).
Chapter 2. Control of vibrations in a stiffened beam

2.2 THEORY

2.2.1 Response of a beam to a harmonic excitation

In this section the response of an arbitrarily terminated beam to a simple harmonic excitation $q(x)e^{j\omega t}$ applied at $x = x_0$ as shown in Figure 2.2 is considered. Left and right boundary conditions are specified as impedance matrices $Z_L$ and $Z_R$.

![Figure 2.2 Beam with an excitation $q$ at location $x_0$.](image)

Following the sign conventions shown in Figure 2.3, the equation of motion for the flexural vibration of the beam shown in Figure 2.2 is

$$EI_{yy} \frac{\partial^4 w}{\partial x^4} + \rho S \frac{\partial^2 w}{\partial t^2} = q(x) e^{j\omega t},$$

(2.1)

where $E$ is Young's modulus of elasticity, $I_{yy}$ is the second moment of area of the beam cross-section, $S$ is the cross-sectional area, $\rho$ is the density and $w$ is the displacement of the beam in the $z$-direction. The general solution to Equation (2.1) is of the form

$$w(x) = Ae^{k_x x} + Be^{-k_x x} + Ce^{j k_x x} + De^{-j k_x x},$$

(2.2)
Figure 2.3  Sign conventions for forces and moments.

On each side of the applied excitation at \( x = x_0 \), a separate solution of Equation (2.1) is required. For \( x < x_0 \) (on the left hand side of the applied excitation),

\[
w_1(x) = A_1e^{k_b x} + B_1e^{-k_b x} + C_1e^{jk_b x} + D_1e^{-jk_b x}, \tag{2.3}
\]

and for \( x > x_0 \) (on the right hand side of the applied excitation), the solution is

\[
w_2(x) = A_2e^{k_b x} + B_2e^{-k_b x} + C_2e^{jk_b x} + D_2e^{-jk_b x}. \tag{2.4}
\]

To find the flexural wave number \( k_b \), the homogeneous form of Equation (2.1) is solved to give the characteristic equation

\[
EI_{yy} k_b^4 - \rho S \omega^2 = 0, \tag{2.5}
\]

so

\[
k_b = \left( \frac{\rho S \omega^2}{EI_{yy}} \right)^{\frac{1}{4}}. \tag{2.6}
\]
Chapter 2. Control of vibrations in a stiffened beam

To solve for the eight unknowns \( A_1, B_1, C_1, D_1, A_2, B_2, C_2 \) and \( D_2 \), eight equations are required, comprising of four boundary condition equations (force and moment conditions at each end of the beam) and four equilibrium equations at the point of application \( x_0 \) of the force or moment.

2.2.2 Boundary conditions at the beam ends

2.2.2.1 Beam boundary impedance

A boundary impedance for a beam may be defined in terms of force and moment impedances because the boundary acts like an external force and moment generator, effectively applying external forces and moments to the beam which are equal to the internal shear force \( F \) and moment \( M \) at the ends of the beam. If a moment and force act simultaneously on any part of a beam, coupling will exist between the moment and force impedances. This is because the moment will result in a lateral displacement as well as a rotation and the lateral force will result in a rotation as well as a lateral displacement. Thus, the local lateral velocity and angular velocity generated at the end of a beam as a result of the end conditions may be written respectively as (Cremer et. al., 1973)

\[
\hat{\omega} = \frac{M}{Z_{mf}} + \frac{F}{Z_f}, \quad (2.7)
\]

and

\[
\hat{\theta} = \frac{M}{Z_{m}} + \frac{F}{Z_{jm}}, \quad (2.8)
\]
where $Z_{mf}$ and $Z_{fm}$ are the coupling impedances and $Z$ and $Z'$ are the force and moment impedances respectively. For harmonic signals $\dot{w} = j\omega w$ and $\dot{\theta} = -j\omega \dot{\theta}$, where the prime indicates differentiation with respect to the $x$-coordinate. Rearranging Equations (2.7) and (2.8) gives

$$
\begin{bmatrix}
F \\
M
\end{bmatrix} =
\begin{bmatrix}
Z_{\dot{w}w} & Z_{\dot{\theta}w} \\
Z_{\dot{w}m} & Z_{\dot{\theta}m}
\end{bmatrix}
\begin{bmatrix}
\dot{w} \\
\dot{\theta}
\end{bmatrix},
$$

or

$$
\begin{bmatrix}
F \\
M
\end{bmatrix} = Z
\begin{bmatrix}
\dot{w} \\
\dot{\theta}
\end{bmatrix},
$$

where

$$
Z =
\begin{bmatrix}
Z_{\dot{w}w} & Z_{\dot{\theta}w} \\
Z_{\dot{w}m} & Z_{\dot{\theta}m}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{Z_m} & \frac{1}{Z_{fm}} \\
\frac{1}{Z_m'} & \frac{1}{Z_f'}
\end{bmatrix}^{-1}
$$

The matrix $Z$ describes the dependence of the shear force and moment at the boundary on the motion of a beam in simple flexure. The relationships between standard support types and corresponding boundary impedances are given in Table 2.1, with cross terms $Z_{m\dot{w}}$ and $Z_{\dot{\theta}w}$ zero for the standard support types shown.
Chapter 2. Control of vibrations in a stiffened beam

Table 2.1

Impedances Corresponding to Standard Terminations

<table>
<thead>
<tr>
<th>End Condition</th>
<th>Representation</th>
<th>Boundary Condition</th>
<th>Impedance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simply Supported</td>
<td>![Simply Supported Diagram]</td>
<td>$w = 0$</td>
<td>$Z_{f\bar{w}} = \infty$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{\partial^3w}{\partial x^3} = 0$</td>
<td>$Z_{m\dot{\theta}} = 0$</td>
</tr>
<tr>
<td>Fixed</td>
<td>![Fixed Diagram]</td>
<td>$w = 0$</td>
<td>$Z_{f\bar{w}} = \infty$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{\partial w}{\partial x} = 0$</td>
<td>$Z_{m\dot{\theta}} = \infty$</td>
</tr>
<tr>
<td>Free</td>
<td>![Free Diagram]</td>
<td>$\frac{\partial^2 w}{\partial x^2} = 0$</td>
<td>$Z_{f\bar{w}} = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{\partial^3 w}{\partial x^3} = 0$</td>
<td>$Z_{m\dot{\theta}} = 0$</td>
</tr>
<tr>
<td>Deflected Spring</td>
<td>![Deflected Spring Diagram]</td>
<td>$\frac{\partial^2 w}{\partial x^2} = 0$</td>
<td>$Z_{f\bar{w}} = \frac{K_D}{\omega}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$EI \frac{\partial^3 w}{\partial x^3} = -K_D w$</td>
<td>$Z_{m\dot{\theta}} = 0$</td>
</tr>
<tr>
<td>Torsion Spring</td>
<td>![Torsion Spring Diagram]</td>
<td>$w = 0$</td>
<td>$Z_{f\bar{w}} = \infty$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$EI \frac{\partial^2 w}{\partial x^2} = K_T \frac{\partial w}{\partial x}$</td>
<td>$Z_{m\dot{\theta}} = -j\frac{K_T}{\omega}$</td>
</tr>
<tr>
<td>Mass</td>
<td>![Mass Diagram]</td>
<td>$\frac{\partial^2 w}{\partial x^2} = 0$</td>
<td>$Z_{f\bar{w}} = -j\omega m$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$EI \frac{\partial^3 w}{\partial x^3} = -m \frac{\partial^2 w}{\partial t^2}$</td>
<td>$Z_{m\dot{\theta}} = 0$</td>
</tr>
<tr>
<td>Dashpot</td>
<td>![Dashpot Diagram]</td>
<td>$\frac{\partial^2 w}{\partial x^2} = 0$</td>
<td>$Z_{f\bar{w}} = -c$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$EI \frac{\partial^3 w}{\partial x^3} = -\frac{\partial w}{\partial t}$</td>
<td>$Z_{m\dot{\theta}} = 0$</td>
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</table>
Chapter 2. Control of vibrations in a stiffened beam

2.2.2.2 Equivalent boundary impedance of an infinite beam

The displacement amplitude function for a right travelling wave in an infinite beam can be written as

\[ w(x) = Be^{-k_b x} + De^{-jk_b x} \]  \hspace{1cm} (2.12)

so that

\[ \dot{w}(x) = j\omega Be^{-k_b x} + j\omega De^{-jk_b x} \]  \hspace{1cm} (2.13)

and

\[ \ddot{\theta}(x) = j\omega k_b Be^{-k_b x} - \omega k_b De^{-jk_b x} \]  \hspace{1cm} (2.14)

where \( k_b \) is the flexural wave number of the beam and \( B \) and \( D \) are constants. Equations (2.13) and (2.14) can be written in matrix form as

\[
\begin{bmatrix}
\dot{w}(x) \\
\ddot{\theta}(x)
\end{bmatrix} = 
\begin{bmatrix}
 j\omega & j\omega \\
 j\omega k_b & -\omega k_b
\end{bmatrix} 
\begin{bmatrix}
 B e^{-k_b x} \\
 D e^{-jk_b x}
\end{bmatrix}.
\]  \hspace{1cm} (2.15)

Inversion gives

\[
\begin{bmatrix}
 B e^{-k_b x} \\
 D e^{-jk_b x}
\end{bmatrix} = 
\begin{bmatrix}
 (1+j) \\
 (1-j)
\end{bmatrix}
\begin{bmatrix}
 2\omega \\
 2\omega k_b
\end{bmatrix} 
\begin{bmatrix}
 \dot{w}(x) \\
 \ddot{\theta}(x)
\end{bmatrix}.
\]  \hspace{1cm} (2.16)

Differentiating Equation (2.12) to eliminate \( w'' \) and \( w''' \) from \( M = -EI_{yy} w'' \) and \( F = EI_{yy} w'''' \) respectively, the following matrix equation can be established;
Chapter 2. Control of vibrations in a stiffened beam

\[
\begin{bmatrix}
F(x) \\
M(x)
\end{bmatrix} = -EI_{yy} \begin{bmatrix} k_b^3 & -jk_b^3 \\
k_b^2 & -k_b^2
\end{bmatrix} \begin{bmatrix}
Be^{-k_b x} \\
De^{-jk_b x}
\end{bmatrix}.
\]

Combining with Equation (2.16) gives

\[
\begin{bmatrix}
F(x) \\
M(x)
\end{bmatrix} = Z_R \begin{bmatrix}
\ddot{w}(x) \\
\ddot{\theta}(x)
\end{bmatrix}
\]

where

\[
Z_R = \begin{bmatrix}
(1 + j)EI_{yy} k_b^3 / \omega & -EI_{yy} k_b^2 / \omega \\
EI_{yy} k_b^2 / \omega & -(1 - j)EI_{yy} k_b / \omega
\end{bmatrix}
\]

is the impedance matrix corresponding to an infinite end at the right hand end of the beam (Pan and Hansen, 1993b). Similarly, the wave impedance matrix corresponding to an infinite end at the left hand end of the beam is

\[
Z_L = \begin{bmatrix}
-(1 + j)EI_{yy} k_b^3 / \omega & -EI_{yy} k_b^2 / \omega \\
EI_{yy} k_b^2 / \omega & (1 - j)EI_{yy} k_b / \omega
\end{bmatrix}.
\]
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2.2.2.3 Impedance equations

From Equation (2.9) the left hand boundary condition of the beam at \( x = x_L \) can be written as

\[
\begin{bmatrix}
F(x_L) \\
M(x_L)
\end{bmatrix} = 
\begin{bmatrix}
Z_{L_f\theta} & Z_{L_f\theta} \\
Z_{L_m\theta} & Z_{L_m\theta}
\end{bmatrix}
\begin{bmatrix}
\hat{w}_1(x_L) \\
\hat{\theta}_1(x_L)
\end{bmatrix}.
\]  

(2.21)

Replacing the bending moment and shear force with a derivative of the displacement function, the following is obtained,

\[
\begin{bmatrix}
Z_{L_f\theta} & Z_{L_f\theta} \\
Z_{L_m\theta} & Z_{L_m\theta}
\end{bmatrix}
\begin{bmatrix}
\hat{w}_1(x_L) \\
\hat{\theta}_1(x_L)
\end{bmatrix} + EI_{yy}
\begin{bmatrix}
-\ddot{w}_1(x_L) \\
\ddot{w}_1(x_L)
\end{bmatrix} = 0.
\]  

(2.22)

Similarly, for the right hand end of the beam (\( x = x_R \)),

\[
\begin{bmatrix}
Z_{R_f\theta} & Z_{R_f\theta} \\
Z_{R_m\theta} & Z_{R_m\theta}
\end{bmatrix}
\begin{bmatrix}
\hat{w}_2(x_R) \\
\hat{\theta}_2(x_R)
\end{bmatrix} + EI_{yy}
\begin{bmatrix}
-\ddot{w}_2(x_R) \\
\ddot{w}_2(x_R)
\end{bmatrix} = 0.
\]  

(2.23)

Equations (2.3) and (2.4) may be differentiated to produce expressions for \( \hat{w}_1 \), \( \hat{w}_2 \), \( \hat{\theta}_1 \), \( \hat{\theta}_2 \), \( \dot{w}_1 \), \( \dot{w}_2 \), \( \ddot{w}_1 \), \( \ddot{w}_2 \), \( \dddot{w}_1 \), \( \ddot{w}_1 \), \( \dot{w}_2 \), and \( \dddot{w}_2 \), which can be substituted into Equations (2.22) and (2.23) to produce four boundary condition equations.
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2.2.3 Equilibrium conditions at the point of application \((x = x_0)\) of a force or moment

Requiring that the displacement and gradient be continuous at any point along the beam, the first two equilibrium conditions which must be satisfied at \(x = x_0\) are

\[ w_1 = w_2 \quad (2.24) \]

and

\[ \frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial x}. \quad (2.25) \]

The form of the excitation \(q(x)\) will affect the higher order equilibrium conditions at \(x = x_0\). In the following sections the equilibrium conditions corresponding to the beam excited by a point force and a concentrated moment acting about an axis parallel to the \(y\)-axis are discussed. These two types of excitation are induced by an actuator placed between a stiffener flange and the beam.

2.2.3.1 Response of a beam to a point force

To begin, \(q(x)\) in Equation (2.1) is replaced by \(q(x) = F_0 \delta(x - x_0)\), where \(F_0\) is the amplitude of a simple harmonic point force acting perpendicular to the beam at position \(x_0\) and \(\delta\) is the Dirac delta function. The first two boundary conditions at \(x = x_0\) are given by Equations (2.24) and (2.25). The second and third order boundary conditions are obtained by twice integrating Equation (2.1) with respect to \(x\) between the limits \((x - x_0)\) and \((x + x_0)\):

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Chapter 2. Control of vibrations in a stiffened beam

\[ \frac{\partial^2 w_1}{\partial x^2} = \frac{\partial^2 w_2}{\partial x^2} \]  \hspace{1cm} (2.26)

and

\[ \frac{\partial^3 w_1}{\partial x^3} - \frac{\partial^3 w_2}{\partial x^3} = \frac{-F_0}{EI_{yy}}. \]  \hspace{1cm} (2.27)

### 2.2.3.2 Response of a beam to a concentrated moment

The excitation represented by \( q(x) \) in Equation (2.1) is replaced by a concentrated moment \( M_0 \) acting at locations \( x_0 \). The excitation \( q(x) \) is replaced by \( q(x) = \frac{\partial M_0}{\partial x} \delta(x - x_0) \). The second and third order boundary conditions at \( x = x_0 \) are

\[ \frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_2}{\partial x^2} = \frac{-M_0}{EI_{yy}} \]  \hspace{1cm} (2.28)

and

\[ \frac{\partial^3 w_1}{\partial x^3} = \frac{\partial^3 w_2}{\partial x^3}. \]  \hspace{1cm} (2.29)

Taking two boundary conditions at each end of the beam from Equations (2.22) and (2.23), the two equilibrium condition Equations (2.24) and (2.25), and two further equilibrium conditions from (2.26) - (2.29), there are eight equations in the eight unknowns \( A_1, B_1, C_1 \).
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$D_1, A_2, B_2, C_2$ and $D_2$. These can be written in the form $\alpha X = B$. The solution vectors $X = [A_1 B_1 C_1 D_1 A_2 B_2 C_2 D_2]^T = \alpha^{-1}B$ can be used to characterise the response of a beam to simple harmonic excitation by a point force or a concentrated moment.

2.2.4 Mass loading of the angle stiffener

The mass of the angle stiffener could be significant and may be taken into account as follows. Given a beam with an arbitrary excitation $q$ at location $x = x_0$ and an angle stiffener at location $x = x_1$, as shown in Figure 2.4, three eigenfunction solutions of Equation (2.1) would be required; one applying over the range $x < x_0$, the second for $x < x_1$ and the third applying when $x > x_1$.

![Figure 2.4](image_url)  

**Figure 2.4** Beam with an excitation $q$ and an angle stiffener.
Chapter 2. Control of vibrations in a stiffened beam

At the stiffener location \( x = x_1 \), the first two equilibrium conditions would be similar to Equations (2.24) and (2.25). The second and third order equilibrium equations would be

\[
\frac{\partial^2 w_1}{\partial x^2} = \frac{\partial^2 w_2}{\partial x^2} \quad (2.30)
\]

and

\[
\frac{\partial^3 w_1}{\partial x^3} - \frac{\partial^3 w_2}{\partial x^3} = \frac{-m_a}{EI_{yy}} \frac{\partial^2 w}{\partial t^2} \quad (2.31)
\]

The beam used in experiments for this work was quite thick, and the mass loading of the stiffener made no measurable difference to the vibration response of the beam (see Section 2.4). The mass loading of the stiffener has therefore not been taken into account in this chapter; however, the effects of both the mass and the stiffness of the angle stiffener are considered in the chapters dealing with plates and cylinders (Chapters 3 and 4).

2.2.5 Minimising vibration using piezoceramic actuators and angle stiffeners

2.2.5.1 One control source and one angle stiffener

For any force or moment excitation, the eight boundary and equilibrium equations in eight unknowns can be written in the form \( \alpha X = B \), where

\[
B = F = \begin{bmatrix} 0, 0, 0, 0, 0, 0, \frac{-F_0}{EI_{yy} k_b} \end{bmatrix}^T \quad (2.32)
\]
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for a force excitation and

\[
B = M = \begin{bmatrix} 0, 0, 0, 0, 0, \frac{-M_0}{EI_y k_0^2} \end{bmatrix}^T
\]  \hspace{1cm} (2.33)

for a moment excitation. The coefficient matrix \( \alpha \) is given by Equation (2.34), where

\[
\beta_L = e^{k_\xi x}, \quad \beta_0 = e^{k_\xi x_0}, \quad \beta_R = e^{k_\xi x_R}, \quad \beta_L^j = e^{j k_\xi x_L}, \quad \beta_0^j = e^{j k_\xi x_0}, \quad \beta_R^j = e^{j k_\xi x_R}, \quad Q_1 = \frac{EI_y k_0^2}{j \omega} \quad \text{and} \quad Q_2 = \frac{EI_y k_0^2}{j \omega}.
\]

Figure 2.5 shows the resultant forces and moments applied to the beam by the angle stiffener and piezoceramic stack (shown in Figure 2.1), with a primary force \( F_p \) at \( x = x_p \). Control forces \( F_1 \) and \( F_2 \) act at \( x = x_1 \) and \( x = x_2 \) respectively, with the concentrated moment \( M_1 \) also acting at \( x = x_1 \). An error sensor is located at axial location \( x = x_e \).

![Figure 2.5](image_url)  

**Figure 2.5** Forces and moment applied to the beam by a point force primary source and a piezoceramic stack and angle stiffener control source.
Chapter 2. Control of vibrations in a stiffened beam

\[
\begin{bmatrix}
\alpha - \\
\end{bmatrix}
\begin{bmatrix}
\mathbb{J} \omega \beta_L (Z_{L,\text{m}} - k_L Z_{L,\text{th}} + Q_L) \\
\mathbb{J} \omega \beta_L (Z_{L,\text{i}} - k_I Z_{L,\text{th}} - Q_I) \\
0 \\
0 \\
\beta_0 - j/\beta_0 \\
\beta_0 + j/\beta_0 \\
\beta_0 - j/\beta_0 \\
-\beta_0 - j/\beta_0 \\
0 \\
0 \\
-\beta_0 - j/\beta_0 \\
-\beta_0 - j/\beta_0 \\
\mathbb{J} \omega \beta_R (Z_{R,\text{m}} - k_R Z_{R,\text{th}} + Q_R) \\
\mathbb{J} \omega \beta_R (Z_{R,\text{i}} - k_I Z_{R,\text{th}} - Q_I) \\
0 \\
0 \\
-\beta_0 + j/\beta_0 \\
-\beta_0 + j/\beta_0 \\
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Chapter 2. Control of vibrations in a stiffened beam

The equation in the displacement unknowns for the primary (excitation) point force is

\[ \alpha_p X_p = F_p , \]  \hspace{1cm} (2.35)

or

\[ X_p = \alpha_p^{-1} F_p , \]  \hspace{1cm} (2.36)

where \( F_p \) is defined by Equation (2.32) with \( F_0 = F_p \). \( \alpha_p \) is the matrix of boundary condition coefficients for the primary force and \( X_p \) is the boundary eigenvector for the primary force.

Similarly

\[ X_1 = \alpha_1^{-1} F_1 , \]  \hspace{1cm} (2.37)

and

\[ X_2 = \alpha_2^{-1} F_2 , \]  \hspace{1cm} (2.38)

where \( X_1 \) and \( X_2 \) are the boundary eigenvectors for the two control forces. In addition,

\[ X_m = \alpha_m^{-1} M , \]  \hspace{1cm} (2.39)

where \( X_m \) is the boundary eigenvector for the control moment and \( M \) is defined by Equation (2.33). At the error sensor \( (x = x_e) \), the displacement due to each force and the moment is given by, for \( z = p, 1, 2 \) and \( m \),

\[ w_z(x_e) = X_z^T E(x_e) , \]  \hspace{1cm} (2.40)
where $X^T$ indicates the transpose of the matrix $X$ and

\[
E(x_e) = \begin{bmatrix}
0 \\
0 \\
0 \\
e^{k_0 x_e} \\
e^{-k_0 x_e} \\
e^{j k_0 x_e} \\
e^{-j k_0 x_e}
\end{bmatrix}.
\] (2.41)

By summation of the displacement equations (Equation (2.40)), the total displacement resulting from the primary and control excitations is

\[
w(x_e) = w_p(x_e) + w_1(x_e) + w_2(x_e) + w_m(x_e) \\
= X_p^T E(x_e) + X_1^T E(x_e) + X_2^T E(x_e) + X_m^T E(x_e) .
\] (2.42)

Substituting Equations (2.36) - (2.39) into (2.42),

\[
w(x_e) = (\alpha_p^{-1} F_p)^T E(x_e) + (\alpha_1^{-1} F_1)^T E(x_e) + (\alpha_2^{-1} F_2)^T E(x_e) + (\alpha_m^{-1} M)^T E(x_e) \\
= \frac{F_p}{k_p El_{yy}} (\alpha_p^{-1})_{i,8}^T E(x_e) + \frac{F_1}{k_1 El_{yy}} (\alpha_1^{-1})_{i,8}^T E(x_e) \\
+ \frac{F_2}{k_2 El_{yy}} (\alpha_2^{-1})_{i,8}^T E(x_e) - \frac{M_1}{k_1 El_{yy}} (\alpha_m^{-1})_{i,7}^T E(x_e) .
\] (2.43)

Defining the transpose of the eighth column of the inverse of $\alpha_p$ as $P = (\alpha_p^{-1})_{i,8}^T$ (and similarly for $A = (\alpha_1^{-1})_{i,8}^T$, $B = (\alpha_2^{-1})_{i,8}^T$, and $C = (\alpha_m^{-1})_{i,7}^T$) is the transpose of the seventh
Chapter 2. Control of vibrations in a stiffened beam

column of the inverse of $\alpha_m$, and setting $w(x_e) = 0$ to find the optimal control force, Equation (2.43) can be re-written as

$$-AE(x_e)F_1 - BE(x_e)F_2 + k_bCE(x_e)M_1 = PE(x_e)F_p.$$  

Analysis of the forces applied by the stack and angle stiffener gives $F_1 = -F_2 = F_s$ and $M_1 = -aF_s$, where $F_s$ is the force applied by the piezoceramic stack (Figure 2.6) and is positive as shown.

![Control forces and moment in terms of piezoceramic stack force $F_s$.](image)

**Figure 2.6** Control forces and moment in terms of piezoceramic stack force $F_s$.

Substituting these quantities into Equation (2.44), the optimal control force $F_s$ can be written

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Chapter 2. Control of vibrations in a stiffened beam

as

$$F_s = \frac{PE(x_c)}{BE(x_c) - AE(x_c) - k_{\mu C}E(x_c)} F_p,$$

(2.45)

or

$$F_s = \frac{PE(x_c)}{(B - A - k_{\mu C})E(x_c)} F_p.$$

(2.46)

2.2.5.2 Two control sources and two angle stiffeners

If a second control source and angle stiffener are introduced at some location $x'_c$ downstream from the first, Equation (2.42) becomes

$$w(x_c) = w_p(x_c) + w_1(x_c) + w_2(x_c) + w_m(x_c) + w'_1(x_c) + w'_2(x_c) + w'_m(x_c) = X_p^T E(x_c) + X_1^T E(x_c) + X_2^T E(x_c) + X_m^T E(x_c) + X'_1^T E(x_c) + X'_2^T E(x_c) + X'_m^T E(x_c),$$

(2.47)

where the prime refers to the values associated with the second control source. The control signal for the second source required to minimise vibration at the same error sensor location $x_e$ can be calculated by

$$F'_s = \frac{PE(x_c)F_p - (B - A - k_{\mu C})E(x_c)F_s}{(B - A - k_{\mu C})E(x_c)},$$

(2.48)

In this way, the magnitude of the first control source signal can be limited to some arbitrary maximum value, and the second control source used to maintain optimal control.
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2.2.5.3 Two error sensors

If a second error sensor is introduced at some location $x_e'$ downstream from the first, and a single control source driven in an attempt to optimally reduce vibration at both locations, Equation (2.42) becomes

$$w(x_e) = X_p^T\left[E(x_e) + E(x_e')\right] + X_1^T\left[E(x_e) + E(x_e')\right] + X_2^T\left[E(x_e) + E(x_e')\right] + X_m^T\left[E(x_e) + E(x_e')\right]$$  (2.49)

This has the effect of shifting the effective error sensor location to some location between $x_e$ and $x_e'$. No improvement in overall attenuation is achieved.

An alternative method of utilising a second error sensor would be to introduce the second sensor only when control using the first error sensor was ineffective (see Section 2.3.3). However, this would be a difficult method to implement in practice. Unlike the case where the control source is badly located (see Section 2.3.2), there is no effect on the control source amplitude required to give optimal control when the error sensor is badly located. The only observed effect is a decrease in the attenuation achieved. To use a second error sensor when the frequency of excitation resulted in the first error sensor location being unsuitable would require some method of measuring the reduction in acceleration level achieved downstream from the first error sensor. When the reduction was less than some expected level, the
Chapter 2. Control of vibrations in a stiffened beam

controller would be switched to use the second error sensor. This method is not useful because it requires the addition of even more sensors, and there is no more practical way of determining when a second error sensor should be used in preference to the first. Use of a second error sensor downstream from the first would yield no practical benefit.
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2.3 NUMERICAL RESULTS

The theoretical model developed in the previous section was programmed in Fortran. The program consisted of about 1000 lines and, for a typical set of results, took two or three hours C.P.U. time to run on a DEC 5000/240 computer.

The discussion that follows examines the effect of varying forcing frequency, control source location (which is defined here as the location of the angle-beam joint), error sensor location, and stiffener flange length on the active control of vibration in beams with four sets of end conditions; infinite, fixed, free and simply supported. The beam parameters (including location of the control source, primary source and error sensor) are listed in Table 2.2. These values are adhered to unless otherwise stated. The results obtained are consistent with previous work done by Pan and Hansen (1993a) for the case of a single point force harmonically applied to a beam and the analogous case of active noise control of ducts (Snyder 1990).

Control forces are expressed as a multiple of the primary force, and the acceleration amplitude dB scale reference level is the far field uncontrolled infinite beam acceleration produced by the primary force. In all cases, the control force is assumed to be optimally adjusted to minimise the acceleration at the error sensor location. The flexural wavelength of vibration in a beam is given by

$$\lambda_b = \frac{2\pi}{k_b} .$$

(2.50)
Chapter 2. Control of vibrations in a stiffened beam

Table 2.2
Beam Parameters for Numerical Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam length $L_x$</td>
<td>10.0 m</td>
</tr>
<tr>
<td>Beam width $L_y$</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Beam height $L_z$</td>
<td>0.025 m</td>
</tr>
<tr>
<td>Young's modulus $E$</td>
<td>71.1 GPa</td>
</tr>
<tr>
<td>Primary force location $x_p$</td>
<td>0.0 m</td>
</tr>
<tr>
<td>Control location $x_1$</td>
<td>1.0 m $\times (2.07\lambda_b)^*$</td>
</tr>
<tr>
<td>Stiffener flange length $a$</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Error sensor location $x_e$</td>
<td>2.0 m $\times (4.14\lambda_b)^*$</td>
</tr>
<tr>
<td>Frequency $f$</td>
<td>1000 Hz</td>
</tr>
<tr>
<td>Wavelength $\lambda_b$</td>
<td>0.4824 m $^*$</td>
</tr>
</tbody>
</table>

$^*$ - Applies only when $f = 1000$ Hz.

2.3.1 Acceleration distributions for controlled and uncontrolled cases

Figure 2.7 shows the acceleration amplitude distribution in dB for the uncontrolled beams. The shape of the curve for the infinite beam represents a travelling wave field with an additional decaying evanescent field close to the source. For the beams not terminated anechoically, waves reflected from the ends of the beam cause a standing wave field to exist. It can be seen from the nature of the response that the near field effects become insignificant at less than one wavelength from the point of application of the primary force. Figure 2.8 shows the acceleration amplitude distribution for controlled cases, using the control, primary
and error sensor locations given in Table 2.2. These locations are marked $x_p$, $x_c$, and $x$ respectively on the infinite beam curve. As expected, the curves for the controlled cases (Figure 2.8) dip to a minimum at the error sensor location ($x_e = 4.14 \lambda_b$) where acceleration has been minimised. In all cases, the calculated reduction in acceleration amplitude downstream of the error sensor is over 100 dB. The reduction or increase in acceleration amplitude upstream of the primary force depends on the control source location, as discussed below.

\[ \text{Figure 2.7} \quad \text{Acceleration distributions for the uncontrolled cases.} \]
2.3.2 Effect of variations in forcing frequency, stiffener flange length and control source location on the control force

Figure 2.9 shows the effect of varying the forcing frequency on the magnitude of the control force required to minimise the beam vibration at the error sensor location. The control source is located 1 metre from the primary source and the error sensor 2 metres from the primary source (Table 2.2).

The minima on the curves for fixed, free and simply supported beams occur at resonance frequencies, when control is easier (Pan at al., 1992). At these frequencies the control force amplitude becomes small but not zero. The maxima in Figure 2.9 occur when the relative spacing between primary and control sources is given by $x = (c + n\lambda_p/2)$ for integer $n$ and...
constant \( c \). This effect is illustrated by Figure 2.10 which shows the control force amplitude as a function of separation between primary and control sources, with a constant error sensor location - control source separation of 1 metre (2.07\( \lambda_b \)).

![Control force amplitude for optimal control as a function of frequency.](image)

**Figure 2.9** Control force amplitude for optimal control as a function of frequency.

The maxima occur because of the difficulty in controlling the flexural vibration when the effective control source location is at a node of the standing wave caused by reflection from the terminations. The constant \( c \) represents the distance (in wavelengths) between the primary source and the first node in the standing wave in the direction of the control source. This constant changes with frequency and termination type. For the fixed and free support beams, at \( f = 1000 \) Hz, \( c \) is approximately zero. Note that the effective control location is somewhere between the beam-angle connection and the actuator, and represents the location...


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Figure 2.10  Control force amplitude for optimal control as a function of control location.

of the combined effect of the two forces and the moment as discussed.

Figure 2.11 shows the phase of the control source relative to the primary source as a function of frequency, again using the control, primary and error sensor locations given in Table 2.2. Apart from the infinite beam case, the control source is either in phase or $180^\circ$ out of phase with the primary source, and this holds for all locations of control source, error sensor and all angle sizes. In the case of the infinite beam, the phase cycles through $180^\circ$ with increasing frequency. The difference is due to the formation of standing waves on beams with end conditions other than infinite. When a standing wave is formed because of the reflection from the beam termination, the vibration, and hence the required control force, is in phase
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with the excitation.

![Graph showing control force phase for optimal control as a function of control location.](image)

**Figure 2.11** Control force phase for optimal control as a function of control location.

Figure 2.12 shows the control force amplitude plotted as a function of increasing stiffener flange length in wavelengths $a/\lambda_b$ (see Figure 2.6 for definition of stiffener flange length). The exponential decrease in control force magnitude with increasing stiffener flange length can be attributed to the increasing size of the angle relative to the flexural wavelength. When the wavelength is large compared to the stiffener flange length, the two control forces operating in opposite directions tend to cancel. The exponential decrease in control force amplitude with increasing stiffener flange length can also be seen in Figure 2.9 where the relative control force amplitude is plotted as a function of frequency. As the excitation frequency increases, the stiffener flange length relative to the flexural wavelength increases,
Figure 2.12  Control force amplitude for optimal control as a function of flange length.

and the control force amplitude decreases. It is difficult to see this effect on the fixed, free and simply supported case figures because of the super-position of the peaks and troughs, but at the frequencies where the peaks and troughs almost coincide (e.g. at 1150 Hz on the free beam figure) some flattening out of the curve is seen. The effect is much clearer on the infinite beam curve where the peaks and troughs are absent.
2.3.3 Effect of variations in forcing frequency, control source location and error sensor location on the attenuation of acceleration level

Figure 2.13 shows the variation in the mean attenuation of acceleration level upstream of the primary source for each beam as a function of the excitation frequency, and Figure 2.14 gives the corresponding results for mean attenuation of acceleration level downstream of the error sensor.

Figure 2.15 shows the variation in the mean attenuation of acceleration level upstream of the primary source for each beam as a function of separation between the primary and control sources. The separation between the error sensor location and the control source location is constant ($2.07\lambda_b$). The minima in the attenuation curve for the free, fixed and simply supported beams correspond to control source locations where maxima in the control force curve occur (Figure 2.10). Maximum attenuation is achieved in all cases with a control source - primary source separation of ($0.23 + n\lambda_b/2$). This value is independent of the beam length and the excitation frequency. Noting also that the maxima occur for the infinite beam as well as the finite beams, it may be concluded that the maxima are a result of the standing wave field established between the primary and control sources only. This standing wave is present for the infinite beam case as well as the other terminations (see Figure 2.8). The effective control location is to the right of the defined control source location (which is the position of the stiffener-beam joint), so the effective control source-primary source separation giving these maxima is about ($0.25 + n\lambda_b/2$). This result is consistent with analogous cases; for a beam excited by point forces only (Pan and Hansen, 1993$a$), and also for sound fields in ducts (Snyder, 1990).
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Figure 2.13  Mean attenuation upstream of the primary source as a function of frequency.

Figure 2.14  Mean attenuation downstream of the error sensor as a function of frequency.
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Figure 2.15 Mean attenuation upstream of the primary source as a function of control source - primary source separation.

Figure 2.16 shows the mean attenuation of acceleration level downstream of the error sensor as a function of separation between primary and control sources. The control source locations giving the best results upstream of the primary source also give high attenuation downstream of the error sensor. Every second minimum occurs at a location corresponding to a maximum in the control force curve (see Figure 2.10), and again these are separated by half a wavelength. Thus it may be concluded that the best attenuation is achieved when the control effort is a minimum. The additional minima occur at control source-primary source separations of \((d + n\lambda_f/2)\), where \(d\) is a constant dependent on frequency and termination type.
Figure 2.16  Mean attenuation downstream of the error sensor as a function of control source - primary source separation.

Figures 2.17 and 2.18 show the mean attenuation upstream of the primary source and downstream of the error sensor respectively as a function of the separation between the control source and error sensor. There is no change in the attenuation upstream of the control source with changing error sensor location outside of the control source near field (i.e. for separations between the control source and error sensor of over $0.75\lambda_b$). Downstream of the error sensor, mean attenuation increases with increasing separation between the error sensor and control source location at the rate of around 50 dB per wavelength separation. The minima in the curves for fixed, free and simply supported beams correspond to separations in the error sensor and control source location of $(d+n\lambda_b)/2$, where $d$ is the constant dependent on frequency and termination type previously defined. The dips in the curves are represented correctly in the figures; finer frequency resolution would not cause them to be significantly lower.
Figure 2.17  Mean attenuation upstream of the primary source as a function of error sensor - control source separation.

Figure 2.18  Mean attenuation downstream of the error sensor as a function of error sensor - control source separation.
2.3.4 Effect of a second angle stiffener and control source on the attenuation of acceleration level.

The results given in Section 2.3.3 indicate that there are some control source locations where significantly less attenuation can be achieved. Figure 2.19 shows the variation in amplitudes of two control sources driven to optimally control vibration at a single error sensor, as a function of control source location, at an excitation frequency of 1000 Hz. The first control source amplitude is limited to 1.5 times the primary source amplitude, and the second control source amplitude and phase calculated by the method given in Section 2.2.5.2. The second control source is located 0.15m downstream from the first, and the error sensor is located 1.0m downstream from the first control source. The control source amplitude for a single control source is also shown on Figure 2.19 for comparison. At this frequency, there are no control source locations at which control using the two control sources requires large control signals.

![Figure 2.19](image_url)  
**Figure 2.19** Control source amplitudes for one and two control sources as a function of control source - primary source separation for the beam with fixed ends.
Figure 2.20 gives the control source amplitudes for two control sources as a function of frequency. It can be seen that there are some lower frequencies where the amplitude of the second control source required for optimal control is high. This can be overcome by using a different spacing between the first and second control sources.

Figure 2.21 shows that the minima in attenuation upstream of the primary sources observed at half-wavelength intervals in control source - primary source separation using one control source only are eliminated by the introduction of the second control source. When the location of a single control source is such that a large control signal is required to control vibration at the error sensor, the vibration level upstream of the primary source is increased severely. As these large control signals are not required when two control sources are used (Figure 2.19), this problem is overcome. The corresponding minima in attenuation downstream of the error sensors are also eliminated by the introduction of the second control
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source (Figure 2.22). The minima that remain when using two control sources (at control source - primary source separations given by \((d + n\lambda_B/2)\), where \(d\) is the constant defined in Section 2.3.3, are a consequence of the error sensor location and cannot be overcome simply. The difficulties involved in effectively introducing a second error sensor downstream from the first are discussed in Section 2.2.5.3.

\[ \text{Figure 2.21} \quad \text{Mean attenuation upstream of the primary source as a function of control location for the beam with fixed ends.} \]

\[ \text{Figure 2.22} \quad \text{Mean attenuation downstream of the error sensor as a function of control location for the beam with fixed ends.} \]
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2.4 EXPERIMENTAL PROCEDURE

2.4.1 Impedance of an experimental termination

Before experimental results and theoretical calculations can be compared, it is necessary to know the impedance of the beam termination used in the experiment. Traditionally, classical terminations (such as the "free end") have been used in theoretical work, and modelled experimentally (e.g. by wire supports). For greater accuracy, the impedances used in the theoretical results that follow were calculated using a method similar to that developed by Fuller et al (1990), except that coupling impedances are initially included in the analysis.

Beam end conditions may be characterised by impedance matrices $Z_L$ and $Z_R$ corresponding respectively to the left and right ends of the beam (see Section 2.2.2.1);

$$Z_L = \begin{bmatrix} Z_{Lf} & Z_{L0} \\ Z_{Lm} & Z_{L0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_{Lf}} & \frac{1}{Z_{Lm}} \end{bmatrix}^{-1} \quad (2.51)$$

and

$$Z_R = \begin{bmatrix} Z_{Rf} & Z_{R0} \\ Z_{Rm} & Z_{R0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_{Rf}} & \frac{1}{Z_{Rm}} \end{bmatrix}^{-1} \quad (2.52)$$

These impedances are used in the theoretical analysis which characterises the response $X$ of an arbitrarily terminated beam to an applied force or moment applied at some position $x_0$ (Section 2.2). The impedance terms become part of the coefficient matrix $\alpha$ in Equation
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(2.34). In the analysis that follows Equation (2.34), the matrix \( \alpha \) is inverted and multiplied by a vector \( F \) or \( M \). Both column vectors consist of seven zero elements and one non-zero force or moment term, which occurs in either the seventh or eight position. The only columns of importance in the inverse matrix \( \alpha^{-1} \) are the seventh and eight columns, as all other columns will be multiplied by a zero element in the vector \( F \) or \( M \). The practical implication of this result is that only the larger elements of the first four rows of the \( \alpha \) matrix will affect the solution vector \( X \). The result is that the accuracy of the solution vector \( X \) can be exactly maintained with the off-diagonal coupling elements \( Z_{m\bar{v}} \) and \( Z_{f\bar{f}} \) of the impedance matrix set to zero. This simplification is justified by examples rather than by formal proof, as inverting the complex matrix \( \alpha \) symbolically is not practical.

Once the impedance matrix has been approximated by the equivalent matrix with just two unknowns, determining the unknown equivalent impedance of a given beam termination from experimental data is possible. Beginning with the beam shown in Figure 2.1, the unknown termination at the left hand end may be described by the equivalent impedance matrix

\[
Z_L = \begin{bmatrix} Z_{L_1} & 0 \\ 0 & Z_{L_2} \end{bmatrix} = \begin{bmatrix} Z_{L_{f\bar{v}}} & Z_{L_{f\bar{f}}} \\ Z_{L_{m\bar{v}}} & Z_{L_{m\bar{f}}} \end{bmatrix}.
\]

(2.53)
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The right hand termination may be such that the impedance values are known, or it may be the same unknown termination used on the left hand end, in which case (following the sign conventions given in Figure 2.3) the equivalent impedance matrix $Z_R$ is given by

$$
Z_R = \begin{bmatrix}
Z_{R1} & 0 \\
0 & Z_{R2}
\end{bmatrix} = \begin{bmatrix}
-Z_{L1} & 0 \\
0 & -Z_{L2}
\end{bmatrix}.
$$

(2.54)

The method that follows will be the same regardless of whether a known or unknown termination is used at the right hand end. Setting the coupling terms in Equations (2.7) and (2.8) to zero for our equivalent case, substituting for $Z_{L1}$ and $Z_{L2}$ and rearranging,

$$
Z_{L1} = \frac{F}{\dot{w}}
$$

(2.55)

and

$$
Z_{L2} = \frac{M}{\dot{\theta}}.
$$

(2.56)

For harmonic signals $\dot{w} = j\omega w$ and $\dot{\theta} = -j\omega \dot{\theta}$. Replacing the bending moment and shear force with derivatives of the displacement function gives

$$
Z_{L1} = \frac{EI y w_1''(x_L)}{j\omega w_1'(x_L)}
$$

(2.57)

and

$$
Z_{L2} = \frac{EI y w_1'(x_L)}{j\omega w_1'(x_L)}.
$$

(2.58)
All that remains is to find the displacement and derivatives required in Equations (2.57) and (2.58). The accelerations $a_1, a_2, a_3, \ldots, a_n$ (relative to an arbitrary reference signal) are measured at $n$ positions $x_1, x_2, x_3, \ldots, x_n$ such that $x_L < x_i < x_0$. For $i = 1$ to $n$,

$$a_{ie}(x_i) = -\omega^2 w_{ie}(x_i) = A_{1e} e^{k_{0}x_i} + B_{1e} e^{-k_{0}x_i} + C_{1e} e^{j\omega x_i} + D_{1e} e^{-j\omega x_i},$$

(2.59)

where the subscript $e$ denotes experimentally obtained values. Writing in matrix form,

$$
\begin{bmatrix}
-a_{1e} \\
\omega^2 \\
-a_{2e} \\
\omega^2 \\
-a_{3e} \\
\omega^2 \\
\vdots \\
-a_{ne} \\
\omega^2
\end{bmatrix} = 
\begin{bmatrix}
\beta_1 \\
\beta_1^{-1} \\
\beta_1^{j} \\
\beta_1^{j} \\
\beta_2 \\
\beta_2^{-1} \\
\beta_2^{j} \\
\beta_2^{j} \\
\vdots \\
\beta_n \\
\beta_n^{-1} \\
\beta_n^{j} \\
\beta_n^{j}
\end{bmatrix}
\begin{bmatrix}
A_{1e} \\
B_{1e} \\
C_{1e} \\
D_{1e}
\end{bmatrix},
$$

(2.60)

where $\beta_i = e^{k_{0}x_i}$, $\beta_i^{-1} = e^{-k_{0}x_i}$, $\beta_i^{j} = e^{j\omega x_i}$, and $\beta_i^{j} = e^{-j\omega x_i}$. Rearranging gives

$$
\begin{bmatrix}
A_{1e} \\
B_{1e} \\
C_{1e} \\
D_{1e}
\end{bmatrix} = 
\begin{bmatrix}
\beta_1 \\
\beta_1^{-1} \\
\beta_1^{j} \\
\beta_1^{j} \\
\beta_2 \\
\beta_2^{-1} \\
\beta_2^{j} \\
\beta_2^{j} \\
\vdots \\
\beta_n \\
\beta_n^{-1} \\
\beta_n^{j} \\
\beta_n^{j}
\end{bmatrix}
\begin{bmatrix}
-a_{1e} \\
\omega^2 \\
-a_{2e} \\
\omega^2 \\
-a_{3e} \\
\omega^2 \\
\vdots \\
-a_{ne} \\
\omega^2
\end{bmatrix},
$$

(2.61)
where the inversion operator represents the generalised inverse or pseudo-inverse. Equation (2.61) represents a system of \( n \) equations in four unknowns. If \( n = 4 \), the system is determined, but the solution \((A_{1e}B_{1e}C_{1e}D_{1e})\) is extremely sensitive to errors in the measured accelerations. The error is significantly reduced if an overdetermined system is used \((n > 4)\), as will be shown.

**Table 2.3**

**Beam Parameters for Impedance Accuracy Calculations**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam length ( L_x )</td>
<td>10.0 m</td>
</tr>
<tr>
<td>Beam width ( L_y )</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Beam height ( L_z )</td>
<td>0.025 m</td>
</tr>
<tr>
<td>Young's modulus ( E )</td>
<td>71.1 GPa</td>
</tr>
<tr>
<td>Excitation force location ( x_0 )</td>
<td>0.0 m</td>
</tr>
<tr>
<td>Excitation frequency ( f )</td>
<td>1000 Hz</td>
</tr>
<tr>
<td>Wavelength ( \lambda_b )</td>
<td>0.4824 m</td>
</tr>
</tbody>
</table>

Let \( w_e \) be the displacement calculated from the constants \( A_{1e}, B_{1e}, C_{1e}, D_{1e} \). For the beam described by the parameters of Table 2.3 with end conditions modelled as infinite, the error induced in the displacement \( w_e \), given an initial error in the real and imaginary parts of the accelerations \((a_{i}, i = 1,n)\) of 10\%, is plotted as a function of the number of acceleration measurements \( n \) in Figure 2.23. For the two frequencies shown, \( n \geq 10 \) provides a reasonable accuracy. The higher the frequency, the greater the number of acceleration measurements that will be required. Measurements do not need to be made simultaneously, so a couple of accelerometers can be used and placed at each of the measurement positions in turn.
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Once calculated, $A_{ie}$, $B_{ie}$, $C_{ie}$ and $D_{ie}$ are substituted into Equation (2.3) and differentiation is carried out to find $w_{ie}(x_L)$, $w_{ie}'(x_L)$, $w_{ie}''(x_L)$ and $w_{ie}'''(x_L)$. Equations (2.57) and (2.58) can then be used to find the equivalent impedances $Z_{L1}$ and $Z_{L2}$.

It should be noted that use of the "equivalent" impedance matrix obtained by eliminating the off-diagonal elements from the impedance matrix is only valid for analyses similar to that followed in this chapter. It is not claimed that the resulting impedance matrix closely approximates the real impedance values of the termination in general circumstances. However, the numerical answers for all derivatives of displacement (and hence acceleration etc.), calculated at any point along the beam, are correct to the accuracy of the calculating

---

**Figure 2.23** Error in displacement calculated using "measured" impedances with a mean experimental error of 10%.
program (eight figures in the work done for this chapter) when compared to the corresponding
derivatives obtained by using the "exact" impedance matrix. Furthermore, it is found that the
most significant elements of the equivalent impedance matrix so approximated are at least
similar if not precisely the same as the corresponding elements of the exact impedance
matrix.

The accuracy of this method has been tested with a variety of cases. The "exact" impedance
matrices and the corresponding approximations calculated using the method described are
given in Table 2.4. All of these examples utilise a right hand impedance corresponding to an
ideally infinite beam, and approximations are made for the various test cases at the left hand
end of the beam. The parameters characterising the beam are given in Table 2.3. For all the
test cases, all derivatives of displacement calculated using the "exact" and equivalent
impedance matrices are either identical, or are insignificantly small.

It would be expected that the simplification might fail when the original matrix had large
elements on the off-diagonal, but this is not the case, as shown by the first two examples in
Table 2.4. The third example shows an approximation for an impedance matrix with four
complex elements. Examples 4, 5 and 6 show the exact and equivalent matrices for the ideal
infinite, free and fixed beam impedances respectively.
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Table 2.4

Impedance Matrices and Corresponding Equivalent Matrices

<table>
<thead>
<tr>
<th>E.g. No.</th>
<th>&quot;Exact&quot; Matrix $Z_L$</th>
<th>Corresponding Equivalent Matrix (with zero off-diagonal elements)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\begin{bmatrix} 0 &amp; -10^{100} - 10^{100}j \ 10^{100} + 10^{100}j &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1.61\times10^{19} - 9.30\times10^{18}j &amp; 0 \ 0 &amp; 2.44\times10^{15} - 5.00\times10^{16}j \end{bmatrix}$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{bmatrix} 0 &amp; -10^{100} \ 10^{100} &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -2.07\times10^{19} - 1.27\times10^{18}j &amp; 0 \ 0 &amp; 1.01\times10^{17} - 4.28\times10^{16}j \end{bmatrix}$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 5 + 6j &amp; 7 + 8j \ 1 + 2j &amp; 3 + 4j \end{bmatrix}$</td>
<td>$\begin{bmatrix} 222.4 + 10.89j &amp; 0 \ 0 &amp; 4.963 + 6.101j \end{bmatrix}$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} -1628 - 1628j &amp; -125.0 \ 125.0 &amp; 9.596 - 9.596j \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1628.0 &amp; 0 \ 0 &amp; 9.596 \end{bmatrix}$</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 5.6\times10^{-14} - 1.3\times10^{-13}j &amp; 0 \ 0 &amp; -1.5\times10^{-15} - 2.8\times10^{-16}j \end{bmatrix}$</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} 10^{100} &amp; 0 \ 0 &amp; 10^{100} \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1.38\times10^{19} - 8.48\times10^{17}j &amp; 0 \ 0 &amp; 5.00\times10^{15} - 8.13\times10^{16}j \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Four different beam terminations were used in the experimental verification of the beam model. For the simple support, the wire support and the anechoic termination, classical impedances are given in Table 2.5 (with the wire support modelled as a free end and the anechoic termination modelled as an infinite end). The fourth termination used was a simply supported termination mounted on vibration isolators, for which a theoretical impedance is
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not easily calculated. Using the method outlined, new impedance matrices were calculated from the experimentally measured uncontrolled acceleration distribution for each end condition. These matrices are also listed in Table 2.5, and differ considerably from the classical theoretical impedances for the three modelled terminations.

Table 2.5
Classical and Calculated Impedance Matrices

<table>
<thead>
<tr>
<th>Termination</th>
<th>Classical Impedance Matrix $Z_i$</th>
<th>Calculated Impedance Matrix $Z_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire Supported</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -109 + 124j &amp; 0 \ 0 &amp; 3.25 + 6.30j \end{bmatrix}$</td>
</tr>
<tr>
<td>Simply Supported</td>
<td>$\begin{bmatrix} \infty &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 3961 + 522j &amp; 0 \ 0 &amp; 8.32 + 8.21j \end{bmatrix}$</td>
</tr>
<tr>
<td>Anechoic</td>
<td>$\begin{bmatrix} -834 - 834j &amp; 125.0 \ -125.0 &amp; 18.7 - 18.7j \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1440 - 1061j &amp; 0 \ 0 &amp; 5.46 + 7.36j \end{bmatrix}$</td>
</tr>
<tr>
<td>Vibration Isolated</td>
<td>None</td>
<td>$\begin{bmatrix} -539 - 834j &amp; 0 \ 0 &amp; 6.75 + 7.71j \end{bmatrix}$</td>
</tr>
</tbody>
</table>

2.4.2 Relating control signal and control force

To relate experimental results to theoretical calculations, it is necessary to determine the input voltage required to drive the piezoceramic stack actuator to produce a unit output force $F_s$. To begin, the stiffness of the stiffener is approximated by modelling the stiffener (Figure 2.24) as two beams (Figure 2.25).
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Figure 2.24 Stiffener showing dimensions.

Figure 2.25 Stiffener flanges modelled as beams; (a) horizontal stiffener flange, and (b) vertical stiffener flange.
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The first beam is characterised by a length \( x_a \), with a load \( F_s \) at one end and a rigid support at the other. The support provides a vertical reaction \( R = F_s \) and moment reaction \( M_1 = aF_s \) as shown. Taking the origin of coordinates to be the support end, the equation for the bending moment in this beam is

\[
M = M_1 - Rx = F_s(a - x)
\]

\[
= EI_{yy} w''(0) = F_s(a - x),
\]

(2.62)

where \( w_a(x) \) is deflection at position \( x \), \( E \) is Young's modulus and \( I_{yy} \) is the second moment of area about the \( y \)-axis. Integrating twice and using the boundary conditions \( w_a(0) = 0 \) and \( w'_a(x) = 0 \),

\[
w_a(x) = \frac{F_s}{EI_{yy}} \left( \frac{ax^2}{2} - \frac{x^3}{6} \right),
\]

\[
\therefore w_a(a) = \frac{F_s a^3}{3EI_{yy}}.
\]

(2.63)

Denoting the deflection of the beam of length \( b \) in Figure 2.25(b) by \( w_b(x) \), and using the reaction \( M = aF_s \) as shown, the bending moment equation is

\[
M_2 = EI_{yy} w''_b(x) = bF_s.
\]

(2.64)

Integrating and using the boundary condition \( w'_b(0) = 0 \),

\[
EI_{yy} w'_b(b) = F_s b^2.
\]

(2.65)
The vertical total deflection of the stiffener at the position of the actuator force $F_s$ is approximately $w = w_a(a) + aw_b'(b)$ (Figure 2.26). Thus

$$w = \frac{F_s}{EI_{yy}} \left( \frac{a^3}{3} + ab^2 \right). \quad (2.66)$$

Stiffness is defined as force per unit displacement, so the equivalent stiffness of the stiffener is

$$k_s = \frac{F_s}{w} = \frac{EI_{yy}}{(a^3/3 + ab^2)}. \quad (2.67)$$

Now the piezoceramic stack voltage required to produce a resultant force $F_s$ is calculated. The maximum force $F_{\text{max}}$ generated by the actuator with an input voltage $V_0$ is
Chapter 2. Control of vibrations in a stiffened beam

\[ F_{\text{max}} = k_a \Delta L_0 \left(1 - \frac{k_a}{k_a + k_s}\right) = \Delta L_0 \left(\frac{k_a k_s}{k_a + k_s}\right), \]  

(2.68)

where \( \Delta L_0 \) is the nominal expansion of the actuator, \( k_a \) is the spring constant of the actuator and \( k_s \) is the spring constant of the stiffener. The applied potential difference \( V \) to achieve a resultant force \( F_s \) is given by

\[ \frac{V}{V_0} = \frac{F_s}{F_{\text{max}}}, \]  

(2.69)

or

\[ V = F_s \times \left[\frac{V_0 (k_a + k_s)}{\Delta L_0 k_a k_s}\right]. \]  

(2.70)

Equation (2.67) can be used to find \( k_s \), which can then be substituted into Equation (2.70) along with data for a stack actuator to find the required potential difference input for a unit force output; alternatively, Equation (2.70) can be rearranged to find the force resulting from a given input potential.

2.4.3 Test procedure

A steel stiffener was bolted tightly to an aluminium beam described by the dimensions given in Table 2.6. The piezoceramic actuator with the characteristics listed in Table 2.7 was placed between the stiffener flange and the beam.
Chapter 2. Control of vibrations in a stiffened beam

Table 2.6
Beam Parameters for Experimental Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam length $L_x$</td>
<td>3.9 m</td>
</tr>
<tr>
<td>Beam width $L_y$</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Beam height $L_z$</td>
<td>0.025 m</td>
</tr>
<tr>
<td>Young's modulus $E$</td>
<td>71.1 GPa</td>
</tr>
<tr>
<td>Primary force location $x_p$</td>
<td>0.0 m</td>
</tr>
<tr>
<td>Control location $x_1$</td>
<td>0.5 m (=0.53λ₁)</td>
</tr>
<tr>
<td>Error sensor location $x_e$</td>
<td>1.0 m (= 1.06λ₁)</td>
</tr>
<tr>
<td>Excitation frequency $f$</td>
<td>263 Hz</td>
</tr>
<tr>
<td>Wavelength $λ_b$</td>
<td>0.94 m</td>
</tr>
</tbody>
</table>

Table 2.7
Angle and Actuator Parameters for Experimental Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiffener flange length $a$</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Angle height $b$</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Angle width $d$</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Angle thickness $t$</td>
<td>0.01 m</td>
</tr>
<tr>
<td>Young's modulus $E$</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Physik Instrumente Translator No.</td>
<td>P244.20</td>
</tr>
<tr>
<td>Actuator spring constant $k_T$</td>
<td>38 N/μm</td>
</tr>
<tr>
<td>Nominal expansion of actuator $ΔL_0$</td>
<td>20 μm</td>
</tr>
<tr>
<td>Nominal voltage $V_0$</td>
<td>-1000 V</td>
</tr>
</tbody>
</table>
Chapter 2. Control of vibrations in a stiffened beam

The actuator was attached only at one end to ensure that no external tensile force was applied to it, as the type of actuator used is weak in tension. The primary source, control source and error sensor locations and the excitation frequency are also given in Table 2.6. The beam was mounted with the larger width dimension in the vertical plane, and excited in the horizontal plane.

Figure 2.27  Experimental arrangement for the active control of vibration in a beam.
The complete experimental arrangement is given in Figure 2.27. The primary signal was produced by a signal analyser and amplified to drive the electrodynamic shaker (Figure 2.28). The shaker acted on the beam through a force transducer, and the magnitude of the primary force was recorded using an oscilloscope.
Figure 2.29 Control system.

The error signal from the accelerometer (Figure 2.29) was passed to another oscilloscope. The amplitude and phase of the primary signal were adjusted using an instrumentation amplifier and a phase shifter to produce the control signal, which drove the piezoceramic actuator. The control signal was adjusted to optimally minimise the acceleration measured by the error sensor accelerometer. The control signal was also recorded using an oscilloscope.
The acceleration was measured at 5 cm intervals along the beam (Figure 2.30). The accelerometer signals were read in turn through a 40 channel multiplexer and passed to a Hewlett-Packard type 35665A signal analyser. The frequency response function was used to analyse the data. The magnitude and phase of the acceleration were recorded on a personal computer, which was also used to switch the recorded channel on the multiplexer. The acceleration output of the force transducer at the primary location was used as the reference signal for the frequency response analysis. Accelerometer readings were taken initially once the error sensor signal had been optimally reduced, and again with the control amplifier switched off (the uncontrolled case).
Chapter 2. Control of vibrations in a stiffened beam

Figures 2.31 - 2.33 show photographs of the experimental equipment. In Figure 2.31, the beam is located in front of the signal generating and recording equipment. The electromagnetic shaker primary source, angle stiffener and accelerometers can be seen connected to the beam. The piezoceramic stack actuator and angle stiffener are shown close-up in Figure 2.32, and one end of the beam supported by the wire termination is shown in Figure 2.33.

Figure 2.31  Experimental equipment for the active control of beam vibration.
Chapter 2. Control of vibrations in a stiffened beam

Figure 2.32 Piezoceramic stack actuator mounted between the beam and the flange of the angle stiffener.

Figure 2.33 Beam termination with wire support.
2.5 EXPERIMENTAL RESULTS

To verify the theoretical model, the experiment described in Section 2.4 was repeated for four different terminations. For the simple support, the wire support and the anechoic termination, classical impedances are given in Table 2.5 (with the wire support modelled as a free end and the anechoic termination modelled as an infinite end). The fourth termination used was a simply supported termination mounted on vibration isolators, for which there is no classical theoretical impedance. Using the method described in Section 2.4.1, new impedance matrices were calculated from the experimental acceleration distribution for each end condition. These matrices are also listed in Table 2.5, and differ considerably from the classical theoretical impedances for the three modelled terminations.

Figure 2.34 shows the controlled and uncontrolled acceleration distributions corresponding to the classical impedance, as well as the corresponding curves for the calculated impedances with the experimental results overlaid, for the anechoic termination. Figures 2.35 and 2.36 show similar results for the wire supported and simply supported beams. These figures show that agreement is far closer between the experimental results and the theoretical results obtained with the calculated impedances than between the experiment and the theoretical results obtained using the classical impedance values. The classical impedance matrices do not model accurately the real supports used in this experiment.
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Figure 2.34  Acceleration distributions for the anechoically terminated beam.

Figure 2.35  Acceleration distributions for the wire supported beam.

Figure 2.36  Acceleration distributions for the simply supported beam.

(a) Uncontrolled case; (b) Controlled case. —— Theoretical results, classical impedance; —— Theoretical results, calculated impedance; • Experimental data.

\( x_p \) = primary source location; \( x_c \) = control source location; \( x_e \) = error sensor location.
Figure 2.37 Acceleration distributions for the vibration isolated beam.

(a) Uncontrolled case; (b) Controlled case. — Theoretical results; • Experimental data.

$x_p =$ primary source location; $x_c =$ control source location; $x_e =$ error sensor location.

Figure 2.37 shows the calculated impedance curves with experimental data for the vibration isolated beam, but no classical impedance curves are given for this case as there is no known classical impedance for this arrangement. The theoretical curves corresponding to calculated impedances shown in the figures for the uncontrolled and controlled cases and for all end conditions show very good agreement with the experimental data.

The main area of difference between the calculated impedance theoretical acceleration curves and the experimental data is the location of the first minima to the right of the primary force. In the uncontrolled cases, the first minima in the theoretical curve occurs slightly to the left of the experimental minima (except in the anechoic termination case). In the controlled cases, the first minima in the theoretical curves are slightly to the right of the experimental results.
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All of these minima occur within half a wavelength of the primary source (and in the controlled cases, also of the control source), and this may indicate that the theory is limited in approximating the effect of the near field.

Table 2.8 compares the theoretical and experimental control force relative to the primary force for each end condition. The primary force was measured using a force transducer placed between the primary shaker and the beam. The control force magnitude was estimated by the procedure outlined above, using Equations (2.60) and (2.63). Again, agreement is good, although it should be said that the control force calculation is dependent on a number of assumptions and should only be used as an order of magnitude calculation.

<table>
<thead>
<tr>
<th>End Condition</th>
<th>Theoretical Amplitude*</th>
<th>Experimental Amplitude*</th>
<th>Theoretical Phase*</th>
<th>Experimental Phase*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire Support</td>
<td>5.20</td>
<td>4.93</td>
<td>0 or 180</td>
<td>0 or 180</td>
</tr>
<tr>
<td>Simple Support</td>
<td>1.69</td>
<td>1.72</td>
<td>0 or 180</td>
<td>0 or 180</td>
</tr>
<tr>
<td>Anechoic</td>
<td>2.43</td>
<td>2.53</td>
<td>0 or 180</td>
<td>0 or 180</td>
</tr>
<tr>
<td>Vibration Isolated</td>
<td>2.32</td>
<td>2.38</td>
<td>0 or 180</td>
<td>0 or 180</td>
</tr>
</tbody>
</table>

*Control force amplitude and phase are expressed relative to the primary force.
Chapter 2. Control of vibrations in a stiffened beam

2.6 SUMMARY

A theoretical model has been developed to describe the vibration response of an arbitrarily terminated beam to a range of excitation types, and in particular to describe the vibration response of beams to a point force primary excitation source and angle stiffener and piezoceramic stack control source. The numerical results indicate that flexural vibrations in beams can be actively controlled using the piezoceramic stack actuator and angle stiffener control source. Numerical results also indicate:

(1) The magnitude of the control source required for optimal control generally decreases with increasing stiffener flange length (see Figure 2.6 for definition of stiffener flange length) and increasing frequency.

(2) The control source amplitude required for optimal control is less when the beam is excited at a resonance frequency.

(3) When there is reflection from the beam terminations, the optimum control force is either in phase or $180^\circ$ out of phase with the primary source. When there is no reflection from beam terminations (the infinite beam case), the control source phase cycles through $180^\circ$ as the excitation frequency is increased.
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(4) For all but the infinite beam, maxima occur in the control source amplitude required for optimal control when the separation between control and primary sources is given by \( x = (c + n\lambda_x/2) \) where \( n \) is an integer and \( c \) is a constant dependent on frequency and termination type. These maxima occur when the control source is located at a node in the standing wave generated by reflection from the beam terminations. Minima in the mean attenuation of acceleration level downstream of the error sensor occur when the control source is located at a node in the standing wave.

(5) Increasing the separation between the primary and control source does not improve attenuation.

(6) The amount of attenuation achieved downstream of the error sensor increases with increasing separation between the error sensor and the control source.

(7) When the error sensor is located at a node in the standing wave that exists in finite beams, the attenuation achieved is less than that achieved with the error sensor located away from a node. Locating the error sensor at a node does not affect the control source required for optimal control.

(8) It is possible to achieve reductions in acceleration level upstream of the primary
source as well as the desired reduction downstream of the error sensor. The maximum mean attenuation in acceleration level upstream of the primary source is theoretically achieved with the separation between primary and effective control source locations given by $x = (0.25 + n\lambda_c/2)$ for $n = 1, 2, 3...$

(9) For error sensor locations outside the control source near field, the mean attenuation of acceleration level upstream of the primary source does not depend on error sensor location.

(10) A second control source can be used to overcome the difficulty in controlling vibration when the first control source is located at a node in a standing wave. The magnitude of the first control source can be arbitrarily limited and the second control source used when the limit is reached. The maxima in control source amplitude and the minima in attenuation that occur when the first control source is located at a standing wave node are eliminated in this way.

(11) There is no practical method of using a second error sensor to eliminate the minima in attenuation that occur when the first error sensor is located at a standing wave node.

The theoretical model outlined was verified experimentally for the four end conditions tested
Chapter 2. Control of vibrations in a stiffened beam

in this paper. The impedance corresponding to each termination was first calculated from experimental data. Comparison between experimental results and theoretical predictions showed that:

(1) The accuracy of the theoretical model when compared to the experimental results is very high, both in predicting the control source amplitude and phase required relative to the primary source, and in determining the acceleration distribution occurring along the beam.

(2) The impedances calculated from experimental measurements give more accurate results than the "classical" impedances corresponding to each termination.

(3) The theoretical model accurately predicts the amount of attenuation that can be achieved experimentally.
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CHAPTER 3. FEEDFORWARD ACTIVE CONTROL OF FLEXURAL VIBRATION IN A PLATE USING PIEZOCERAMIC ACTUATORS AND AN ANGLE STIFFENER

3.1 INTRODUCTION

In this chapter, the active control of flexural vibration in plates using as control sources piezoceramic actuators placed between a stiffener flange and the plate surface is investigated. The plate is rectangular with an angle stiffener mounted across the smaller dimension. The classical equation of motion for the flexural vibration of a plate is used to develop a theoretical model for the plate with primary point sources and an angle stiffener and control actuators (Section 3.2). The effective control signal is a combination of the effects of the point forces at the base of the actuators, and the reaction line force and line moment at the base of the stiffener (Section 3.2.5).

Figure 3.1 Plate showing primary sources, angle stiffener, piezoceramic stack control actuators and error sensors.
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The displacement at a point is the sum of the displacements due to each of the primary source and control source forces and moments. Optimal control is achieved by minimising the total mean square displacement at the location of the line of error sensors downstream of the control sources.

The theoretical analysis considers two different sets of plate supports. In both cases, the sides of the plate are modelled as simply supported and the left hand end is modelled as free. In the first case, the right hand end is modelled as infinite and in the second the right hand end is modelled as free. The influence of the control source location, the error sensor location and the excitation frequency on the control source amplitude and achievable attenuation are investigated, and the physical reasons for each observation are explained (Section 3.3). The effect of introducing a second angle stiffener and set of control sources is also examined.

A modal analysis of the plate is performed to show that the stiffener significantly affects the vibration response of the plate. Experimental verification of the theoretical model is performed for the semi-infinite plate with and without active vibration control. The experimental methods are described in Section 3.4. Experimental results are compared with theoretical predictions for the vibration of the plate with and without active vibration control (Section 3.5).
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3.2 THEORY

3.2.1 Response of a plate to a harmonic excitation

The response of the plate shown in Figure 3.2 to a simple harmonic excitation \( q(x,y)e^{j\omega t} \) is considered. The edges of the plate at \( y = 0 \) and \( y = L_y \) are modelled as simply supported.

![Figure 3.2 Plate with excitation \( q \) at location \( (x_0, y_0) \).]

Following the sign conventions shown in Figure 3.3, the equation of motion for the flexural vibration of the plate shown in Figure 3.2 is (Leissa, 1969)

\[
D_h \nabla^4 W(x, y, t) + \rho h \frac{\partial^2 W(x, y, t)}{\partial t^2} = q(x, y) e^{j\omega t},
\]

(3.1)

where \( D_h = \frac{Eh^3}{12(1-v^2)} \) is the flexural rigidity, \( E \) is Young's modulus of elasticity, \( h \) is the plate thickness, \( v \) is Poisson's ratio, \( \rho \) is the plate density, \( t \) is time, \( w \) is displacement, and \( \nabla^4 = \nabla^2 \nabla^2 \) is the square of the Laplacian operator.
Figure 3.3  Sign conventions for forces and moments. Conventions for moments in the $y$-plane are similar.

As the edges of the plate at $y = 0$ and $y = L_y$ are simply supported, the following harmonic series solution in $y$ can be assumed for the plate vibrational displacement (Pan and Hansen, 1994);

$$w(x, y, t) = \sum_{n=1}^{\infty} w_n(x) \sin \frac{n\pi y}{L_y} e^{i\omega t}, \quad (3.2)$$

where $n$ is a mode number and $\omega$ is the angular frequency. Each eigenfunction $w_n(x)$ can be expressed in terms of modal wavenumbers $k_{in}$ as follows:

$$w_n(x) = A_n e^{k_{1n}x} + B_n e^{k_{2n}x} + C_n e^{k_{3n}x} + D_n e^{k_{4n}x}. \quad (3.3)$$

On each side of an applied force or moment at $x = x_0$, the eigenfunction is a different linear combination of the terms $e^{k_{in}x}$ ($i = 1,2,3,4$). For $x < x_0$,

$$w_{1n}(x) = A_{1n} e^{k_{1n}x} + B_{1n} e^{k_{2n}x} + C_{1n} e^{k_{3n}x} + D_{1n} e^{k_{4n}x}, \quad (3.4)$$
Chapter 3. Control of vibrations in a stiffened plate

and for \( x > x_0 \),

\[
\begin{align*}
  w_{2n}(x) &= A_{2n} e^{k_1 x} + B_{2n} e^{k_2 x} + C_{2n} e^{k_3 x} + D_{2n} e^{k_4 x},
\end{align*}
\]

(3.5)

To find the modal wavenumbers \( k_{int} \), the homogeneous form of Equation (3.1) is multiplied by \( \sin \frac{n \pi y}{L_y} \) and integrated with respect to \( y \) to give

\[
\frac{d^4 w_n(x)}{dx^4} - 2 \left( \frac{n \pi}{L_y} \right)^2 \frac{d^2 w_n(x)}{dx^2} + \left( \frac{n \pi}{L_y} \right)^4 - \frac{\rho h \omega^2}{D_h} \right) w_n(x) = 0 .
\]

(3.6)

The corresponding characteristic equation is

\[
k_n^4 + 2 \left( \frac{n \pi}{L_y} \right)^2 k_n^2 + \left( \frac{n \pi}{L_y} \right)^4 - \frac{\rho h \omega^2}{D_h} = 0 ,
\]

(3.7)

which has the roots

\[
k_{1n,2n} = \pm \left[ \left( \frac{n \pi}{L_y} \right)^2 - \left( \frac{\rho h \omega^2}{D_h} \right) \right]^\frac{1}{2}.
\]

(3.8)

and

\[
k_{3n,4n} = \pm \left[ \left( \frac{n \pi}{L_y} \right)^2 - \left( \frac{\rho h \omega^2}{D_h} \right) \right]^\frac{1}{2}.
\]

(3.9)

To solve for the eight unknown constants \( A_{1n}, B_{1n}, C_{1n}, D_{1n}, A_{2n}, B_{2n}, C_{2n} \) and \( D_{2n} \), eight equations are required, comprising of four boundary condition equations (force and moment
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conditions at each end of the plate) and four equilibrium condition equations at the point of application \(x_0\) of the force or moment, for each cross-plate mode \(n\) of the plate vibration.

3.2.2 Boundary conditions at the plate ends

For the purposes of this work, two sets of boundary conditions will be examined; those corresponding to a plate with free ends, and to a semi-infinite plate with the end at \(x = 0\) modelled as free. Both plates will be modelled with simply supported sides (at \(y = 0\) and \(y = L_y\)).

3.2.2.1 Free end conditions

In terms of displacement, the bending moment boundary condition \(M_x(0,y) = 0\) for a free end at \(x = 0\) becomes (Pan and Hansen, 1994):

\[
\begin{align*}
&\left[ k_{1n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] A_{1n} + \left[ k_{2n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] B_{1n} \\
&\quad + \left[ k_{3n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] C_{1n} + \left[ k_{4n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] D_{1n} = 0.
\end{align*}
\]  

(3.10)

The corresponding equation for a free end at \(x = L_x\) is

\[
\begin{align*}
&\left[ k_{2n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] A_{2n} e^{k_{2n}L_x} + \left[ k_{3n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] B_{2n} e^{k_{3n}L_x} \\
&\quad + \left[ k_{4n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] C_{2n} e^{k_{4n}L_x} + \left[ k_{5n}^2 - \nu \left( \frac{n\pi}{L_y} \right)^2 \right] D_{2n} e^{k_{5n}L_x} = 0.
\end{align*}
\]  

(3.11)
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The free end condition also requires that the net vertical force at the end be zero. This condition yields the following equation in terms of the displacement unknowns for the end \( x = 0 \),

\[
\begin{align*}
\left[ k_{1n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{1n} \right] A_{1n} + \left[ k_{2n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{2n} \right] B_{1n} &= 0, \\
\left[ k_{3n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{3n} \right] C_{1n} + \left[ k_{4n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{4n} \right] D_{1n} &= 0.
\end{align*}
\] (3.12)

and for \( x = L_x \),

\[
\begin{align*}
\left[ k_{1n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{1n} \right] A_{1n} e^{k_{1n}L_x} + \left[ k_{2n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{2n} \right] B_{1n} e^{k_{2n}L_x} + \\
\left[ k_{3n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{3n} \right] C_{1n} e^{k_{3n}L_x} + \left[ k_{4n}^3 - (2 - \nu) \left( \frac{n\pi}{L_y} \right)^2 k_{4n} \right] D_{1n} e^{k_{4n}L_x} &= 0.
\end{align*}
\] (3.13)

3.2.2.2 Infinite end conditions

An infinite end produces no reflections, so the boundary conditions corresponding to an infinite end at \( x = L_x \) are simply

\[ A_{2n} = 0 \] (3.14)

and

\[ C_{2n} = 0. \] (3.15)
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3.2.3 Equilibrium conditions at the point of application \((x = x_0)\) of a force or moment

Requiring that the displacement and gradient in each direction be continuous at any point on the plate, the first two equilibrium conditions which must be satisfied at \(x = x_0\) are

\[
\begin{align*}
  w_{1n} &= w_{2n} \\
  \frac{\partial w_{1n}}{\partial x} &= \frac{\partial w_{2n}}{\partial x}.
\end{align*}
\] (3.16)

The form of the excitation \(q(x,y)\) will affect the higher order equilibrium conditions at \(x = x_0\). In the following sections the equilibrium conditions corresponding to the plate excited by a point force, a line force parallel to the \(y\)-axis, and line moments acting about an axis parallel to the \(y\)-axis are discussed. These three types of excitation are induced by an actuator placed between a stiffener flange and the plate.

3.2.3.1 Response of a plate to a point force

The response of the plate to a simple harmonic point force \(F_0\) acting normal to the plate at position \((x_0,y_0)\) is considered. The excitation \(q(x,y)\) in Equation (3.1) is replaced by \(q(x,y) = F_0 \delta(x - x_0) \delta(y - y_0)\), where \(\delta\) is the Dirac delta function. The second and third order boundary conditions at \(x = x_0\) are (Pan and Hansen, 1994):

\[
\frac{\partial^2 w_{1n}}{\partial x^2} = \frac{\partial^2 w_{2n}}{\partial x^2},
\] (3.18)
Chapter 3. Control of vibrations in a stiffened plate

and

\[
\frac{\partial^3 w_{1n}}{\partial x^3} - \frac{\partial^3 w_{2n}}{\partial x^3} = - \left( \frac{2F_0}{L_y D_h} \right) \sin \frac{n\pi y_0}{L_y} .
\]  
(3.19)

3.2.3.2 Response of a plate to a distributed line force parallel to the y-axis

Instead of a single point force, the excitation represented by \( q(x,y) \) in Equation (3.1) is replaced by an array of \( N \) equally spaced point forces distributed along a line parallel to the y-axis between \( y_1 \) and \( y_2 \). These forces act at locations \((x_0, y_k), k = 1, N\) and each has a magnitude of \( F_0/N \), so \( q(x,y) \) in Equation (3.1) is replaced by \( q(x,y) = \left( \frac{F_0}{N} \right) \sum_{k=1}^{N} \delta(x - x_0)\delta(y - y_k) \). The second and third order boundary conditions at \( x = x_0 \) are

\[
\frac{\partial^3 w_{1n}}{\partial x^2} = \frac{\partial^3 w_{2n}}{\partial x^2}
\]  
(3.20)

and

\[
\frac{\partial^3 w_{1n}}{\partial x^3} - \frac{\partial^3 w_{2n}}{\partial x^3} = \left( \frac{2F_0}{n\pi(y_2 - y_1)D_h} \right) \left( \cos \frac{n\pi y_2}{L_y} - \cos \frac{n\pi y_1}{L_y} \right) .
\]  
(3.21)

3.2.3.3 Response of a plate to a distributed line moment parallel to the y-axis

The excitation represented by \( q(x,y) \) in Equation (3.1) is replaced by a distributed line moment \( M_0 \) per unit length acting along a line parallel to the y-axis between the locations \((x, y)\) and \((x_0, y_2)\). The excitation term \( q(x,y) \) in Equation (3.1) is replaced by \( q(x,y) = \frac{\partial M}{\partial x} = M_0 \delta'(x - x_0)\left[ h(y - y_1) - h(y - y_2) \right] \). The second and third order boundary conditions at \( x = x_0 \) are
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\[ \frac{\partial^2 w_{1n}}{\partial x^2} - \frac{\partial^2 w_{2n}}{\partial x^2} = \frac{2M_0}{n\pi D_h} \left( \cos \frac{n\pi y_2}{L_y} - \cos \frac{n\pi y_1}{L_y} \right) \] 

(3.22)

and

\[ \frac{\partial^3 w_{1n}}{\partial x^3} = \frac{\partial^3 w_{2n}}{\partial x^3} . \] 

(3.23)

Taking two boundary conditions at each end of the plate from Equations (3.10) - (3.15), the two equilibrium condition Equations (3.16) and (3.17), and two further equilibrium conditions from Equations (3.18) - (3.23), eight equations in the eight unknowns \( A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 \) are obtained. These can be written in the form \( \alpha X = B \). The solution vectors \( X = [A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2]^T = \alpha^{-1}B \) can be used to characterise the response of a plate to simple harmonic excitation by a single point force, a distributed line force along a line parallel to the y-axis or a distributed line moment about a line parallel to the y-axis.

3.2.4 Modelling the effects of the angle stiffener

The mass and stiffness of the angle stiffener may be significant and are taken into account as follows. Given a plate with an arbitrary excitation \( q \) at axial position \( x = x_0 \) and an angle stiffener extending across the width of the plate at axial position \( x = x_1 \), as shown in Figure 3.4, three eigenfunction solutions of Equation (3.1) are now required.
For $x < x_0$,

$$w_{1n}(x) = A_{1n}e^{k_{1n}x} + B_{1n}e^{k_{2n}x} + C_{1n}e^{k_{3n}x} + D_{1n}e^{k_{4n}x},$$

(3.24)

for $x_0 < x < x_1$,

$$w_{2n}(x) = A_{2n}e^{k_{1n}x} + B_{2n}e^{k_{2n}x} + C_{2n}e^{k_{3n}x} + D_{2n}e^{k_{4n}x},$$

(3.25)

and for $x > x_1$,

$$w_{3n}(x) = A_{3n}e^{k_{1n}x} + B_{3n}e^{k_{2n}x} + C_{3n}e^{k_{3n}x} + D_{3n}e^{k_{4n}x}.$$  

(3.26)

These eigenfunctions allow for reflection at the stiffener location. To solve for $w_{1n}(x)$, $w_{2n}(x)$ and $w_{3n}(x)$, twelve equations in the twelve unknowns $A_{1n}$, $B_{1n}$, $C_{1n}$, $D_{1n}$, $A_{2n}$, $B_{2n}$, $C_{2n}$, $D_{2n}$, $A_{3n}$, $B_{3n}$, $C_{3n}$ and $D_{3n}$ are now required. In addition to the eight equilibrium conditions at $x = x_0$, which depend on the form of the excitation $q$, and the boundary conditions at each end of the plate, the equilibrium conditions which must be satisfied at the stiffener location $x = x_1$ are

$$w_{2n} = w_{3n},$$

(3.27)
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\[
\frac{\partial w_{2n}}{\partial x} = \frac{\partial w_{3n}}{\partial x}, \quad (3.28)
\]

\[
\frac{\partial^2 w_{2n}}{\partial x^2} = \frac{\partial^2 w_{3n}}{\partial x^2}, \quad (3.29)
\]

and

\[
\frac{\partial^3 w_{2n}}{\partial x^3} - \frac{\partial^3 w_{3n}}{\partial x^3} = \frac{2}{n\pi L D_h} \left( K_a w + m_a \frac{\partial^2 w}{\partial t^2} \right) \left( \cos(n\pi) - 1 \right), \quad (3.30)
\]

where \( K_a \) is the stiffness and \( m_a \) the mass per unit length of the stiffener. If the angle stiffener is very rigid compared to the plate, Equations (3.27) and (3.30) can be replaced by the two conditions

\[
w_2 = 0 \quad (3.31)
\]

and

\[
w_3 = 0. \quad (3.32)
\]

3.2.5 Minimising vibration using piezoceramic actuators and an angle stiffener

For any force or moment excitation, the twelve boundary and equilibrium equations in twelve unknowns can be written in the form \( aX = B \), where \( X = [A_{1n}, B_{1n}, C_{1n}, D_{1n}, A_{2n}, B_{2n}, C_{2n}, D_{2n}, A_{3n}, B_{3n}, C_{3n}, D_{3n}]^T \) and \( B \) is a column vector. When the excitation position is to the left of the stiffener location, i.e. \( x_0 < x_1 \), \( B \) has a non zero excitation term in the seventh row for excitation
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by a line moment about a line parallel to the y-axis or the eighth row otherwise. For a semi-
infinite plate with the end at \( x = 0 \) free, \( \alpha \) is given by Equation (3.33). If the excitation position
is to the right of the stiffener location, i.e. \( x_0 > x_1 \), then a similar analysis is followed, resulting
in an excitation vector \( B \) with the non zero term in the eleventh row for excitation by a line
moment about a line parallel to the y-axis or the twelfth row otherwise, and \( \alpha \) is given by
Equation (3.34). For the plate with both ends modelled as free, the equations corresponding to
rows 3 and 4 of the matrix \( \alpha \) are replaced by Equations (3.11) and (3.13). For both sets of
boundary conditions, the matrix equations \( \alpha X = B \) can be solved for \( X \) for any type of excitation
and the result can be used with Equations (3.2) and (3.24) - (3.26) to calculate the corresponding
plate response.

Figure 3.5 shows the semi-infinite plate with primary forces \( F_{p1} \) and \( F_{p2} \) located at \( x = x_p, y = y_{p1} \) and \( y = y_{p2} \), control actuators at \( x = x_c \) and a line of error sensors at \( x = x_e \). Figure 3.6 shows
the resultant forces and moments applied to the plate by the control actuators. Control forces \( F_{c1}, \)
\( F_{c2} \) and \( F_{c3} \) act at \( (x_{c2}, y_{ci}, i = 1,3) \), with the distributed line force \( F_c \) and distributed line moment
\( M_c \) acting about a line parallel to the y-axis at \( (x_{c1}, y = 0 \text{ to } y = L_y) \).
Figure 3.5 Semi-infinite plate showing primary forces, control actuators, angle stiffener and line of error sensors.

Figure 3.6 Angle stiffener and control actuators showing control forces and moment.
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\[
x = \begin{bmatrix}
  k_{1n}^2 - \nu \left( \frac{m}{L_y} \right)^2 & k_{2n}^2 - \nu \left( \frac{m}{L_y} \right)^2 & k_{3n}^2 - \nu \left( \frac{m}{L_y} \right)^2 & k_{4n}^2 - \nu \left( \frac{m}{L_y} \right)^2 \\
  k_{1n}^3 - (2 - \nu) \left( \frac{m}{L_y} \right)^2 k_{1n} & k_{2n}^3 - (2 - \nu) \left( \frac{m}{L_y} \right)^2 k_{2n} & k_{3n}^3 - (2 - \nu) \left( \frac{m}{L_y} \right)^2 k_{3n} & k_{4n}^3 - (2 - \nu) \left( \frac{m}{L_y} \right)^2 k_{4n}
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  e^{k_{1n} \eta_0} & e^{k_{2n} \eta_0} & e^{k_{3n} \eta_0} & e^{k_{4n} \eta_0} \\
  k_{1n} e^{k_{1n} \eta_0} & k_{2n} e^{k_{2n} \eta_0} & k_{3n} e^{k_{3n} \eta_0} & k_{4n} e^{k_{4n} \eta_0} \\
  k_{1n}^2 e^{k_{1n} \eta_0} & k_{2n}^2 e^{k_{2n} \eta_0} & k_{3n}^2 e^{k_{3n} \eta_0} & k_{4n}^2 e^{k_{4n} \eta_0} \\
  k_{1n}^3 e^{k_{1n} \eta_0} & k_{2n}^3 e^{k_{2n} \eta_0} & k_{3n}^3 e^{k_{3n} \eta_0} & k_{4n}^3 e^{k_{4n} \eta_0}
\end{bmatrix}
\]

\[
(3.33)
\]
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\[
\begin{bmatrix}
  k_{1n}^2 - \nu \left( \frac{m}{L} \right)^2 & k_{2n}^2 - \nu \left( \frac{m}{L} \right)^2 & k_{3n}^2 - \nu \left( \frac{m}{L} \right)^2 & k_{4n}^2 - \nu \left( \frac{m}{L} \right)^2 \\
  k_{1n}^3 - (2 - \nu) \nu \left( \frac{m}{L} \right)^2 k_{1n} & k_{2n}^3 - (2 - \nu) \nu \left( \frac{m}{L} \right)^2 k_{2n} & k_{3n}^3 - (2 - \nu) \nu \left( \frac{m}{L} \right)^2 k_{3n} & k_{4n}^3 - (2 - \nu) \nu \left( \frac{m}{L} \right)^2 k_{4n}
\end{bmatrix}
\]

\[
x =
\begin{bmatrix}
  e^{k_{1n}r_1} & e^{k_{2n}r_1} & e^{k_{3n}r_1} & e^{k_{4n}r_1} \\
  k_{1n} e^{k_{1n}r_1} & k_{2n} e^{k_{2n}r_1} & k_{3n} e^{k_{3n}r_1} & k_{4n} e^{k_{4n}r_1} \\
  k_{1n}^2 e^{k_{1n}r_1} & k_{2n}^2 e^{k_{2n}r_1} & k_{3n}^2 e^{k_{3n}r_1} & k_{4n}^2 e^{k_{4n}r_1} \\
  e^{k_{1n}r_1} & e^{k_{2n}r_0} & e^{k_{3n}r_0} & e^{k_{4n}r_0} \\
  -k_{1n} e^{k_{1n}r_1} & -k_{2n} e^{k_{2n}r_1} & -k_{3n} e^{k_{3n}r_1} & -k_{4n} e^{k_{4n}r_1} \\
  -k_{1n}^2 e^{k_{1n}r_1} & -k_{2n}^2 e^{k_{2n}r_1} & -k_{3n}^2 e^{k_{3n}r_1} & -k_{4n}^2 e^{k_{4n}r_1} \\
  e^{k_{1n}r_0} & e^{k_{2n}r_0} & e^{k_{3n}r_0} & e^{k_{4n}r_0} \\
  -e^{k_{1n}r_0} & -e^{k_{2n}r_0} & -e^{k_{3n}r_0} & -e^{k_{4n}r_0} \\
  -k_{1n} e^{k_{3n}r_0} & -k_{2n} e^{k_{3n}r_0} & -k_{3n} e^{k_{3n}r_0} & -k_{4n} e^{k_{3n}r_0} \\
  -k_{1n}^2 e^{k_{3n}r_0} & -k_{2n}^2 e^{k_{3n}r_0} & -k_{3n}^2 e^{k_{3n}r_0} & -k_{4n}^2 e^{k_{3n}r_0} \\
  k_{1n} e^{k_{1n}r_0} & k_{2n} e^{k_{2n}r_0} & k_{3n} e^{k_{3n}r_0} & k_{4n} e^{k_{4n}r_0} \\
  k_{1n}^2 e^{k_{1n}r_0} & k_{2n}^2 e^{k_{2n}r_0} & k_{3n}^2 e^{k_{3n}r_0} & k_{4n}^2 e^{k_{4n}r_0} \\
  k_{1n}^3 e^{k_{1n}r_0} & k_{2n}^3 e^{k_{2n}r_0} & k_{3n}^3 e^{k_{3n}r_0} & k_{4n}^3 e^{k_{4n}r_0}
\end{bmatrix}
\]

(3.34)
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The plate response amplitude at any location \((x,y)\) to a particular excitation located at \(x_0\), with an angle stiffener across the width of the plate at location \(x_1\) is

\[
w(x,y) = \sum_{n=1}^{\infty} \left[ X^T E(x) \right] \sin \frac{n\pi y}{L_y},
\]

(3.35)

where, for \(x < x_0\) and \(x < x_1\),

\[
E(x) = \begin{bmatrix} e^{k_{1w}x} & e^{k_{2w}x} & e^{k_{3w}x} & e^{k_{4w}x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,
\]

(3.36)

for \(x_0 < x < x_1\) or \(x_1 < x < x_0\),

\[
E(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & e^{k_{1w}x} & e^{k_{2w}x} & e^{k_{3w}x} & e^{k_{4w}x} & 0 & 0 & 0 \end{bmatrix}^T,
\]

(3.37)

and for \(x > x_0\) and \(x > x_1\),

\[
E(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{k_{1w}x} & e^{k_{2w}x} & e^{k_{3w}x} & e^{k_{4w}x} \end{bmatrix}^T.
\]

(3.38)

By summation of the displacement equations calculated for each force and moment, the total plate displacement resulting from the primary and control excitations is

\[
w = \sum_{i=1}^{2} \left( W_{F_p i} \right) + \sum_{i=1}^{3} \left( w_{F_0 i} \right) + w_{F_i} + w_{M_i}
\]

\[
+ \sum_{i=1}^{2} \left( X_{F_p i}^T E(x) \right) + \sum_{i=1}^{3} \left( X_{F_0 i}^T E(x) \right) + X_{F_c i}^T E(x) + X_{M_c i}^T E(x) \right) \sin
\]

(3.39)

where the subscripts \(F_p\), \(F_c\), \(F\) and \(M_c\) on \(w\) and \(X\) refer to the corresponding excitation force or moment.
As the excitation vector $B$ has a non-zero element in one row only, the solution vector $X$ can be written in terms of a single column of the inverse matrix $\mathbf{a}^{-1}$:

$$X = (\mathbf{a}^{-1})_{k,i} B_i, \quad (\mathbf{a}^{-1})_{k,11} B_{11}, \quad \text{or} \quad (\mathbf{a}^{-1})_{k,12} B_{12}, \quad k = 1,12,$$

(3.40)

where $(\mathbf{a}^{-1})_{k,i}$ is the $k^{th}$ element in the $i^{th}$ column of the inverse of $\mathbf{a}$ and $B_i$ is the $i^{th}$ element (the non-zero element) of $B$. The value taken by $i$ depends on the form and location of the excitation, as discussed previously.

### 3.2.5.1 Control sources driven by the same signal

If the three control actuators are driven by the same signal, then the actuator forces are $F_{c1} = F_{c2} = F_{c3} = -F_s$, say. Also, if the angle stiffener is rigid compared to the plate, the line force $F_c = 3F_s$ and the line moment $M_c = -3aF_s$, where $a$ is the width of the stiffener flange. If the two primary shakers are driven by the same signal, then the primary forces are $F_{p1} = F_{p2} = F_{p3}$, say. Defining

$$F_{pi} = \frac{-2}{L_pD_h} \left[ \mathbf{a}^{-1} \right]_{i,8}^T (i = 1,2),$$

(3.41)

$$F_{ci} = \frac{-2}{L_pD_h} \left[ \mathbf{a}^{-1} \right]_{i,12}^T (i = 1,3),$$

(3.42)

$$F_c = \frac{-2}{\pi L_pD_h} \left[ \mathbf{a}^{-1} \right]_{12}^T,$$

(3.43)
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and

\[ M_C = \frac{2}{\pi D_h} \alpha^{-1}_M^{T} \]

(3.44)

and substituting Equations (3.41)-(3.44) into Equation (3.39) and rearranging gives

\[
\begin{align*}
    w(x,y) & = \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{2} -F_{pi} \sin \frac{n\gamma_{xi}}{L_y} \right] F_{p} + \\
    & \left[ \left( \sum_{i=1}^{3} -F_{ci} \sin \frac{n\gamma_{yi}}{L_y} \right) + \left( \frac{3F_C - 3x_d M_C}{n} \right) (1 - \cos n\pi) \right] E(x) \sin \frac{n\gamma}{L_y},
\end{align*}
\]

(3.45)

or

\[
\begin{align*}
    w(x,y) & = w_p(x,y) F_p + w_s(x,y) F_s,
\end{align*}
\]

(3.46)

where

\[
\begin{align*}
    w_p(x,y) & = \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{2} -F_{pi} \sin \frac{n\gamma_{xi}}{L_y} \right] E(x) \sin \frac{n\gamma}{L_y} \\
    & (3.47)
\end{align*}
\]

and

\[
\begin{align*}
    w_s(x,y) & = \sum_{n=1}^{\infty} \left[ \left( \sum_{i=1}^{3} -F_{ci} \sin \frac{n\gamma_{yi}}{L_y} \right) + \left( \frac{3F_C - 3x_d M_C}{n} \right) (1 - \cos n\pi) \right] E(x) \sin \frac{n\gamma}{L_y}.
\end{align*}
\]

(3.48)

The radial acceleration at the line \( x = x_e \) is to be minimised. The mean square of the
displacement defined in Equation (3.46) is integrated over the width of the plate:

\[ \int_{0}^{L_y} |w(x,y)|^2 dy = \int_{0}^{L_y} \left| F_p w_p(x,y) + F_s w_s(x,y) \right|^2 dy . \] (3.49)

Noting that \(|z|^2 = z \overline{z}\) (where \(\overline{z}\) is the complex conjugate of \(z\)), and writing \(F_s = F_{sr} + F_{sj} j\),

\[ \int_{0}^{L_y} |w(x,y)|^2 dy = \int_{0}^{L_y} \left[ |F_{sr}|^2 |w_p|^2 + F_p w_p \overline{w_s} (F_{sr} - F_{sj} j) + \overline{F_p w_p} w_s (F_{sr} + F_{sj} j) + (F_{sr}^2 + F_{sj}^2) |w_s|^2 \right] dy . \] (3.50)

The partial derivatives of Equation (3.50) with respect to the real and imaginary components of the control force are taken and set equal to zero to find

\[ \frac{\partial}{\partial F_{sr}} \int_{0}^{L_y} \left[ F_p w_p \overline{w_s} + F_p w_p w_s + 2F_{sr} |w_s|^2 \right] dy = 0 \] (3.51)

and

\[ \frac{\partial}{\partial F_{sj}} \int_{0}^{L_y} \left[ F_p w_p \overline{w_s} - F_p w_p w_s + 2F_{sj} |w_s|^2 \right] dy = 0 . \] (3.52)

Adding Equations (3.51) and (3.52) gives

\[ \int_{0}^{L_y} \left[ F_p w_p \overline{w_s} + F_s |w_s|^2 \right] dy = 0 . \] (3.53)

The optimal control force \(F_s\) required to minimise normal acceleration at the ring of error sensors...
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can thus be calculated by

\[
F_s = -F_p \frac{\int_0^{L_y} w_p(x_e,0) w(x_e,0) dy}{\int_0^{L_y} |w(x_e,0)|^2 dy}.
\]  
(3.54)

3.2.5.2 Control sources driven independently

If the three control actuators are driven independently, then a similar analysis is followed; however, three equations instead of one result from integrating the mean square of the displacement defined in Equation (3.46) and setting the partial derivatives of the integration with respect to the real and imaginary components of each control force equal to zero. The optimal control forces \(F_{s1}, F_{s2}\) and \(F_{s3}\) required to minimise acceleration at the line of error sensors can be calculated by

\[
\begin{bmatrix}
F_{s1} \\
F_{s2} \\
F_{s3}
\end{bmatrix} = -F_p
\begin{bmatrix}
\int_0^{L_y} w_1 w_1 dy & \int_0^{L_y} w_1 w_2 dy & \int_0^{L_y} w_1 w_3 dy \\
\int_0^{L_y} w_2 w_1 dy & \int_0^{L_y} w_2 w_2 dy & \int_0^{L_y} w_2 w_3 dy \\
\int_0^{L_y} w_3 w_1 dy & \int_0^{L_y} w_3 w_2 dy & \int_0^{L_y} w_3 w_3 dy
\end{bmatrix}^{-1}
\begin{bmatrix}
\int_0^{L_y} w_p w_1 dy \\
\int_0^{L_y} w_p w_2 dy \\
\int_0^{L_y} w_p w_3 dy
\end{bmatrix}.
\]  
(3.55)
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3.2.5.3 Two angle stiffeners and two sets of control sources

If a second set of three control sources and an additional angle stiffener are introduced at some location $\mathbf{x}_c$ downstream from the first, and a prime used to denote values associated with the second set of control sources, the optimal control forces $F_{s1}'$, $F_{s2}'$ and $F_{s3}'$ required to minimise acceleration at the line of error sensors can be calculated by

\[
\begin{bmatrix}
F_{s1}' \\
F_{s2}' \\
F_{s3}'
\end{bmatrix} = \frac{1}{L_y} \begin{bmatrix}
\int_{0}^{L_x} w_1'w_1'dy \\
\int_{0}^{L_x} w_2'w_1'dy \\
\int_{0}^{L_x} w_3'w_1'dy
\end{bmatrix} \begin{bmatrix}
L_y \\
L_y \\
L_y
\end{bmatrix}^{-1} \begin{bmatrix}
\int_{0}^{L_x} \left( F_p w_p + F_{s1} w_1 + F_{s2} w_2 + F_{s3} w_3 \right) w_1'dy \\
\int_{0}^{L_x} \left( F_p w_p + F_{s1} w_1 + F_{s2} w_2 + F_{s3} w_3 \right) w_2'dy \\
\int_{0}^{L_x} \left( F_p w_p + F_{s1} w_1 + F_{s2} w_2 + F_{s3} w_3 \right) w_3'dy
\end{bmatrix}
\]

(3.56)

3.2.5.4 Discrete error sensors

If the sum of the squares of the vibration amplitude measured at $Q$ discrete points $(x_q, y_q)$, $q = 1, Q$ is used as the error signal instead of the integral over the plate width at location $x_c$, Equation (3.55) becomes

\[
\begin{bmatrix}
F_{s1}' \\
F_{s2}' \\
F_{s3}'
\end{bmatrix} = -F_p \begin{bmatrix}
\sum_{q=1}^{Q} w_1 w_1 \\
\sum_{q=1}^{Q} w_2 w_1 \\
\sum_{q=1}^{Q} w_3 w_1 \\
\sum_{q=1}^{Q} w_1 w_2 \\
\sum_{q=1}^{Q} w_2 w_2 \\
\sum_{q=1}^{Q} w_3 w_2 \\
\sum_{q=1}^{Q} w_1 w_3 \\
\sum_{q=1}^{Q} w_2 w_3 \\
\sum_{q=1}^{Q} w_3 w_3
\end{bmatrix} \begin{bmatrix}
\sum_{q=1}^{Q} w_p w_1 \\
\sum_{q=1}^{Q} w_p w_2 \\
\sum_{q=1}^{Q} w_p w_3
\end{bmatrix}^{-1}
\]

(3.57)
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where

\[ w_i = w_i(x_e, y_{qe}) \tag{3.58} \]
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3.3 NUMERICAL RESULTS

The theoretical model developed in the previous section was programmed in Fortran. The program consisted of about 1800 lines and, for a typical set of results, took two or three hours C.P.U. time to run on a SPARC-20 computer.

The discussion that follows examines the effect of varying forcing frequency, control source location and error sensor location on the active control of vibration in plates with two sets of boundary conditions. In both models the sides of the plate at $y = 0$ and $y = L_y$ are modelled as pinned and the end at $x = 0$ is modelled as free. In one model, the end at $x = L_x$ is also modelled as free and in the second model the plate is modelled as semi-infinite in the $x$-direction. The plate parameters (including location of the control source, primary source and error sensor) are listed in Table 3.1. These values are adhered to unless otherwise stated. The stiffener was assumed to be very stiff in comparison to the plate.

Control forces are expressed as a multiple of the primary force, and the acceleration amplitude dB scale reference level is the far field uncontrolled infinite plate acceleration produced by the primary sources only. In all cases, the control forces are assumed to be optimally adjusted to minimise the acceleration at the line of error sensors. The flexural wavelength of vibration in a plate is given by

$$\lambda_b = \frac{1}{f} \sqrt{\frac{E_h}{12\rho(1 - \nu^2)}}. \quad (3.59)$$
Table 3.1
Plate Parameters for Numerical and Experimental Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plate length $L_x$</td>
<td>2.0 m</td>
</tr>
<tr>
<td>Plate width $L_y$</td>
<td>0.50 m</td>
</tr>
<tr>
<td>Plate thickness $h$</td>
<td>0.003 m</td>
</tr>
<tr>
<td>Young's modulus $E$</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Primary source location $x_p$</td>
<td>0.025 m</td>
</tr>
<tr>
<td>Primary source locations $y_{p1}$, $y_{p2}$</td>
<td>0.17m, 0.33m</td>
</tr>
<tr>
<td>Control source location $x_1$</td>
<td>0.5 m (=1.39$\lambda_b$)</td>
</tr>
<tr>
<td>Control source locations $y_{c1}$, $y_{c2}$, $y_{c3}$</td>
<td>0.08m, 0.25m, 0.42m</td>
</tr>
<tr>
<td>Stiffener flange length $a$</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Error sensor location $x_e$</td>
<td>1.0 m (= 2.79$\lambda_b$)</td>
</tr>
<tr>
<td>Excitation frequency $f$</td>
<td>230 Hz</td>
</tr>
<tr>
<td>Wavelength $\lambda_b$</td>
<td>0.359 m</td>
</tr>
</tbody>
</table>

* - Applies only when $f = 230$ Hz.

3.3.1 Acceleration distributions for controlled and uncontrolled cases

Figures 3.7 and 3.8 show the uncontrolled acceleration amplitude distribution in dB for the semi-infinite and finite plates. The shape of the curve downstream of the angle stiffener location ($x_{c1} = 0.5$m) represents a travelling wave field with an additional decaying evanescent field close to the source. Waves reflected from the stiffener and the plate ends cause standing wave fields to exist, both upstream and downstream of the angle stiffener for the finite plate and upstream of the stiffner for the semi-infinite plate.
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Figure 3.7 Uncontrolled semi-infinite plate acceleration distribution. The edges $y = 0$ and $y = 0.5$ are simply supported and the end at $x = 0$ is free.

Figure 3.8 Uncontrolled finite plate acceleration distribution. Edges $y = 0$ and $y = 0.5$ are simply supported and the ends at $x = 0$ and $x = 2.0$ are free.
Chapter 3. Control of vibrations in a stiffened plate

It can be seen from the nature of the response that the near field effects become insignificant at less than 0.2m (½ wavelength) from the plate ends and the stiffener location.

Figures 3.9 and 3.10 show the controlled acceleration amplitude distributions for the semi-infinite and finite plates with the three control sources driven by the same signal. The acceleration level is less downstream of the error sensor location than at the error sensor location ($x_e = 1.0m$). This is consistent with previous work dealing with minimisation of vibration at a line across a plate using control sources driven by the same signal (Pan and Hansen, 1994). The calculated reduction in acceleration amplitude downstream of the error sensor is over 30 dB for the semi-infinite plate and over 20 dB for the finite plate.

Figures 3.11 and 3.12 show the controlled acceleration amplitude distributions for the semi-infinite and finite plates with the three control sources driven independently. The acceleration level is at a minimum at the error sensor location ($x_e = 1.0m$). The calculated reduction in acceleration amplitude downstream of the error sensor is around 45 dB for the semi infinite plate and over 40 dB for the finite plate. The slope of the attenuation curve as a function of location in the $x$-direction is greater for the case with control sources driven independently. Independently driven control sources require less room to deliver a given level of attenuation than control sources driven by the same signal.
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Figure 3.9 Controlled semi-infinite plate acceleration distribution - control sources driven by the same signal.

Figure 3.10 Controlled finite plate acceleration distribution - control sources driven by the same signal.
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**Figure 3.11** Controlled semi-infinite plate acceleration distribution - control sources driven independently.

**Figure 3.12** Controlled finite plate acceleration distribution - control sources driven independently.
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3.3.2 Effect of variations in forcing frequency, control source location and error sensor location on the control forces

Figure 3.13 shows the effect of varying the forcing frequency on the magnitude of the control force(s). The average control force amplitude for the case where control sources are driven independently is generally slightly lower than the control force amplitude for the case when the control sources are driven by the same signal, as expected, because the energy input to the plate is more efficient with independently driven control sources. The independently driven control force amplitude is greater at some higher frequencies where control sources driven by the same signal do not control the vibration well. This will be discussed later.

The maxima at frequencies of 38 Hz, 96 Hz, 209 Hz and 380 Hz on Figure 3.13 occur when the relative spacing between primary and control sources is given by \( x = (c + nx_s) \) for integer \( n \) and some constant \( c \), where \( x_s \) is the spacing between axial nodes. This effect is illustrated by Figure 3.14 which shows the control force magnitude as a function of separation between primary and control sources, with a constant error sensor location - control source separation of 0.5 metres (1.39\( \lambda_p \)). The maxima occur because of the difficulty in controlling the flexural vibration when the control location corresponds to a node in the standing wave field developed by reflections from the free end and the stiffener. The constant \( c \) is frequency dependent and represents the distance (in wavelengths) between the primary source and the first node in the standing wave in the direction of the control source. The axial separation between nodes \( x_s \) is discussed further in Section 4.3.
Three control sources and two primary sources were used.

Figure 3.13  Mean control source amplitude for optimal control as a function of frequency.
Three control sources and two primary sources were used.
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Figure 3.14 Mean control source amplitude for optimal control as a function of primary source - control source separation. Three control sources and two primary sources were used.
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Figure 3.15 shows that the location of the line of error sensors does not significantly affect the control source amplitude, provided the error sensor location is outside of the control source near field.

The phase of the control source relative to the primary source is zero at all frequencies and for all locations of control source and error sensor, for independent control and for the case with control sources driven by a common signal. This result is due to the formation of the standing waves between the plate end(s) and stiffener location. When a standing wave is formed, the vibration, and hence the required control force, is in phase with the excitation.

3.3.3 Effect of variations in forcing frequency, control source location and error sensor location on the attenuation of acceleration level

Figure 3.16 shows the variation in the mean attenuation of acceleration level downstream of the error sensors as a function of frequency. For the semi-infinite plate, the main minima in the curve occur at the same frequencies as the maxima in the control source amplitude plot (Figure 3.13); that is, where the control location is at a node in the standing wave generated by waves reflected from the stiffener. This effect is not seen clearly in the plots for the finite plate, where reflections from the downstream end of the plate cause rapid variation in the achievable attenuation. For both plates, little attenuation is achieved with all of the control sources driven by the same signal above 280 Hz, which is the cut-on frequency for the second higher order cross-plate mode. Independently driven control sources can cope with the higher order cross-plate modes.
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Figure 3.15 Mean control source amplitude for optimal control as a function of error sensor-control source separation. Three control sources and two primary sources were used.
Figure 3.16  Mean attenuation downstream of the line of error sensors as a function of frequency.
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The minima in the plot showing the variation in attenuation with separation distance between the primary and control sources on the semi-infinite plate (Figure 3.17(a)) correspond to maxima in the control source amplitude with control source location plot (Figure 3.14(a)), which occur as a result of the standing wave field set up between the stiffener and the upstream end of the plate. Similarly, every second minima in the plot for the finite plate (Figure 3.17(b)) corresponds to a maxima in Figure 3.14(b). The other minima in Figure 3.17(b) occur when the control location is at a node in the standing wave generated by reflections from the downstream plate end.

Figure 3.18 shows the mean attenuation downstream of the error sensor as a function of the separation between the control source and the line of error sensors. For the cases where the control sources are driven by a common signal, attenuation increases with increasing separation between the line of error sensors and the control location at the rate of about 18 dB per wavelength separation, up to a maximum of about 85 dB above four wavelengths separation. The achievable attenuation increases with greater separation between the control sources and the error sensors because the near field component of the control excitation diminishes with greater separation. The maximum is limited by the accuracy of the calculations. The maximum attenuation achievable experimentally would of course be lower. When the control sources are driven independently, the slope of the attenuation curve is greater.
Figure 3.17  Mean attenuation downstream of the line of error sensors as a function of control source - primary source separation.
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Figure 3.18  Mean attenuation downstream of the line of error sensors as a function of error sensor - control source separation.

(a) Semi - infinite plate.

(b) Finite plate.

Control sources driven by the same signal; Control sources driven independently.
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3.3.4 Number of control sources required for optimal control

Table 3.2 shows the amount of attenuation of acceleration level achieved downstream of the error sensors with various numbers of control sources. The control sources are located at a single axial location. The across-plate locations are given by \( y_{c1} = 0.08 \), \( y_{c2} = 0.25 \), \( y_{c3} = 0.33 \), \( y_{c4} = 0.40 \), and \( y_{c5} = 0.42 \). The other locations of primary sources, control sources and error sensors and the plate dimensions used were those given in Table 3.1. The results given are for the semi-infinite plate.

<table>
<thead>
<tr>
<th>Number of Control Sources</th>
<th>Mean Attenuation (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.54</td>
</tr>
<tr>
<td>2</td>
<td>28.2</td>
</tr>
<tr>
<td>3</td>
<td>44.4</td>
</tr>
<tr>
<td>4</td>
<td>44.4</td>
</tr>
<tr>
<td>5</td>
<td>44.4</td>
</tr>
</tbody>
</table>

3.3.5 Effect of a second angle stiffener and set of control sources on the attenuation of acceleration level

The results given in Section 3.3.3 indicate that there are some control source locations where significantly less attenuation can be achieved. Figure 3.19 shows the variation in mean control source amplitude using six control sources located in two sets of three, driven to optimally
control vibration at a single line of error sensors, as a function of control source location. The control source signals are calculated by the method described in Section 3.2.5.3. The second angle stiffener and set of control sources is located 0.15m downstream from the first, and the line of error sensors is located 0.5m downstream from the first control source. The mean control source amplitude using a single set of control sources is also shown in Figure 3.19 for comparison. Figure 3.20 shows the mean attenuation of acceleration level downstream of the line of error sensors as a function of control location, using one and two sets of control sources. At this frequency, there are no control source locations at which control using the two sets of control sources is difficult. Figure 3.20 shows that good control can be achieved at any frequency using two sets of control actuators.

Figure 3.19 Mean control source amplitude for optimal control using one and two sets of independently driven control sources as a function of frequency for the semi-infinite plate.
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Figure 3.20 Mean attenuation downstream of the line of error sensors using one and two sets of independently driven control sources as a function of control source -primary source separation for the semi-infinite plate.

Figure 3.21 Mean attenuation downstream of the line of error sensors using one and two sets of independently driven control sources as a function of frequency for the semi-infinite plate.
Chapter 3. Control of vibrations in a stiffened plate

3.3.6 Number of error sensors required for optimal control

Table 3.3 shows the control source amplitude and amount of attenuation of acceleration level achieved downstream of the error sensors with various numbers of error sensors. The error sensors were located at axial location \( x_e = 1.0 \text{m} \) and unevenly spaced across-plate locations. The other locations of primary sources and control sources and the plate dimensions used were those given in Table 3.1. The results given are for the semi-infinite plate.

<table>
<thead>
<tr>
<th>Number of Error Sensors</th>
<th>Mean Control Source Amplitude*</th>
<th>Mean Attenuation (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.83446</td>
<td>0.24127</td>
</tr>
<tr>
<td>2</td>
<td>1.5046</td>
<td>6.6589</td>
</tr>
<tr>
<td>3</td>
<td>0.74635</td>
<td>44.306</td>
</tr>
<tr>
<td>4</td>
<td>0.74636</td>
<td>44.329</td>
</tr>
<tr>
<td>5</td>
<td>0.74636</td>
<td>44.307</td>
</tr>
<tr>
<td>6</td>
<td>0.74636</td>
<td>44.372</td>
</tr>
<tr>
<td>7</td>
<td>0.74636</td>
<td>44.401</td>
</tr>
<tr>
<td>8</td>
<td>0.74636</td>
<td>44.467</td>
</tr>
<tr>
<td>9</td>
<td>0.74635</td>
<td>44.527</td>
</tr>
<tr>
<td>10</td>
<td>0.74635</td>
<td>44.576</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0.74635</td>
<td>44.626</td>
</tr>
</tbody>
</table>

*Mean control source amplitude is expressed relative to the primary source amplitude. Three control sources and two primary sources were used.
3.4 EXPERIMENTAL PROCEDURE

3.4.1 Modal analysis

A modal analysis was performed on the plate to be used in the vibration control experiment. The software package "PC Modal" was used to perform the analysis. The modal analysis experimental arrangement is given in Figure 3.22. A Brüel and Kjær type 8202 impact hammer and type 2032 signal analyser were used in the modal analysis. The dimensions of the plate were the same as those given in Section 3.3 (see Table 3.1). The plate model consisted of 96 nodes dividing the plate into 10cm squares. The analysis was performed for the two cases with and without the angle stiffener attached to the plate.

Figure 3.22 Experimental arrangement for the modal analysis of the plate.
3.4.2 Active vibration control

A steel stiffener was bolted tightly to a plate of the same dimensions described in the Section 3.3 (see Table 3.1). Three piezoceramic actuators were placed between the stiffener flange and the plate. The actuators were attached only at one end to ensure that no external tensile force were applied to them, as the type of actuator used is weak in tension. The primary source, control source and error sensor locations and the excitation frequency are also given in Table 3.1. The plate was mounted horizontally and excited in the vertical plane.

The complete experimental arrangement is given in Figure 3.23. The experimental equipment can be divided into three functional groups; the primary excitation system, the control system and the acceleration measurement system.

The primary signal was produced by a signal analyser and amplified to drive the electrodynamic shakers (Figure 3.24). The shakers acted on the plate through force transducers, and the magnitudes of the primary forces were recorded using an oscilloscope.

The error signals from the line of six accelerometers (Figure 3.25) were passed to a transputer controller. The controller determined the control signals to drive the piezoceramic actuators, optimally minimising the acceleration measured by the error sensors. The control signals were also recorded on an oscilloscope.
Figure 3.23  Experimental arrangement for the active control of vibration in the plate.

The acceleration was measured at 10 or 15 cm intervals along the plate in four lines equally spaced across the plate (Figure 3.26). The accelerometer signals were read in turn through a 40 channel multiplexer connected to a Hewlett-Packard type 35665A signal analyser. The frequency
response function was used to analyse the data. The magnitude and phase of the acceleration were recorded on a personal computer, which was also used to switch the recorded channel on the multiplexer. The acceleration output of the force transducer at one of the primary locations was used as the reference signal for the frequency response analysis. Accelerometer readings were taken initially once the error sensor signals had been optimally reduced, and again with the control amplifiers switched off (the uncontrolled case). The experiment was repeated with the three control actuators driven by a common control signal.
Figure 3.25  Control system.
Figure 3.26 Acceleration measurement. Not all of the accelerometer - multiplexer connections are shown.
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Figures 3.27 - 3.29 show the photographs of the experimental equipment. In Figure 3.27, the plate can be seen in the foreground with the signal generating and recording equipment in the background. The plate is simply supported along the two long edges and the far end is mounted in a diverging sandbox termination, approximating a semi-infinite end. The near end is free. Two electromagnetic shaker primary sources can be seen, as well as the angle stiffener mounted across the plate and accelerometers mounted at various positions on the plate. The plate is shown closer-up in Figure 3.28, with the primary sources removed. The piezoceramic stack actuators are shown in Figure 3.29, mounted between the stiffener flange and the plate surface.

**Figure 3.27** Experimental equipment for the active vibration control of plate vibration.
Chapter 3. Control of vibrations in a stiffened plate

Figure 3.28  Accelerometers and angle stiffener mounted on the plate.

Figure 3.29  Piezoceramic stack actuators mounted between the plate and the flange of the angle stiffener.
3.5 EXPERIMENTAL RESULTS

3.5.1 Modal analysis

Figures 3.30 - 3.33 show the differences between the vibration response of the plate with and without the angle stiffener attached for the $n,1$ modes ($n = 1,4$). The presence of the angle stiffener makes a significant difference to the mode shapes.

**Figure 3.30**  The 1,1 mode for the unstiffened and stiffened plate.

**Figure 3.31**  The 2,1 mode for the unstiffened and stiffened plate.
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Figure 3.32 The 3,1 mode for the unstiffened and stiffened plate.

Figure 3.33 The 4,1 mode for the unstiffened and stiffened plate.

Figure 3.34 The 2,3 and 3,2 modes for the unstiffened plate.
Chapter 3. Control of vibrations in a stiffened plate

3.5.2 Active vibration control

Figure 3.36 shows the theoretical and experimental acceleration distributions for each of the four lines where accelerometers were placed in the experiment for the semi-infinite plate. For both the uncontrolled case and the controlled case with control actuators driven by the same signal, the experimental results and theoretical curves are in close agreement. For the controlled case with three independently driven control sources, the theoretical analysis predicted greater reduction in acceleration level than was achieved experimentally. An error analysis showed that a very small error (a tenth of a percent) in the control signal would produce a decrease in attenuation corresponding to the difference between the experimental and theoretical data.
Chapter 3. Control of vibrations in a stiffened plate

Figure 3.36 Acceleration distributions for the semi-infinite plate.

(1) Uncontrolled case; (2) Controlled case. — Theoretical results and □ experimental data for the case with three control sources driven by the same signal; — Theoretical results and ■ experimental data for the case with independently driven control sources.

(a) y = 0.1m; (b) y = 0.2m; (c) y = 0.3m; (d) y = 0.4m.

x_p = primary source location, x_c = control source location, x_s = sensor location.
3.6 SUMMARY

A theoretical model has been developed to describe the vibration response of a stiffened plate to a range of excitation types, and in particular to describe the vibration response of stiffened plates to point force primary excitation sources and angle stiffener and piezoceramic stack control sources. The numerical results indicate that flexural vibrations in plates can be actively controlled using piezoceramic stack actuators placed between the flange of an angle stiffener and the plate surface. Numerical results also indicate:

(1) The mean amplitude of the control forces required for optimal control generally decreases with increasing frequency.

(2) The optimum control forces are either in phase or 180° out of phase with the primary sources. This is true for the semi-infinite plate as well as the finite plate, because a standing wave is generated by the vibration reflections from the finite end and the angle stiffener.

(3) Maxima occur in the mean control source amplitude required for optimal control when the separation between control and primary forces is given by \( x = (c + nx_s) \) where \( n \) is an integer, \( c \) is a constant dependent on frequency, and \( x_s \) is the axial separation between standing wave nodes. These maxima occur when the control sources are located at a nodal line in the standing wave generated by reflection from the plate termination and the angle stiffener. Minima in the mean attenuation of acceleration level downstream of the
error sensor occur when the control sources are located at a nodal line in the standing wave.

(4) Increasing the separation between the primary and control sources does not improve attenuation.

(5) The amount of attenuation achieved downstream of the error sensors increases with increasing separation between the error sensors and the control sources.

(6) When the line of error sensors is located at a nodal line in the standing wave that exists in finite plates, the attenuation achieved is less than that achieved with the error sensors located away from a node. Locating the error sensors at a node does not affect the amplitude of the control sources required for optimal control.

(7) A second set of control sources can be used to overcome the difficulty in controlling vibration when the first set of control sources is located at a nodal line in a standing wave. The maxima in mean control source amplitude and the minima in attenuation that occur when the first set of control sources are located at a standing wave nodal line are eliminated in this way.

(8) At low frequencies, there is very little difference in the mean control effort required for optimal control and the mean attenuation downstream of the line of error sensors
achieved between control using independent control sources and control sources driven by a common signal. When higher order across-plate modes become significant, very little attenuation is achieved with control sources driven by a common signal. Good reduction in acceleration level is achieved with independently driven control sources right across the frequency range considered.

(9) Numerical results indicated that three control actuators and three error sensors were sufficient for optimally controlling vibration in the plate considered at the frequency considered.

The theoretical model outlined was verified experimentally for the plate with simply supported sides, one end free and the other end anechoically terminated. A modal analysis of the plate indicated that the anechoic termination allowed some reflection and so did not exactly model the ideal infinite end, and that the angle stiffener made a significant difference to the vibration response of the plate. Comparison between experimental results and theoretical predictions for the vibration of the plate with and without active vibration control showed that:

(1) The theoretical model accurately predicted the vibration response of the plate for the uncontrolled case and the case with control sources driven by the same signal. The angle stiffener reflected more of the vibration and transmitted less than the theoretical model predicted.
Chapter 3. Control of vibrations in a stiffened plate

(2) The theoretical model predicted more attenuation that could be achieved experimentally for the case with independently driven control sources. An error analysis indicated that an error in the control source signal of 0.1% would produce a decrease in attenuation corresponding to the difference between the theoretical prediction and the experimental result. Nevertheless, around 25 dB attenuation was achieved experimentally for the case with independently driven control sources.
Chapter 4. Control of vibrations in a stiffened cylinder

CHAPTER 4. FEEDFORWARD ACTIVE CONTROL OF FLEXURAL VIBRATION IN A CYLINDER USING PIEZOCERAMIC ACTUATORS AND AN ANGLE STIFFENER

4.1 INTRODUCTION

In this chapter, the active control of flexural vibration in cylindrical shells using as control sources piezoceramic actuators placed between the flange of a ring stiffener and the shell surface is investigated. The classical equations of motion for the vibration of a shell developed by Flügge (1960) are used to develop a theoretical model for the shell with primary point sources and a ring stiffener and control actuators (Section 4.2).

Figure 4.1 Cylinder showing primary sources, ring stiffener, piezoceramic stack control actuators and error sensors.
Chapter 4. Control of vibrations in a stiffened cylinder

The effective control signal is a combination of the effects of the point forces at the base of the actuators, and the reaction line force and line moment in a ring at the base of the stiffener (Section 4.2.6). The displacement at a point is the sum of the displacements due to each of the primary source and control source forces and moments. Optimal control is achieved by minimising the total mean square displacement at the location of the ring of error sensors downstream of the control sources.

The theoretical analysis considers two different sets of cylinder supports. In both cases, the left hand end is modelled as free. In the first case, the right hand end is modelled as infinite and in the second the right hand end is modelled as free. The influence of the control source locations, the location of the ring of error sensors and the excitation frequency on the control source amplitude and achievable attenuation are investigated, and the physical reasons for each observation are explained (Section 4.3).

A modal analysis of the cylinder is performed to show that the ring stiffener significantly affects the vibration response of the cylinder. Experimental verification of the theoretical model is performed for the simply supported cylinder with and without active vibration control. The experimental methods are described in Section 4.4. Experimental results are compared with theoretical predictions for the vibration of the cylinder with and without active vibration control (Section 4.5).
Chapter 4. Control of vibrations in a stiffened cylinder

4.2 THEORY

4.2.1 The differential equations of motion for a cylindrical shell and the general solution

The differential equations governing the vibration of a cylindrical shell are different from the equations of motion for beams and plates, for two main reasons. First, unlike the cases of the equations of motion for beams and plates, there is no universally accepted version of the equations of motion for the vibration of a cylindrical shell, and second, rather than there being one equation for the transverse vibration of a beam or plate, there are three simultaneous equations to be considered for the coupled vibrations in the radial, axial and tangential directions.

Because of the complex nature of the derivation of the equations of motion from stress-strain relationships, different researchers have derived slightly different equations of motion for shells. Leissa (1973a) lists and describes the derivation of the main theories. The simplest form was given by Donnell-Mushtari, and other versions include a variety of complicating terms. Perhaps the most popular version was that developed by Flügge (1960), but including inertia terms (see Section 1.2.1.4).

The response of the cylindrical shell shown in Figure 4.2 to simple harmonic excitations \( q_x e^{j\omega t} \), \( q_\theta e^{j\omega t} \) and \( q_r e^{j\omega t} \) in the axial \((x)\), tangential \((\theta)\) and radial \((r)\) directions respectively is considered. The end of the cylinder at \( x = 0 \) is modelled as simply supported.
Chapter 4. Control of vibrations in a stiffened cylinder

Following the sign conventions given in Figure 4.3, the Flügge equations of motion for the response of the cylindrical shell shown in Figure 4.2 are

\[
R^2 \frac{\partial^2 u}{\partial x^2} + \frac{(1 - v) \partial^2 u}{\partial \theta^2} - \frac{pR^2(1 - v^2) \partial^2 u}{\partial t^2} + \frac{R(1 + v) \partial^2 v}{\partial x \partial \theta} + \frac{Rv \partial w}{\partial x} + \left( \frac{(1 - v) \partial^2 u}{2 \partial \theta^2} - \frac{R^3 \partial^3 w}{2 \partial x^3} + \frac{R(1 - v) \partial^3 w}{2 \partial x \partial \theta^2} \right) = - \frac{(1 - v^2)}{Eh} q(x, \theta) e^{j\omega t} \tag{4.1}
\]

\[
\frac{R(1 + v) \partial^2 u}{2 \partial x \partial \theta} + \frac{R^2(1 - v) \partial^2 v}{2 \partial x^2} + \frac{\partial^2 v}{\partial \theta^2} - \frac{pR^2(1 - v^2) \partial^2 v}{E \partial t^2} + \frac{\partial w}{\partial x} + \frac{3R^2(1 - v) \partial^3 v}{2 \partial x^2} - \frac{3R^2(3 - v) \partial^3 w}{2 \partial x \partial \theta} = - \frac{(1 - v^2)}{Eh} q(x, \theta) e^{j\omega t} \tag{4.2}
\]

and

\[
R^3 \frac{\partial^3 u}{\partial x^3} - \frac{R^2(3 - v) \partial^3 v}{2 \partial x^2 \partial \theta} + w + \frac{\partial^2 w}{\partial \theta^2} + \left( \frac{(1 - v) \partial^3 u}{2 \partial x \partial \theta^2} + \frac{\partial R^2(1 - v^2) \partial^2 w}{E \partial t^2} \right) = \frac{(1 - v^2)}{Eh} q(x, \theta) e^{j\omega t}, \tag{4.3}
\]
where $R$ is the shell radius, $h$ is the shell thickness, $E$ is Young’s modulus of elasticity, $\nu$ is Poisson’s ratio, $\rho$ is the shell density, $\xi = h^2/12R^2$, $\nabla^4 = \nabla^2 \nabla^2$ is the square of the modified Laplacian operator $\nabla^2 = R^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}$, and $u(x, \theta, t), v(x, \theta, t)$ and $w(x, \theta, t)$ are the displacements in the axial, tangential and radial directions respectively.

As the cylinder is closed, the following harmonic series solutions in $\theta$ can be assumed for the cylinder vibrational displacements in the $x$, $y$ and $z$ directions:

$$u(x, \theta, t) = \sum_{n=1}^{\infty} u_n(x) \cos(n\theta) \ e^{j\omega t}, \quad (4.4)$$

$$v(x, \theta, t) = \sum_{n=1}^{\infty} v_n(x) \sin(n\theta) \ e^{j\omega t} \quad (4.5)$$

and

$$w(x, \theta, t) = \sum_{n=1}^{\infty} w_n(x) \cos(n\theta) \ e^{j\omega t}, \quad (4.6)$$
where $n$ is a mode number and $\omega$ is the angular frequency. Each of the eigenfunctions $u_n(x)$, $v_n(x)$ and $w_n(x)$ can be expressed in terms of modal wavenumbers $k_s$ as follows (Forsberg, 1964):

$$u_n(x) = \sum_{s=1}^{8} \beta_{sn} A_{sn} e^{k_{sn}x}, \quad (4.7)$$

$$v_n(x) = \sum_{s=1}^{8} \gamma_{sn} A_{sn} e^{k_{sn}x}, \quad (4.8)$$

and

$$w_n(x) = \sum_{s=1}^{8} A_{sn} e^{k_{sn}x}, \quad (4.9)$$

where $A_{sn}$, $\beta_{sn}$ and $\gamma_{sn}$ are arbitrary constants.

### 4.2.2 Determining the wavenumbers $k$ and constants $\beta$ and $\gamma$

Substitution of Equations (4.4) - (4.9) into the homogeneous forms of Equations (4.1) - (4.3) yields

$$\sum_{n=0}^{m} \sum_{s=1}^{8} \left[ \beta_{sn} \left( R^2 k_{sn}^2 - \frac{(1 - \nu)}{2} n^2 + \frac{(1 - \nu^2)}{2} R^2 \omega^2 - \xi \frac{(1 - \nu)}{2} n^2 \right) + \frac{(1 + \nu)}{2} R k_{sn} \right] A_{sn} e^{k_{sn}x} \cos(n\theta) e^{iot} = 0, \quad (4.10)$$
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\[ \sum_{n=0}^{\infty} \sum_{s=1}^{8} \left[ \beta_{s n} \left( \frac{(1 - \nu) R n k_{s n}}{2} \right) + \gamma_{s n} \left( \frac{(1 - \nu) R^2 k^2_{s n}}{2} - n^2 + \frac{\rho (1 - \nu^2) R^2 \omega^2}{E} \right) \right] A_{s n} e^{k_{s n} \xi} \cos(n\theta) e^{j\omega t} = 0 \]  

(4.11)

and

\[ \sum_{n=0}^{\infty} \sum_{s=1}^{8} \left[ \beta_{s n} \left( \frac{(1 - \nu) R^2 k^2_{s n}}{2} - \frac{(1 - \nu) R n^2 k_{s n}}{2} \right) + \gamma_{s n} \left( n^2 \frac{(3 - \nu) R^2 k^2_{s n}}{2} \right) \right] A_{s n} e^{k_{s n} \xi} \cos(n\theta) e^{j\omega t} = 0 . \]  

(4.12)

For a non-trivial solution valid over the surface of the cylinder, \( A_{s n} e^{k_{s n} \xi} \cos(n\theta) e^{j\omega t} \) is not zero, and Equations (4.10) - (4.12) can be re-written equivalently in the matrix form

\[ CA = 0 , \]  

(4.13)

where \( A = [\beta_{s n}, \gamma_{s n}, 1]^T (s = 1, 8) \) and \( C \) contains the remainder of the coefficients. For homogeneous boundary conditions, the determinant of \( C \) must be non-zero for each \( n \), leading to an eighth-order algebraic equation for \( k_{s n} \);

\[ g_{s8} k_{s n}^8 + g_{s6} k_{s n}^6 + g_{s4} k_{s n}^4 + g_{s2} k_{s n}^2 + g_{s0} = 0 , \]  

(4.14)

where

\[ g_{s8} = \xi KR^6 - \xi KR^6 , \]  

(4.15)

\[ g_{s6} = \xi HR^6 + GKR^4 - 2KR^4 n^2 + \xi F^2 R^2 + C^2 n^2 R^4 - 2\xi FCR^3 n - 2\xi DR^4 Kn^2 - \xi vHR^6 + 2R^4 K \] ,  

(4.16)
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\[ g_{sd} = JKR^2 + \nu^2 KR^2 + \xi(2n^2R^2 + GK) + GHR^4 - 2FnR^2 + \xi R^2G - 2C^2n^4R^2 - 2\xi FC DRn^3 + 2FCnR + 2CR^3n^2 - \xi D^2R^2n^4K + 2\nu HR^4 + 2DR^3n^2K - 2\xi HDR^4n^2) \]  
(4.17)

\[ g_{s2} = JHR^2 + JGK + n^2R^2 + JC^2n^2 - 2C\nu Rn^2 - \nu^2 HR^4 + 2\nu DHR^2n^2 + \xi(-2n^2GHR^2 - 2FnG - \xi HD^2R^2n^4 + 2CDRn^4) \]  
(4.18)

and

\[ g_{s0} = JGH + Gn^2, \]  
(4.19)

where \( B = \rho \left( \frac{1 - \nu}{E} \right) R^2 \omega^2 \), \( C = \frac{(1 + \nu) R}{2} \), \( D = \frac{(1 - \nu)}{2} \), \( F = \frac{(3 - \nu)}{2} R^2 n \), \( G = B - Dn^2 - \xi Dn^2 \), \( H = -n^2 + B \), \( J = 1 - B + \xi n^4 + \xi - 2\xi n^2 \) and \( K = DR^2 + \xi F/n \). As found by Forsberg (1964), all solutions of Equation (4.14) are of the form

\[ k = \pm a, \pm jb, \pm (c \pm jd) \].  
(4.20)

where \( a, b, c \) and \( d \) are real quantities. This is different to the form of the solutions given by Flügge, because the inertia terms have been included here.

The constants \( \beta_{sn} \) and \( \gamma_{sn} \) can now be found from any two of Equations (4.10) - (4.12). Rearranging Equations (4.11) and (4.12) gives, for \( n \geq 0 \),

\[ \beta_{sn} = \frac{(\xi R^4 k_m^4 - 2\xi R^2 k_m^2 n^2 - J)(k_m^2 + H) + (n - \xi F k_m^2)^2}{(\xi R^3 k_m^4 + \xi DRk_m n^2 - \nu Rk_m)(k_m^2 + H) - (n - \xi F k_m^2)(Ck_m n)} \]  
(4.21)
and

\[ \gamma_{sn} = \frac{(n - \xi Fk_{sn}^2) + (Ck_{sn} n)\beta_{sn}}{(Kk_{sn}^2 + H)}. \]  

(4.22)

The constants \( \beta_{sn} \) and \( \gamma_{sn} \) depend only on \( \omega, \rho, \nu, E, h \) and \( R \). Note also that \( \gamma_{s0} = 0 \), as the \( n = 0 \) mode is a purely transverse expansion-contraction mode (see Figure 4.4).

![Figure 4.4 Circumferential modes of vibration.](image)

On each side of an applied force or moment at \( x = x_0 \), each eigenfunction is a different linear combination of the terms \( e^{k_{ix}} \) (\( i = 1,4 \)). For \( x < x_0 \),

\[ u_{in}(x) = \sum_{i=1}^{8} \beta_{sn} A_{1sn} e^{k_{ix}}, \]  

(4.23)
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\[ v_{1n}(x) = \sum_{s=1}^{8} \gamma_{sn} A_{1sn} e^{k_{sn}x}, \]  
\[ w_{1n}(x) = \sum_{s=1}^{8} A_{1sn} e^{k_{sn}x}. \]

For \( x > x_0 \),

\[ u_{2n}(x) = \sum_{s=1}^{8} \beta_{sn} A_{2sn} e^{k_{sn}x}, \]
\[ v_{2n}(x) = \sum_{s=1}^{8} \gamma_{sn} A_{2sn} e^{k_{sn}x}, \]
\[ w_{2n}(x) = \sum_{s=1}^{8} A_{2sn} e^{k_{sn}x}. \]

To solve for the sixteen unknowns \( A_{1sn} \) and \( A_{2sn} \), for \( s = 1, 8 \), sixteen equations are required, comprising eight boundary conditions (four conditions at each end of the cylinder) and eight equilibrium conditions at the point of application \( x_0 \) of the force or moment, for each circumferential mode \( n \) of the cylinder vibration.

4.2.3 Boundary conditions at the cylinder ends

For the purposes of this work, two sets of boundary conditions will be examined; those
corresponding to a shell with simply supported ends, and those corresponding to a semi-infinite shell with the end at \( x = 0 \) modelled as simply supported.

### 4.2.3.1 Simply supported end conditions

The four boundary conditions corresponding to a simple support are \( u = 0, \, v = 0, \, w = 0 \) and \( M_x = 0 \) (Leissa 1973a), where \( M_x \) is the moment resultant in the \( x \)-plane and is given by

\[
M_x = \frac{Eh^3}{12(1 - \nu^2)} \left[ -\frac{\partial^2 w}{\partial x^2} + \frac{v}{R^2} \frac{\partial v}{\partial \theta} - \frac{v}{R^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{R} \frac{\partial u}{\partial x} \right]. \tag{4.29}
\]

In terms of the displacement unknowns, these boundary conditions for a simply supported end at \( x = 0 \) are

\[
\sum_{s = 1}^{8} \beta_{sn} A_{1sn} = 0, \tag{4.30}
\]

\[
\sum_{s = 1}^{8} \gamma_{sn} A_{1sn} = 0, \tag{4.31}
\]

\[
\sum_{s = 1}^{8} A_{1sn} = 0 \tag{4.32}
\]

and

\[
\sum_{s = 1}^{8} \left[ \frac{v}{R^2} (\gamma_{sn} + n) + \frac{\beta_{sn} k_{sn} - k_{sn}^2}{R} A_{1sn} \right] = 0 \tag{4.33}
\]
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The corresponding boundary conditions for a simply supported end at \( x = L_x \) are

\[
\sum_{s = 1}^{8} \beta_{sn} A_{2sn} e^{k_{sn}L_x} = 0 ,
\]

(4.34)

\[
\sum_{s = 1}^{8} \gamma_{sn} A_{2sn} e^{k_{sn}L_x} = 0 ,
\]

(4.35)

\[
\sum_{s = 1}^{8} A_{2sn} e^{k_{sn}L_x} = 0
\]

(4.36)

and

\[
\sum_{s = 1}^{8} \left[ \frac{\nu n}{R^2} (\gamma_{sn} + n) + \frac{\beta_{sn} k_{sn} - k_{sn}^2}{R^2} \right] A_{2sn} e^{k_{sn}L_x} = 0 .
\]

(4.37)

4.2.3.2 Infinite end conditions

An infinite end produces no reflections, so the boundary conditions corresponding to an infinite end at \( x = L_x \) are

\[
A_{21n} = 0 ,
\]

(4.38)

\[
A_{23n} = 0 ,
\]

(4.39)

\[
A_{25n} = 0
\]

(4.40)

and

\[
A_{27n} = 0 .
\]

(4.41)
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4.2.4 Equilibrium conditions at the point of application of a force or moment

Requiring that the displacement and gradient in each direction be continuous at any point in the cylinder wall, the first six equilibrium conditions at \( x = x_0 \) which must be satisfied are

\[
\begin{align*}
\frac{\partial u_{1n}}{\partial x} &= \frac{\partial u_{2n}}{\partial x}, \\
\frac{\partial v_{1n}}{\partial x} &= \frac{\partial v_{2n}}{\partial x}, \\
\frac{\partial w_{1n}}{\partial x} &= \frac{\partial w_{2n}}{\partial x}.
\end{align*}
\]

The form of the excitation \( q_r(x, \theta) \) will affect the higher order equilibrium conditions at \( x = x_0 \).

In the following sections the response of the shell to a point force, a circumferential line force and a circumferential line moment is discussed.

4.2.4.1 Response of a shell to a radially acting point force

The response of the shell to a simple harmonic point force \( F_0 \) acting normal to the shell at position \((x_0, \theta_0)\) is considered. The excitation \( q_r(x, \theta) \) in Equation (4.3) is replaced by

\[
q_r(x, \theta) = RF_0 \delta(x - x_0) \delta(\theta - \theta_0),
\]

where \( \delta \) is the Dirac delta function. Replacing \( u, v \) and \( w \) by...
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Equations (4.4) - (4.6), dividing by $e^{j\omega t}$ and multiplying by $\cos(n\theta)$, Equation (4.3) becomes

$$\frac{\xi R^2(3 - v)}{2} n v_n''(x) + w_n(x) + \xi R^4 w_n^{(iv)}(x) - 2\xi R^2 n^2 w_n''(x) + \xi n^4 \omega^2 w_n(x)$$

$$- \frac{\rho R^2 (1 - v^2)}{E} \omega^2 w_n(x) + \xi \omega^2 w_n(x) - 2\xi n^2 \omega^2 w_n(x) \cos^2(n\theta)$$

$$- \frac{R(1 - v^2)}{Eh} \delta(x - x_0) \delta(\theta - \theta_0) \cos(n\theta) \right].$$

The integral with respect to $\theta$ around the circumference of the cylinder is taken, noting that

$$\int_{0}^{2\pi} \delta(\theta - \theta_0) \cos(n\theta) \, d\theta = \cos(n\theta) \right].$$

To find

$$\pi \left[ R v u_n''(x) - \xi R^3 \frac{\omega^2}{2} n u_n''(x) + n v_n(x)$$

$$\frac{\xi R^2(3 - v)}{2} n v_n''(x) + w_n(x) + \xi R^4 w_n^{(iv)}(x) - 2\xi R^2 n^2 w_n''(x) + \xi n^4 \omega^2 w_n(x)$$

$$- \frac{\rho R^2 (1 - v^2)}{E} \omega^2 w_n(x) + \xi \omega^2 w_n(x) - 2\xi n^2 \omega^2 w_n(x) \right]$$

$$- \frac{R(1 - v^2)}{Eh} \delta(x - x_0) \cos(n\theta) \right].$$

Next, the integral with respect to $x$ is taken between the limits $x_0 - \delta$ and $\xi + \delta$, using the conditions
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\[ \int_{x_0-\delta}^{x_0+\delta} w_n(x) \, dx = 0 , \quad (4.51) \]

\[ \int_{x_0-\delta}^{x_0+\delta} w'_n(x) \, dx = 0 \quad (4.52) \]

and

\[ \int_{x_0-\delta}^{x_0+\delta} w''_n(x) \, dx = 0 \quad (4.53) \]

(similarly for \( u_n(x) \) and \( v_n(x) \)) as \( \delta \to 0 \), to find

\[ \pi \left[ \xi R^4 w'''_n(x) - \xi R^3 u'''_n(x) \right]_{x_0-\delta}^{x_0+\delta} = \frac{R(1 - \nu^2)}{Eh} F_0 \cos(n\theta_0) , \quad (4.54) \]

or

\[ \left[ \frac{\partial^2 u_{1n}}{\partial x^2} - \frac{\partial^2 u_{2n}}{\partial x^2} \right] - R \left[ \frac{\partial^3 w_{1n}}{\partial x^3} - \frac{\partial^3 w_{2n}}{\partial x^3} \right] = \frac{(1 - \nu^2)}{\xi \pi R^2 Eh} F_0 \cos(n\theta_0) \quad (4.55) \]

Finally, the integral with respect to \( x \) is taken again between the limits \( x_0 - \delta \) and \( x_0 + \delta \) to find

the second order equilibrium condition

\[ \frac{\partial^2 w_{1n}}{\partial x^2} = \frac{\partial^2 w_{2n}}{\partial x^2} . \quad (4.56) \]
4.2.4.2 Response of a shell to a circumferentially distributed line force

Instead of a point force, the excitation represented by \(q_r(x, \theta)\) in Equation (4.3) is replaced by an array of \(N\) equally spaced point forces distributed along an arc parallel to the \(\theta\)-axis between \(\theta_1\) and \(\theta_2\). These forces act at locations \((\chi_k, \theta_k, k = 1, \ldots, N)\) and each has a magnitude of \(F/N\), so \(q_r(x, \theta)\) in Equation (4.3) is replaced by \(q_r(x, \theta) = \left(\frac{RF_0}{N}\right) \sum_{k=1}^{N} \delta(x - x_0)\delta(\theta - \theta_k)\). Following the method of Section (4.2.4.1), and using the relation

\[
\lim_{N \to \infty} \left[ \sum_{k=1}^{N} \left(\frac{\theta_2 - \theta_1}{N}\right) \cos(n\theta_k) \right] = \int_{\theta_1}^{\theta_2} \cos(n\theta) \, d\theta,
\]

(4.57)

the second and third order equilibrium conditions at \(x = x_0\) are

\[
\frac{\partial^2 w_{in}}{\partial x^2} = \frac{\partial^2 w_{2n}}{\partial x^2}
\]

(4.58)

and

\[
\left[ \frac{\partial^2 u_{in}}{\partial x^2} - \frac{\partial^2 u_{2n}}{\partial x^2} \right] - R \left[ \frac{\partial^3 w_{in}}{\partial x^3} - \frac{\partial^3 w_{2n}}{\partial x^3} \right] = \frac{(1 - \nu^2)}{\xi \pi n (\theta_2 - \theta_1) R^2 Eh} F_0 \left[ \sin(n\theta_2) - \sin(n\theta_1) \right].
\]

(4.59)

4.2.4.3 Response of a shell to a circumferentially distributed line moment

The excitation represented by \(q_r(x, \theta)\) in Equation (4.3) is replaced by a distributed line moment \(M_0\) per unit length acting along an arc parallel to the \(\theta\)-axis between \(\theta_1\) and \(\theta_2\). The excitation \(q_r(x, \theta)\) is replaced by \(q_r(x, \theta) = \frac{\partial M}{\partial x} = R M_0 \left[ \delta'(x - x_0) \right] [h(\theta - \theta_1) - h(\theta - \theta_2)],\) where \(h\) is the
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unit step function. Following the method of Section (4.2.4.1), and using the relation

\[
\int_0^{2\pi} h(\theta - \theta_1) \cos(n\theta) \, d\theta = \frac{\sin(n\theta_1)}{n}, \tag{4.60}
\]

Equation (4.50) becomes

\[
\pi \left[ R v \frac{d^3 u}{dx^3}(x) - \xi R^3 \frac{d^3 u}{dx^3}(x) - \xi \frac{R(1 - v)}{2} n^2 \frac{d^2 u}{dx^2}(x) + n v_n(x) \right] \\
\xi \frac{R^2(3 - v)}{2} n v_n''(x) + w_n(x) + \xi R^4 \frac{d^4 w}{dx^4}(x) - 2\xi R^2 n^2 w_n''(x) + \xi n^4 w_n' \bigg|_{x_0 - \delta}^{x_0 + \delta} \\
= \frac{\rho R^2(1 - v^2)}{E} \omega^2 w_n(x) + \xi w_n(x) - 2\xi n^2 w_n(x) \\
- \frac{R(1 - v^2)}{nEh} \delta'(x - x_0) \left[ \sin(n\theta_2) - \sin(n\theta_1) \right]. \tag{4.61}
\]

Next, the integral with respect to \(x\) is taken between the limits \(x_0 - \delta\) and \(x_0 + \delta\) to find

\[
\pi \left[ \xi R^4 \frac{d^4 w}{dx^4}(x) - \xi R^3 \frac{d^3 u}{dx^3}(x) \right]_{x_0 - \delta}^{x_0 + \delta} = \frac{R(1 - v^2)}{nEh} M_0 \delta(x - x_0) \left[ \sin(n\theta_2) - \sin(n\theta_1) \right]. \tag{4.62}
\]

or

\[
\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2} \right] - R \left[ \frac{\partial^3 w_1}{\partial x^3} - \frac{\partial^3 w_2}{\partial x^3} \right] \\
= \frac{(1 - v^2)}{\xi \pi n R^2 Eh} M_0 \delta(x - x_0) \left[ \sin(n\theta_2) - \sin(n\theta_1) \right]. \tag{4.63}
\]

The integral with respect to \(x\) is taken again between the limits \(x_0 - \delta\) and \(x_0 + \delta\) to find the second order equilibrium condition

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\[
\frac{\partial^2 w_{1n}}{\partial x^2} - \frac{\partial^2 w_{2n}}{\partial x^2} = - \frac{(1 - v^2)}{\xi \pi n R^3 E h} M_0 \left[ \sin(n\theta_2) - \sin(n\theta_1) \right].
\] (4.64)

Differentiation gives

\[
\frac{\partial^3 w_{1n}}{\partial x^3} = \frac{\partial^3 w_{2n}}{\partial x^3}.
\] (4.65)

Taking four boundary conditions at each end of the shell from Equations (4.30) - (4.41), the six equilibrium condition Equations (4.42) - (4.47), and two further equilibrium conditions from Equations (4.55), (4.56), (4.58), (4.59), (4.64) and (4.65), sixteen equations in the sixteen unknowns \(A_{1n}\) and \(A_{2n}\) for \(s = 1, 8\) are obtained. These can be written in the form \(AX = B\).

The solution vectors \(X = [A_{11n} A_{12n} A_{13n} \ldots A_{18n} A_{21n} A_{22n} \ldots A_{28n}]^T = A^{-1}B\) can be used to characterise the response of a cylindrical shell to simple harmonic excitation by a single point force, a circumferentially distributed line force or a circumferentially distributed line moment.

4.2.5 Modelling the effects of the angle stiffener

The mass and stiffness of the angle stiffener may be significant. Given a cylindrical shell with some excitation \(q_r\) at axial position \(x = x_0\) and an angle stiffener extending around the circumference of the cylinder at axial position \(x = x_1\), as shown in Figure 4.5, three eigenfunction solutions of Equations (4.1) - (4.3) are now required.
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For $x < x_0$

$$w_{1n}(x) = \sum_{s=1}^{8} A_{1sn} e^{k_{sn}x}$$  \hspace{1cm} (4.66)

for $x_0 < x < x_1$,

$$w_{2n}(x) = \sum_{s=1}^{8} A_{2sn} e^{k_{sn}x}$$  \hspace{1cm} (4.67)

and for $x > x_1$,

$$w_{3n}(x) = \sum_{s=1}^{8} A_{3sn} e^{k_{sn}x}$$  \hspace{1cm} (4.68)

and similarly for $u_{in}$ and $v_{in}$ ($i = 1,3$). These eigenfunctions allow for reflection at the stiffener.
location. Twenty four equations in the twenty four unknowns \( A_{isn}; \ i = 1,3; \ s = 1,8 \) are now required. In addition to the eight equilibrium conditions at \( x = x_0 \) which depend on the form of the excitation \( q_s \), and the boundary conditions at each end of the shell, the equilibrium conditions which must be satisfied at the stiffener location \( x = x_1 \) are

\[
u_{2n} = v_{3n}, \tag{4.69}
\]

\[
\frac{\partial u_{2n}}{\partial x} = \frac{\partial u_{3n}}{\partial x}, \tag{4.70}
\]

\[
v_{2n} = v_{3n}, \tag{4.71}
\]

\[
\frac{\partial v_{2n}}{\partial x} = \frac{\partial v_{3n}}{\partial x}, \tag{4.72}
\]

\[
w_{2n} = w_{3n}, \tag{4.73}
\]

\[
\frac{\partial w_{2n}}{\partial x} = \frac{\partial w_{3n}}{\partial x}, \tag{4.74}
\]

\[
\frac{\partial^2 w_{2n}}{\partial x^2} = \frac{\partial^2 w_{3n}}{\partial x^2}, \tag{4.75}
\]

and

\[
\left[ \frac{\partial^2 u_{1n}}{\partial x^2} - \frac{\partial^2 u_{2n}}{\partial x^2} \right] - R \left[ \frac{\partial^3 w_{1n}}{\partial x^3} - \frac{\partial^3 w_{2n}}{\partial x^3} \right] = - \frac{(1 - v^2)}{2 \xi \pi R^2 E h} \left( K_a + m_a \omega^2 \right) w_n(x_0). \tag{4.76}
\]
where \( K_a \) is the stiffness and \( m_a \) the mass per unit length of the stiffener. If the angle stiffener is very rigid compared to the cylinder, Equations (4.73) and (4.76) can be replaced by the following two conditions:

\[
\begin{align*}
    w_{2n} & = 0 \\
    w_{3n} & = 0 
\end{align*}
\]  

(4.77)  
(4.78)

### 4.2.6 Minimising vibration using piezoceramic actuators and an angle stiffener

For any force or moment excitation, the twenty four equations in twenty four unknowns can be written in the form \( aX = B \), where \( X = [A_{11n} A_{12n} A_{13n} \ldots A_{18n} A_{21n} A_{22n} \ldots A_{28n} A_{31n} A_{32n} \ldots A_{38n}]^T \), and \( B \) is a column vector. When the excitation position is to the left of the stiffener location, i.e. \( x < x_1 \), \( B \) has a non zero excitation term in the fifteenth row for excitation by a line moment about an arc parallel to the \( \theta \)-axis or the sixteenth row otherwise. For a simply supported cylindrical shell with the end at \( x = 0 \) free, \( a \) is given by Equation (4.79), except in the case of a distributed moment excitation when row sixteen is replaced by Equation (4.65). If the excitation position is to the right of the stiffener location, i.e. \( x > x_1 \), then a similar analysis is followed, resulting in an excitation vector \( B \) with the non zero term in the twenty third row for excitation by a line moment about an arc parallel to the \( \theta \)-axis or the twenty fourth row otherwise, and \( a \) is given by Equation (4.80), except in the case of a distributed moment excitation when row twenty four is replaced by Equation (4.65). For the cylinder with the right hand end modelled as infinite, the equations corresponding to rows 5-8 of the matrix \( a \) are replaced by Equations (4.38) - (4.41). For both sets of boundary conditions, the matrix equation

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\( \alpha \mathbf{X} = \mathbf{B} \) can be solved for \( \mathbf{X} \) for any type of excitation and the result can be used with Equations (4.4) - (4.9) and (4.66) - (4.68) to calculate the corresponding cylinder response.

Figure 4.6 shows the semi-infinite shell with primary forces \( F_{p1} \) and \( F_{p2} \) located at \( x = x_p, \theta = \theta_{p1} \) and \( \theta = \theta_{p2} \) control actuators at \( x = x_c \) and a line of error sensors at \( x = x_e \). Figure 4.7 shows the resultant forces and moments applied to the cylinder by the control actuators. Control forces \( F_{ci}, i = 1,6 \) act at \( (x_c, \theta_{ci}, i = 1,6) \), with the distributed force \( F_c \) and distributed moment \( M_c \) acting about the circumference parallel to the \( \theta \)-axis at \( (x_c, \theta = 0 \text{ to } \theta = 2\pi) \).
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Figure 4.6 Semi-infinite cylinder showing primary forces, control actuators, angle stiffener and ring of error sensors.

Figure 4.7 Part-stiffener and control actuator showing control forces and moment.
\[\begin{align*}
\beta_{ln} & \quad \ldots \quad \beta_{sn} & 0 & \ldots \\
\gamma_{ln} & \quad \ldots \quad \gamma_{sn} & 0 & \ldots \\
1 & \quad \ldots \quad 1 & 0 & \ldots \\
\frac{vn}{R^2} (\gamma_{ln} + n) + \frac{\beta_{ln} k_{1n} - k_{1n}^2}{R} & \quad \frac{vn}{R^2} (\gamma_{sn} + n) + \frac{\beta_{sn} k_{sn} - k_{sn}^2}{R} & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
\beta_{ln} e^{k_{ln}x_0} & \quad \ldots \quad \beta_{sn} e^{k_{sn}x_0} & -\beta_{ln} e^{k_{ln}x_0} & \ldots \\
\beta_{ln} k_{1n} e^{k_{ln}x_0} & \quad \ldots \quad \beta_{sn} k_{sn} e^{k_{sn}x_0} & -\beta_{ln} k_{1n} e^{k_{ln}x_0} & \ldots \\
\gamma_{ln} e^{k_{ln}x_0} & \quad \ldots \quad \gamma_{sn} e^{k_{sn}x_0} & -\gamma_{ln} e^{k_{ln}x_0} & \ldots \\
\gamma_{ln} k_{1n} e^{k_{ln}x_0} & \quad \ldots \quad \gamma_{sn} k_{sn} e^{k_{sn}x_0} & -\gamma_{ln} k_{1n} e^{k_{ln}x_0} & \ldots \\
e^{k_{ln}x_0} & \quad \ldots \quad e^{k_{sn}x_0} & -e^{k_{ln}x_0} & \ldots \\
k_{1n} e^{k_{ln}x_0} & \quad \ldots \quad k_{sn} e^{k_{sn}x_0} & -k_{1n} e^{k_{ln}x_0} & \ldots \\
k_{1n}^2 e^{k_{ln}x_0} & \quad \ldots \quad k_{sn}^2 e^{k_{sn}x_0} & -k_{1n}^2 e^{k_{ln}x_0} & \ldots \\
\left(\beta_{ln} - Rk_{1n}\right) e^{k_{ln}x_0} & \quad \left(\beta_{sn} - Rk_{sn}\right) e^{k_{sn}x_0} & \left(\beta_{ln} - Rk_{1n}\right) k_{1n} e^{k_{ln}x_0} & \ldots \\
0 & \quad \ldots \quad 0 & \beta_{ln} e^{k_{ln}x_1} & \ldots \\
0 & \quad \ldots \quad 0 & \beta_{ln} k_{1n} e^{k_{ln}x_1} & \ldots \\
0 & \quad \ldots \quad 0 & \gamma_{ln} e^{k_{ln}x_1} & \ldots \\
0 & \quad \ldots \quad 0 & \gamma_{ln} k_{1n} e^{k_{ln}x_1} & \ldots \\
0 & \quad \ldots \quad 0 & e^{k_{ln}x_1} & \ldots \\
0 & \quad \ldots \quad 0 & k_{1n} e^{k_{ln}x_1} & \ldots \\
0 & \quad \ldots \quad 0 & k_{1n}^2 e^{k_{ln}x_1} & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
0 & \quad \ldots \quad 0 & 0 & \ldots \\
\end{align*}\]
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\[
\begin{align*}
\vdots & 
0 & 0 & \vdots & 0 \\
\vdots & 
0 & 0 & \vdots & 0 \\
\vdots & 
0 & 0 & \vdots & 0 \\
\vdots & 
0 & \beta_{1n} e^{k_i L_x} & \vdots & \beta_{8n} e^{k_i L_x} \\
\vdots & 
0 & \gamma_{1n} e^{k_i L_x} & \vdots & \gamma_{8n} e^{k_i L_x} \\
\vdots & 
0 & e^{k_i L_x} & \vdots & e^{k_i L_x} \\
\vdots & 
0 & \left( \frac{\nu n}{R^2} \gamma_{1n} + n \right) + \frac{\beta_{1n}}{R} k_{1n} - k_{1n}^2 & \vdots & \left( \frac{\nu n}{R^2} \gamma_{8n} + n \right) + \frac{\beta_{8n}}{R} k_{8n} - k_{8n}^2 \\
\vdots & 
- \beta_{8n} e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
- \beta_{8n} k_{8n} e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
- \gamma_{8n} e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
- \gamma_{8n} k_{8n} e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
- e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
- k_{8n} e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
- k_{8n}^2 e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
- \beta_{8n} - R k_{8n} \gamma_{8n}^2 e^{k_i L_{n_0}} & 0 & \vdots & 0 \\
\vdots & 
\beta_{8n} e^{k_i L_{1n_1}} & - \beta_{1n} e^{k_i L_{1n_1}} & \vdots & - \beta_{8n} e^{k_i L_{1n_1}} \\
\vdots & 
\beta_{8n} k_{8n} e^{k_i L_{1n_1}} & - \beta_{1n} k_{1n} e^{k_i L_{1n_1}} & \vdots & - \beta_{8n} k_{8n} e^{k_i L_{1n_1}} \\
\vdots & 
\gamma_{8n} e^{k_i L_{1n_1}} & - \gamma_{1n} e^{k_i L_{1n_1}} & \vdots & - \gamma_{8n} e^{k_i L_{1n_1}} \\
\vdots & 
\gamma_{8n} k_{8n} e^{k_i L_{1n_1}} & - \gamma_{1n} k_{1n} e^{k_i L_{1n_1}} & \vdots & - \gamma_{8n} k_{8n} e^{k_i L_{1n_1}} \\
\vdots & 
0 & 0 & \vdots & 0 \\
\vdots & 
0 & - e^{k_i L_{1n_1}} & \vdots & - e^{k_i L_{1n_1}} \\
\vdots & 
0 & - k_{1n} e^{k_i L_{1n_1}} & \vdots & - k_{8n} e^{k_i L_{1n_1}} \\
\vdots & 
0 & - k_{1n}^2 e^{k_i L_{1n_1}} & \vdots & - k_{8n}^2 e^{k_i L_{1n_1}} \\
\end{align*}
\]

(4.79)
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\[
\alpha = \begin{bmatrix}
\beta_{1n} & \ldots & \beta_{8n} & 0 & \ldots \\
\gamma_{1n} & \ldots & \gamma_{8n} & 0 & \ldots \\
1 & \ldots & 1 & 0 & \ldots \\
\frac{vn}{R^2}(\gamma_{1n} + n) + \frac{\beta_{1n}k_{1n}}{R} - k_{1n}^2 & \ldots & \frac{vn}{R^2}(\gamma_{8n} + n) + \frac{\beta_{8n}k_{8n}}{R} - k_{8n}^2 & 0 & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
\beta_{1n}e^{k_{1n}x_1} & \ldots & \beta_{8n}e^{k_{8n}x_1} & -\beta_{1n}e^{k_{1n}x_1} & \ldots \\
\beta_{1n}k_{1n}e^{k_{1n}x_1} & \ldots & \beta_{8n}k_{8n}e^{k_{8n}x_1} & -\beta_{1n}k_{1n}e^{k_{1n}x_1} & \ldots \\
\gamma_{1n}e^{k_{1n}x_1} & \ldots & \gamma_{8n}e^{k_{8n}x_1} & -\gamma_{1n}e^{k_{1n}x_1} & \ldots \\
\gamma_{1n}k_{1n}e^{k_{1n}x_1} & \ldots & \gamma_{8n}k_{8n}e^{k_{8n}x_1} & -\gamma_{1n}k_{1n}e^{k_{1n}x_1} & \ldots \\
e^{k_{1n}x_1} & \ldots & e^{k_{8n}x_1} & 0 & \ldots \\
0 & \ldots & 0 & -e^{k_{1n}x_1} & \ldots \\
k_{1n}e^{k_{1n}x_1} & \ldots & k_{8n}e^{k_{8n}x_1} & -k_{1n}e^{k_{1n}x_1} & \ldots \\
k_{1n}^2e^{k_{1n}x_1} & \ldots & k_{8n}^2e^{k_{8n}x_1} & -k_{1n}^2e^{k_{1n}x_1} & \ldots \\
0 & \ldots & 0 & \beta_{1n}e^{k_{1n}x_0} & \ldots \\
0 & \ldots & 0 & \beta_{1n}k_{1n}e^{k_{1n}x_0} & \ldots \\
0 & \ldots & 0 & \gamma_{1n}e^{k_{1n}x_0} & \ldots \\
0 & \ldots & 0 & \gamma_{1n}k_{1n}e^{k_{1n}x_0} & \ldots \\
0 & \ldots & 0 & e^{k_{1n}x_0} & \ldots \\
0 & \ldots & 0 & k_{1n}e^{k_{1n}x_0} & \ldots \\
0 & \ldots & 0 & k_{1n}^2e^{k_{1n}x_0} & \ldots \\
0 & \ldots & 0 & (\beta_{1n} - Rk_{1n})e^{k_{1n}x_0} & \ldots \\
\end{bmatrix}
\]
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\[ \begin{align*}
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \beta_{1n} e^{k_{1n}z} & \quad \beta_{8n} e^{k_{8n}z} \\
\vdots & \quad 0 & \quad 0 & \quad \gamma_{1n} e^{k_{1n}z} & \quad \gamma_{8n} e^{k_{8n}z} \\
\vdots & \quad 0 & \quad 0 & \quad e^{k_{1n}z} & \quad e^{k_{8n}z} \\
\vdots & \quad 0 & \quad 0 & \quad 0 & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad 0 & \quad 0 \\
\vdots & \quad -\beta_{8n} e^{k_{8n}z} & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad -\beta_{8n} k_{8n} e^{k_{8n}z} & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad -\gamma_{8n} e^{k_{8n}z} & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad -\gamma_{8n} k_{8n} e^{k_{8n}z} & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad 0 & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad \ldots & \quad 0 & \quad \ldots & \quad 0 \\
\vdots & \quad -\beta_{8n} e^{k_{8n}z} & \quad -\beta_{1n} e^{k_{1n}0} & \quad \ldots & \quad -\beta_{8n} e^{k_{8n}z} \\
\vdots & \quad -\beta_{8n} k_{8n} e^{k_{8n}0} & \quad -\beta_{1n} k_{1n} e^{k_{1n}0} & \quad \ldots & \quad -\beta_{8n} k_{8n} e^{k_{8n}z} \\
\vdots & \quad -\gamma_{8n} e^{k_{8n}0} & \quad -\gamma_{1n} e^{k_{1n}0} & \quad \ldots & \quad -\gamma_{8n} e^{k_{8n}z} \\
\vdots & \quad -\gamma_{8n} k_{8n} e^{k_{8n}0} & \quad -\gamma_{1n} k_{1n} e^{k_{1n}0} & \quad \ldots & \quad -\gamma_{8n} k_{8n} e^{k_{8n}z} \\
\vdots & \quad e^{k_{8n}0} & \quad e^{k_{1n}0} & \quad \ldots & \quad e^{k_{8n}z} \\
\vdots & \quad k_{8n} e^{k_{8n}0} & \quad -k_{1n} e^{k_{1n}0} & \quad \ldots & \quad k_{8n} e^{k_{8n}z} \\
\vdots & \quad k_{8n}^2 e^{k_{8n}0} & \quad -k_{1n}^2 e^{k_{1n}0} & \quad \ldots & \quad k_{8n}^2 e^{k_{8n}z} \\
\vdots & \quad -\beta_{1n} - R k_{1n} e^{k_{1n}0} & \quad -\beta_{1n} - R k_{1n} e^{k_{1n}0} & \quad \ldots & \quad -\beta_{8n} - R k_{8n} e^{k_{8n}z} \\
\end{align*} \]

\[ (\text{4.80}) \]
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The cylinder response at any location \((x, \theta)\) to a particular excitation located at \(x_0\), with a ring stiffener located at \(x_1\) is (omitting the time dependent terms \(e^{j\omega t}\))

\[
    u(x, \theta) = \sum_{n=1}^{\infty} [X_n^T E(x)] \cos(n\theta), \quad (4.81)
\]

\[
    v(x, \theta) = \sum_{n=1}^{\infty} ([X_n] E(x)) \cos(n\theta), \quad (4.82)
\]

and

\[
    w(x, \theta) = \sum_{n=1}^{\infty} (X^T E(x)) \cos(n\theta), \quad (4.83)
\]

where \(X_\beta = \beta_i X_i\) and \(X_q = \gamma_i X_i\) for \(i = 1, 8\), and for \(x < x_0\) and \(x < x_1\),

\[
    E = \begin{bmatrix}
        e^{k_{i1} x} & e^{k_{i2} x} & \ldots & e^{k_{i8} x} \\
        0 & 0 & \ldots & 0 
    \end{bmatrix}^T.
\]

(4.84)

for \(x_0 < x < x_1\) or \(x_1 < x < x_0\),

\[
    E = \begin{bmatrix}
        0 & 0 & \ldots & 0 & e^{k_{i1} x} & e^{k_{i2} x} & \ldots & e^{k_{i8} x} & 0 & 0 & \ldots & 0 
    \end{bmatrix}^T.
\]

(4.85)

and for \(x > x_0\) and \(x > x_1\),

\[
    E = \begin{bmatrix}
        0 & 0 & \ldots & 0 & e^{k_{i1} x} & e^{k_{i2} x} & \ldots & e^{k_{i8} x} \\
        e^{k_{i1} x} & e^{k_{i2} x} & \ldots & e^{k_{i8} x} & 0 & 0 & \ldots & 0 
    \end{bmatrix}^T.
\]

(4.86)

By summation of the displacements corresponding to each force and moment, the total radial displacement is found to be

\[
    w = \sum_{i=1}^{2} \left[ \sum_{i=1}^{6} \{w_{F_i}\} + w_{F_i} + w_{M_i} \right] + \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{2} X_n^T E(x) + X_n^T E(x) \times X_n^T E(x) \right] \cos(n\theta).
\]

(4.87)
where the subscripts \( F_{pi}, F_{ci}, F_c \) and \( M_c \) on \( w \) and \( X \) refer to the corresponding excitation force or moment.

As the excitation vector \( B \) has a non-zero element in one row only, the solution vector \( X \) can be written in terms of a single column of the inverse \( \alpha^{-1} \):

\[
X = (\alpha^{-1})_{k,15} B_{15}, (\alpha^{-1})_{k,16} B_{16}, (\alpha^{-1})_{k,23} B_{23}, \text{ or } (\alpha^{-1})_{k,24} B_{24}, k = 1,2,4, \quad (4.88)
\]

where \((\alpha^{-1})_{k,i}\) is the \(k\)th element in the \(i\)th column of the inverse of \( \alpha \) and \( B_i \) is the \(i\)th element (the non-zero element) of \( B \). The value taken by \( i \) depends on the form and location of the excitation, as discussed previously.

### 4.2.6.1 Control sources driven by the same signal

If the three control actuators are driven by the same signal, then \( F_{ci} = -F_s \), say, for \( i = 1,6 \). The \(i\)th actuator also generates a distributed force of total magnitude \( F_s \) and a distributed moment of total magnitude \( x_a F_s \) acting along an arc between \((\theta_{i-1} + \theta_i)/2 \) and \((\theta_i + \theta_{i+1})/2 \) where \( x_a \) is the width of the stiffener flange. Additionally, if the primary shakers are driven by the same signal, then \( F_{p1i} = F_{p2i} (i = 1,2) = F_p \), say. Defining

\[
F_{pi} = \frac{(1 - v^2)}{\pi \xi R^2 E h} \left( \alpha_F^{-1} \right)_{p_{k,16}}^{\text{tr}} (i = 1,2), \quad (4.89)
\]

\[
F_{ci} = -\frac{(1 - v^2)}{\pi \xi R^2 E h} \left( \alpha_{Fc}^{-1} \right)_{c_{k,24}}^{\text{tr}} (i = 1,6), \quad (4.90)
\]
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\[
(F_C)_i = \frac{2(1 - \nu^2)}{\pi \xi R^2 E h} \left[ \theta_{i+1} - \theta_{i-1} \right] \left[ \alpha_{i}^{-1}\right]_{i,k,16}^T \quad (i = 1, 6)
\]  

(4.91)

and

\[
(M_C)_i = \frac{(1 - \nu^2)}{\pi \xi R^3 E h} \left[ \alpha_{i}^{-1}\right]_{i,k,15}^T \quad (i = 1, 6)
\]

(4.92)

Substituting Equations (4.89)-(4.92) into Equation (4.87) and rearranging gives

\[
w(x, \theta) = \sum_{n = 1}^{\infty} \left\{ \sum_{i = 1}^{2} (F_Pi \cos (n\theta_{pi})) \left[ F_p + \sum_{i = 1}^{6} (F_Ci \cos (n\theta_{ci})) + \right. \right. \left. \left. \frac{(F_Ci) + x_a (M_Ci)}{n} \left[ \sin \left( \frac{n\theta_{i+1} + n\theta_i}{2} \right) - \sin \left( \frac{n\theta_{i-1} + n\theta_i}{2} \right) \right] \right\} E(x) \cos(n\theta)
\]

or

\[
w(x, \theta) = w_p(x, \theta) F_p + w_s(x, \theta) F_s
\]

(4.93)

(4.94)

where

\[
w_p(x, \theta) = \sum_{n = 1}^{\infty} \left\{ \sum_{i = 1}^{2} (F_Pi \cos (n\theta_{pi})) \left[ E(x) \cos (n\theta) \right]\right\}
\]

(4.95)

and

\[
w_s(x, \theta) = 6 \sum_{n = 1}^{\infty} \left\{ \sum_{i = 1}^{6} (F_Ci \cos (n\theta_{ci})) + \right. \]

\[
\left. \frac{(F_Ci) + x_a (M_Ci)}{n} \left[ \sin \left( \frac{n\theta_{i+1} + n\theta_i}{2} \right) - \sin \left( \frac{n\theta_{i-1} + n\theta_i}{2} \right) \right] F_s \right\} E(x) \cos(n\theta)
\]

(4.96)
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The radial acceleration around the ring at \( x = x_e \) is to be minimised. The mean square of the displacement defined in Equation (4.94) is integrated around the circumference of the cylinder:

\[
2\pi \int_0^{2\pi} |w(x_e, \theta)|^2 d\theta = 2\pi \int_0^{2\pi} |F_p w_p(x_e, \theta) + F_s w_s(x_e, \theta)|^2 d\theta . \tag{4.97}
\]

Noting that \( |z|^2 = z \bar{z} \) (where \( \bar{z} \) is the complex conjugate of \( z \)), and writing \( F_s = F_{sr} + jF_{sj} \),

\[
2\pi \int_0^{2\pi} |w(x_e, \theta)|^2 d\theta = 2\pi \int_0^{2\pi} \left(|F_p|^2 |w_p|^2 + F_p w_p \overline{w_s} (F_{sr} - jF_{sj}) + \overline{F_p} w_p \overline{w_s} (F_{sr} + jF_{sj}) + (F_{sr}^2 + F_{sj}^2) |w_s|^2 \right) d\theta . \tag{4.98}
\]

The partial derivatives of Equation (4.98) with respect to the real and imaginary components of the control force are taken and set equal to zero to find

\[
\frac{\partial (\quad)}{\partial F_{sr}} = 2\pi \int_0^{2\pi} \left[F_p w_p \overline{w_s} + \overline{F_p} w_p w_s + 2F_{sr} |w_s|^2 \right] d\theta = 0 \tag{4.99}
\]

and

\[
\frac{j \partial (\quad)}{\partial F_{sj}} = 2\pi \int_0^{2\pi} \left[F_p w_p \overline{w_s} - \overline{F_p} w_p w_s + 2jF_{sj} |w_s|^2 \right] d\theta = 0 . \tag{4.100}
\]

Adding Equations (4.99) and (4.100) gives

\[
2\pi \int_0^{2\pi} \left[F_p w_p w_s + F_s |w_s|^2 \right] d\theta = 0 . \tag{4.101}
\]
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The optimal control force \( F_s \) required to minimise normal acceleration at the ring of error sensors can thus be calculated by

\[
F_s = -F_p \frac{\int_0^{2\pi} w_p(x, \theta) \overline{w_i(x, \theta)} d\theta}{\int_0^{2\pi} |w_i(x, \theta)|^2 d\theta}.
\]  

(4.102)

4.2.6.2 Control sources driven independently

If the six control actuators are driven independently, then a similar analysis is followed; however, six equations instead of one result from integrating the mean square of the displacement defined in Equation (4.94) and setting the partial derivatives of the integration with respect to the real and imaginary components of each control force equal to zero. The optimal control forces \( F_{si} \), \( i = 1, 6 \), required to minimise acceleration at the ring of error sensors can be calculated by

\[
\begin{bmatrix}
F_{s1} \\
F_{s2} \\
\vdots \\
F_{s6}
\end{bmatrix}
= -F_p \begin{bmatrix}
\int_0^{2\pi} w_1 \overline{w_1} d\theta & \int_0^{2\pi} w_2 \overline{w_1} d\theta & \ldots & \int_0^{2\pi} w_6 \overline{w_1} d\theta \\
\int_0^{2\pi} w_1 \overline{w_2} d\theta & \int_0^{2\pi} w_2 \overline{w_2} d\theta & \ldots & \int_0^{2\pi} w_6 \overline{w_2} d\theta \\
\vdots & \vdots & \ddots & \vdots \\
\int_0^{2\pi} w_1 \overline{w_6} d\theta & \int_0^{2\pi} w_2 \overline{w_6} d\theta & \ldots & \int_0^{2\pi} w_6 \overline{w_6} d\theta
\end{bmatrix}^{-1}
\begin{bmatrix}
\int_0^{2\pi} w_p \overline{w_1} d\theta \\
\int_0^{2\pi} w_p \overline{w_2} d\theta \\
\vdots \\
\int_0^{2\pi} w_p \overline{w_6} d\theta
\end{bmatrix}.
\]  

(4.103)

4.2.6.3 Discrete error sensors

If the sum of the squares of the vibration amplitude measured at \( Q \) discrete points \((x_e, \theta_{qe})\), \( q = 1, Q \) is used as the error signal instead of the integral around the circumference of the cylinder at location \( x_e \), Equation (4.103) becomes

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4.2.7 Natural frequencies

Leissa (1973a) gives the characteristic equation for the free vibration frequencies of a cylinder derived from the Flügge equations of motion:

\[
[\Omega^6 - \left(1 + \frac{1}{2} \alpha^2 \right) (3 + 2v) (n^2 + \lambda^2)^2 + \xi \left( (n^2 + \lambda^2)^2 \right) \right] \Omega^4 + \frac{1}{2} \alpha^2 (3 + 2v) (n^2 + \lambda^2)^2 \left( 4 \xi \left( (n^2 + \lambda^2)^2 \right) - 2 \xi \right] \frac{1}{2} \alpha^2 (3 + 2v) (n^2 + \lambda^2)^2 \Omega^2 - \frac{1}{2} \alpha^2 (3 + 2v) (n^2 + \lambda^2)^2 \xi \left( (n^2 + \lambda^2)^2 \right) + 2(2 - v) \lambda^2 n^2 + n^4 - 2 \xi \alpha^2 \lambda^2 - 6 \lambda^2 n^2 - 2(2 - v) \lambda^2 n^4 + 2n^6 ]
\]

(4.106)

where \( \lambda = m \pi R / L, n \) is the circumferential mode number, \( m \) is the axial mode number, and the frequency parameter \( \Omega \) is given by

\[
\Omega = \sqrt{\frac{\rho (1 - v^2) R^2}{E}} \left( \frac{1}{2} \right)^{\frac{1}{2}}
\]

(4.107)

The natural frequencies \( \omega \) can be obtained from Equation (4.107) by substitution of the real, positive solutions \( \Omega \) from Equation (4.106).
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4.3 NUMERICAL RESULTS

The theoretical model developed in the previous section was programmed in Fortran. The coefficient matrices $a$ (see Equations (4.79) and (4.80)) were close to singular. To obtain an accurate solution, 16-bit data types were required. The program consisted of about 3000 lines and, for a typical set of results, took 2 days C.P.U. time to run on a SPARC-20 computer.

The discussion that follows examines the effect of varying forcing frequency, control source location and error sensor location on the active control of vibration in cylinders with two sets of boundary conditions. In both models the end at $x = 0$ was modelled as simply supported. In one model, the end at $x = L_x$ was also modelled as simply supported and in the second model the cylinder was modelled as semi-infinite in the $x$-direction. The cylinder parameters (including location of the control source, primary source and error sensor) are listed in Table 4.1, and the excitation frequency was 132 Hz. These values are adhered to unless otherwise stated. The stiffener was assumed to be very stiff in comparison to the cylinder.

4.3.1 Acceleration distributions for controlled and uncontrolled cases

Figures 4.8 and 4.9 show the uncontrolled radial acceleration amplitude distribution in dB for the semi-infinite and finite cylinders. The cylinder has been "unrolled" in the figures so that the acceleration distribution can be seen more easily. The shape of the curves is very similar for the two cases, apart from near the end $x = 2.0$m where the acceleration of the finite cylinder is zero. It can be seen from the nature of the response that the near field effects become insignificant within a few centimetres of the points of discontinuity.
Unlike the corresponding plate and beam cases, there is no clear evidence of the existence of standing wave fields either upstream or downstream of the stiffener location on the finite or infinite cylinders. This is because the nodes that occur in the standing wave on a cylinder occur at large separations.
Let \( x \) be the separation between axial nodes in a standing wave. For a beam, \( x = \frac{\lambda_b}{2} \) where \( \lambda_b \) is the flexural wavelength of vibration in a beam given by Equation (2.50), because vibration in a beam is one-dimensional. The path length corresponding to a mode is simply the length of the beam. In the case of a plate, the total path length corresponding to a mode may consist of a combination of plate widths and lengths, and similarly for a cylinder the path length may consist of a combination of cylinder circumferences and lengths. Generally, for plates and cylinders, \( x = \frac{\lambda_g}{2} \), where the flexural wavelength of vibration in a plate is given by Equation (3.59) and in a cylinder is given by Pan and Hansen (1995b):

\[
\lambda_b = \frac{1}{f} \left( \frac{E h^2 \omega^2}{12 p (1 - v^2) R^4} \right)^{\frac{1}{2}}. \tag{4.108}
\]

In the far field of vibration, the eigenfunction describing the dependence of the displacement of a cylinder, plate or beam on axial coordinate \( x \) is given by, for each across-plate mode or circumferential cylinder mode \( n \),

\[
w_n(x) = A_3 e^{j n x} + A_4 e^{-j n x}. \tag{4.109}
\]

Equation (4.109) is similar to Equations (2.2), (3.3) and (4.25), but retaining only the terms with no real (decaying) exponential part. The total displacement \( w_n(x) \) is the sum of the displacement contributed by each mode \( n \). Axial nodes in the displacement amplitude occur when \( w_n(x) \) is at a minimum. Dropping the coefficients \( A_3 \) and \( A_4 \), differentiating Equation (4.109) and setting the result equal to zero gives
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\[ jb_ne^{jb_nx} - jb_ne^{jb_nx} = 0 . \]  \hspace{1cm} (4.110)

Minima in Equation (4.109) occur when

\[ \sin(b_nx) = 0 . \]  \hspace{1cm} (4.111)

Nodes in the standing wave in the axial direction are thus separated by intervals

\[ x_{sn} = \frac{\pi}{b_n} , \]  \hspace{1cm} (4.112)

where \( n \) represents the number of the cross-plate mode on plate structures and circumferential mode on cylindrical structures. The mode number \( n \) has no significance on beams.

Considering the fixed beam described in Section 2.2, the finite plate described in Section 3.2 and the finite cylinder described in Section 4.2, the following table can be established comparing the distance between axial nodes. It can be seen from Table 4.2 that a cylinder of similar radius and thickness to those considered in this chapter would need to be about 25m long before a standing wave node could be observed. The distance between axial nodes on cylinders is typically much larger than for beams or plates.
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### Table 4.2

Comparison of Axial Node Separation in Beams, Plates and Cylinders

<table>
<thead>
<tr>
<th>Mode</th>
<th>Beam ($\lambda_B/2 = 0.2412$ m) $x_m$ (m)</th>
<th>Plate ($\lambda_B/2 = 0.1793$ m) $x_m$ (m)</th>
<th>Cylinder ($\lambda_B/2 = 0.9501$ m) $x_m$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2412</td>
<td>0.1921</td>
<td>23.09</td>
</tr>
<tr>
<td>2</td>
<td>0.2412</td>
<td>0.2573</td>
<td>26.85</td>
</tr>
<tr>
<td>3</td>
<td>0.2412</td>
<td>0.4521</td>
<td>27.93</td>
</tr>
<tr>
<td>4</td>
<td>0.2412</td>
<td>0.1744</td>
<td>28.29</td>
</tr>
<tr>
<td>5</td>
<td>0.2412</td>
<td>0.1205</td>
<td>28.29</td>
</tr>
<tr>
<td>6</td>
<td>0.2412</td>
<td>0.0941</td>
<td>28.29</td>
</tr>
<tr>
<td>7</td>
<td>0.2412</td>
<td>0.0779</td>
<td>28.29</td>
</tr>
<tr>
<td>8</td>
<td>0.2412</td>
<td>0.0667</td>
<td>28.29</td>
</tr>
</tbody>
</table>

Close examination of Figure 4.9 reveals a slight decrease in acceleration amplitude between the stiffener location and the right hand end. This is due to the standing wave effect; the acceleration level begins to curve towards a minimum that would occur a half-wavelength away were the cylinder long enough. There is no decrease in acceleration level to the right of the stiffener location in Figure 4.8 as no standing wave exists downstream of the stiffener on the semi-infinite cylinder.
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Figure 4.8 Uncontrolled semi-infinite cylinder radial acceleration distribution. The end at $x = 0$ is modelled as simply supported.

Figure 4.9 Uncontrolled finite cylinder radial acceleration distribution. The ends at $x = 0$ and $x = 2.0$ are modelled as simply supported.
Chapter 4. Control of vibrations in a stiffened cylinder

Figures 4.10 and 4.11 show the controlled acceleration amplitude distributions for the semi-infinite and finite cylinders with the six control sources driven by the same signal. The acceleration level is only marginally reduced. The calculated reduction in acceleration amplitude downstream of the ring of error sensors is only about 3 dB. Several higher-order circumferential modes contribute significantly to the vibration response of the cylinder so control sources driven by a common signal are not capable of significantly reducing the overall vibration level.

Figures 4.12 and 4.13 show the controlled acceleration amplitude distributions for the semi-infinite and finite cylinders with the six control sources driven independently. The acceleration level is at a minimum at the error sensor location ($x_e = 1.0$ m). The calculated reduction in acceleration amplitude downstream of the error sensor is a little over 40 dB for the semi infinite cylinder and a little under 40 dB for the finite cylinder.

Figures 4.14 - 4.21 show the axial and tangential acceleration distributions corresponding to the radial acceleration distributions given in Figures 4.8, 4.9, 4.12 and 4.13. Only radial vibration at the ring of error sensors is optimally reduced in the controlled cases. The figures show that controlling radial vibration also results in significant reduction of vibration in the axial and tangential directions (between 30 and 40 dB attenuation). Axial acceleration levels are of a similar order as radial acceleration levels, while tangential acceleration is generally a few dB less. In theory, tangential acceleration modes are out of phase with radial and axial acceleration modes, and this can also be seen on the figures.
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Figure 4.10  Controlled semi-infinite cylinder radial acceleration distribution - control sources driven by the same signal.

Figure 4.11  Controlled finite cylinder radial acceleration distribution - control sources driven by the same signal.
Controlled semi-infinite cylinder radial acceleration distribution - control sources driven independently.

Controlled finite cylinder radial acceleration distribution - control sources driven independently.
Figure 4.14  Uncontrolled semi-finite cylinder axial acceleration distribution.

Figure 4.15  Uncontrolled finite cylinder axial acceleration distribution.
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Figure 4.16 Controlled semi-finite cylinder axial acceleration distribution - control sources driven independently.

Figure 4.17 Controlled finite cylinder axial acceleration distribution - control sources driven independently.
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Figure 4.18 Uncontrolled semi-finite cylinder tangential acceleration distribution.

Figure 4.19 Uncontrolled finite cylinder tangential acceleration distribution.
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Figure 4.20 Controlled semi-finite cylinder tangential acceleration distribution - control sources driven independently.

Figure 4.21 Controlled finite cylinder tangential acceleration distribution - control sources driven independently.
4.3.2 Effect of variations in forcing frequency, control source location and error sensor location on the control forces

Figure 4.22 shows the effect of varying the forcing frequency on the magnitude of the control force(s) required to minimise the radial cylinder vibration at the line of error sensors. The mean control source amplitudes for the cases where control sources are driven by the same signal are low and do not vary significantly with frequency. For the cases with control sources driven independently, the mean control source amplitude is larger, and varies a little with frequency, particularly for the finite cylinder case. The corresponding results for beam and plate structures show large maxima in the control source amplitudes that are a result of the control source location corresponding to a standing wave node; this situation does not arise for the cylinder structures where the bending wavelength greatly exceeds the cylinder length.

Figures 4.23 and 4.24 show the effect of the locations of the ring of control sources and the line of error sensors on the control source amplitude required for optimal control. There are fluctuations in the control effort required with different axial locations of control sources and error sensors, again particularly for the finite cylinder, but no large maxima in control effort occur.

These results show that, unlike for the equivalent beam and plate cases, there are no frequencies or axial locations of control sources and error sensors that result in an unrealistically high control effort required to optimally control the vibration at the ring of error sensors.
Figure 4.22  Mean control source amplitude for optimal control as a function of frequency. Six control sources and two primary sources were used.
Figure 4.23  Mean control source amplitude for optimal control as a function of control source - primary source separation. Six control sources and two primary sources were used.
Figure 4.24  Mean control source amplitude for optimal control as a function of error sensor - control source separation. Six control sources and two primary sources were used.
Figure 4.25 shows the dependence of control source amplitude required for optimal control on circumferential control source location.

Figure 4.25(a) shows the amplitude of the second control source required for optimal control, assuming only two control sources are used, and the first is located at $\theta_{c1} = \pi/6$ radians. If the second control source is located near to the first, the control source amplitude required for optimal control tends toward infinity. This also occurs when the second control source is located near $\theta_{c2} = 2\pi - \theta_{c1} = 11\pi/6$.

Figure 4.25(b) shows the amplitude of the third control source required for optimal control, assuming three control sources are used. The first is located at $\theta_{c1} = \pi/6$ and the second at $\theta_{c2} = 3\pi/4$ radians. If the third control source is located near to either of the first two, the control source amplitude required for optimal control becomes large. This also occurs when the third control source is located near $\theta_{c3} = 2\pi - \theta_{c1} = 11\pi/6$ or $2\pi - \theta_{c2} = 5\pi/4$.

Figure 4.25(c) shows the amplitude of the fourth control source required for optimal control, assuming three control sources are used. The first is located at $\theta_{c1} = \pi/6$, the second at $\theta_{c2} = 3\pi/4$ and the third at $\theta_{c3} = 35\pi/24$ radians. If the fourth control source is located near to either of the first three, the control source amplitude required for optimal control becomes large. This also occurs when the fourth control source is located near $\theta_{c4} = 2\pi - \theta_{c1} = 11\pi/6$, $2\pi - \theta_{c2} = 5\pi/4$ or $2\pi - \theta_{c3} = 13\pi/24$. 
Figure 4.25  Control source amplitude for optimal control as a function of circumferential control source location.

(a) Amplitude of the second of two control sources; the first control source was located at $\theta_{c1} = \pi/6$ radians.

(b) Amplitude of the third of three control sources; the first control source was located at $\theta_{c1} = \pi/6$ and the second at $\theta_{c2} = 3\pi/4$ radians.

(c) Amplitude of the fourth of four control sources; the first control source was located at $\theta_{c1} = \pi/6$, the second at $\theta_{c2} = 3\pi/4$ and the third at $\theta_{c3} = 35\pi/24$ radians.
If the circumferential location of an additional control source is $\theta_{i+1}$ and there are $i$ control sources already in place at locations $\theta_j$, then the amplitude of control source $i+1$ will be large when $\cos(\theta_{i+1}) = \cos(\theta_j)$, because at these locations, control source $i+1$ contributes to the same modes as one of the other control sources.

4.3.3 Effect of variations in forcing frequency, control source location and error sensor location on the attenuation of acceleration level

Figure 4.26 shows the variation in the mean attenuation of radial, axial and tangential acceleration level downstream of the ring of error sensors as a function of frequency for the cases with control sources driven independently. There is some variation, particularly for the semi-infinite cylinder below 200 Hz, but overall the amount of attenuation of acceleration in each direction is not greatly dependent on the excitation frequency. Radial acceleration is attenuated slightly more than axial and tangential acceleration, and axial acceleration is also attenuated slightly more than tangential acceleration.

Figure 4.27 shows that little attenuation can be achieved using control sources driven by a common signal. This is because there are many circumferential modes contributing significantly to the vibration of the cylinder, even at low frequencies. The level of attenuation achieved is not greatly dependent on the separation between control sources and primary sources, as indicated by Figure 4.28. Figure 4.29 shows that attenuation of acceleration level increases with increasing separation between control sources and error sensors. Very little attenuation is achieved with the error sensors located close to the control sources.
Figure 4.26  Mean attenuation downstream of the line of error sensors as a function of frequency, with the control actuators driven independently.
Figure 4.27  Mean attenuation downstream of the line of error sensors as a function of frequency, with the control actuators driven by the same signal.
Figure 4.28  Mean attenuation downstream of the line of error sensors as a function of control source - primary source separation.
Figure 4.29  Mean attenuation downstream of the line of error sensors as a function of error sensor - control source separation.
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4.3.4 Number of control sources required for optimal control

Table 4.3 shows the amount of attenuation of acceleration level achieved downstream of the error sensors with various numbers of control sources. The control sources are located at a single axial location. The locations of primary sources, control sources and error sensors and the cylinder dimensions used were those given in Table 4.1. The results given are for the simply supported cylinder.

<table>
<thead>
<tr>
<th>Number of Control Sources</th>
<th>Mean Attenuation (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.8225</td>
</tr>
<tr>
<td>2</td>
<td>10.883</td>
</tr>
<tr>
<td>3</td>
<td>35.902</td>
</tr>
<tr>
<td>4</td>
<td>36.976</td>
</tr>
<tr>
<td>5</td>
<td>36.976</td>
</tr>
<tr>
<td>6</td>
<td>36.976</td>
</tr>
</tbody>
</table>

4.3.5 Number of error sensors required for optimal control

Table 4.4 shows the control source amplitude and amount of attenuation of acceleration level achieved downstream of the error sensors with various numbers of error sensors. The error sensors were located at axial location $x_e = 1.0\text{m}$ and unevenly spaced circumferential locations. The other locations of primary sources and control sources and the cylinder dimensions used were those given in Table 4.1. The results given are for the simply-supported cylinder.
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Table 4.4
Effect of the Number of Error Sensors on Control
Source Amplitude and Mean Attenuation

<table>
<thead>
<tr>
<th>Number of Error Sensors</th>
<th>Mean Control Source Amplitude*</th>
<th>Mean Attenuation of Radial Acceleration (dB)</th>
<th>Mean Attenuation of Axial Acceleration (dB)</th>
<th>Mean Attenuation of Tangential Acceleration (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.14538</td>
<td>-0.62607</td>
<td>-0.93307</td>
<td>-0.82318</td>
</tr>
<tr>
<td>2</td>
<td>0.34850</td>
<td>8.0704</td>
<td>7.4881</td>
<td>6.8480</td>
</tr>
<tr>
<td>3</td>
<td>0.23683</td>
<td>36.035</td>
<td>32.944</td>
<td>32.050</td>
</tr>
<tr>
<td>4</td>
<td>0.24036</td>
<td>36.618</td>
<td>33.699</td>
<td>32.132</td>
</tr>
<tr>
<td>5</td>
<td>0.24036</td>
<td>36.619</td>
<td>33.700</td>
<td>32.131</td>
</tr>
<tr>
<td>6</td>
<td>0.24036</td>
<td>36.622</td>
<td>33.710</td>
<td>32.134</td>
</tr>
<tr>
<td>7</td>
<td>0.24036</td>
<td>36.626</td>
<td>33.713</td>
<td>32.137</td>
</tr>
<tr>
<td>8</td>
<td>0.24036</td>
<td>36.635</td>
<td>33.714</td>
<td>32.139</td>
</tr>
<tr>
<td>9</td>
<td>0.24036</td>
<td>36.644</td>
<td>33.721</td>
<td>32.141</td>
</tr>
<tr>
<td>10</td>
<td>0.24036</td>
<td>36.654</td>
<td>33.727</td>
<td>32.144</td>
</tr>
<tr>
<td>∞</td>
<td>0.24082</td>
<td>36.976</td>
<td>34.925</td>
<td>32.616</td>
</tr>
</tbody>
</table>

*Mean control source amplitude is expressed relative to the primary source amplitude. Six control sources and two primary sources were used.

4.3.6 Natural frequencies

Table 4.5 lists the natural frequencies of the cylinder described in Table 4.1, except that the effects of the ring stiffener have not been included. The experimental results of Section 4.5.1 indicate that the ring stiffener increases the natural frequencies of each mode by a small amount.
Table 4.5

Natural Frequencies of the Unstiffened Cylinder

<table>
<thead>
<tr>
<th>Circumferential Mode Number</th>
<th>Natural Frequency (Hz)</th>
<th>Axial Mode Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1331</td>
<td>1602</td>
</tr>
<tr>
<td>1</td>
<td>321</td>
<td>890</td>
</tr>
<tr>
<td>2</td>
<td>114</td>
<td>392</td>
</tr>
<tr>
<td>3</td>
<td>106</td>
<td>222</td>
</tr>
<tr>
<td>4</td>
<td>179</td>
<td>216</td>
</tr>
<tr>
<td>5</td>
<td>284</td>
<td>299</td>
</tr>
<tr>
<td>6</td>
<td>415</td>
<td>424</td>
</tr>
<tr>
<td>7</td>
<td>571</td>
<td>578</td>
</tr>
<tr>
<td>8</td>
<td>751</td>
<td>757</td>
</tr>
</tbody>
</table>
4.4 EXPERIMENTAL PROCEDURE

4.4.1 Modal analysis

A modal analysis was performed on the cylinder to be used in the vibration control experiment. The software package "PC Modal", a Brüel and Kjær type 8202 impact hammer and type 2032 signal analyser were used in the modal analysis. The modal analysis experimental arrangement is illustrated in Figure 4.30. The dimensions of the cylinder were the same as those given in Section 4.3 (see Table 4.1). The cylinder model consisted of 45 nodes dividing the cylinder into a line of 21 nodes parallel to the $x$-axis and a ring of 25 nodes parallel to the $\theta$-axis. The analysis was performed for the two cases with and without the angle stiffener attached to the cylinder.

![Experimental arrangement for the modal analysis of the cylinder.](image)
Chapter 4. Control of vibrations in a stiffened cylinder

4.4.2 Active vibration control

A steel stiffener was bolted tightly to a cylinder described by the dimensions given in Table 4.1. Six piezoceramic actuators were placed between the stiffener flange and the cylinder wall. The actuators were attached only at one end to ensure that no external tensile force were applied to them, as the type of actuator used is weak in tension. The primary source, control source and error sensor locations and the excitation frequency are given in Table 4.1.

The complete experimental arrangement is shown in Figure 4.31. The primary signal was produced by a signal analyser and amplified to drive the electrodynamic shakers (Figure 4.32). The shakers acted on the shell through force transducers, and the magnitudes of the primary forces were recorded using an oscilloscope.

The error signals from the ring of eight accelerometers (Figure 4.33) were passed to an EZ-ANC controller. The controller determined the control signals to drive the piezoceramic actuators, optimally minimising the acceleration measured by the error sensors. The control signals were also monitored on an oscilloscope.

The acceleration was measured at 10 or 15 cm intervals along the cylinder in four lines at locations $\theta = 0, \pi/6, \pi/4$ and $\pi/2$ (Figure 4.34). The accelerometer signals were read in turn through a 40 channel multiplexer connected to a Hewlett-Packard type 35665A signal analyser, in which the frequency response function was used to analyse the data. The magnitude and phase of the acceleration were recorded on a personal computer, which was also used to switch the
Figure 4.31 Experimental arrangement for the active control of vibration in the cylinder.
Figure 4.32  Primary system.

recorded channel on the multiplexer. The acceleration output of the force transducer at one of the primary source locations was used as the reference signal for the frequency response analysis. Accelerometer readings were taken initially once the error sensor signals had been optimally reduced, and again with the control amplifiers switched off (the uncontrolled case). The
Figure 4.33  Control system.

experiment was repeated with the six control actuators driven by a common control signal.
Figure 4.34 Acceleration measurement. Not all of the accelerometer - multiplexer connections are shown.
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Figures 4.35 - 4.38 show the photographs of the experimental equipment. In Figure 4.35, the cylinder is shown with the signal generating and recording equipment around it. The cylinder is simply supported at each end. The two electromagnetic shaker primary sources can be seen. The accelerometers mounted on the cylinder are shown in Figure 4.36. The ring stiffener can be seen through the open end of the cylinder in Figure 4.37, which also shows the EZ-ANC controllers. The piezoceramic stack actuators are shown in Figure 4.38, mounted at various angles 0 as described earlier in this section.

Figure 4.35 Experimental equipment for the active vibration control of cylinder vibration.
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Figure 4.36 Accelerometers mounted on the cylinder.

Figure 4.37 EZ-ANC controllers and control source amplifiers (foreground) with cylinder showing ring stiffener.
Figure 4.38  Piezoceramic stack actuators and the ring stiffener.
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4.5 EXPERIMENTAL RESULTS

4.5.1 Modal analysis

The following figures show the modal analysis results. Each figure represents one ring around the circumference of the cylinder (located at $x = 0.2$ m) and one line along the cylinder.

Figures 4.39 and 4.40 show the $n,1$ ($n = 2,3,4,5$) modes of vibration of the cylinder without the angle stiffener. These modes were significant on the unstiffened cylinder, but were not found to be significant modes for the cylinder with the ring-stiffener in place. Figures 4.41 - 4.43 compare the $5,2$, $6,2$ and $7,3$ modes for the unstiffened and stiffened cylinder. The presence of the ring-stiffener greatly reduced the vibration amplitude of these modes also. However, the presence of the ring-stiffener did not significantly affect the $n,4$ modes (Figures 4.44 and 4.45), presumably because the stiffener was located at $\frac{1}{4}$ the length of the cylinder from one end, which is the location of a node in an $n,4$ vibration mode. The assumption that the cylinder displacement was limited at the stiffener location is borne out by these results.

It is also of interest that the higher order circumferential modes ($n,5$, $n,6$, $n,7$) are a significant part of the vibration response of the cylinder, as predicted by the theoretical model. The natural frequencies of the unstiffened cylinder also agree closely with the theoretical predictions (see Section 4.3.6).
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Figure 4.39 The 2,1 and 3,1 modes for the unstiffened cylinder.

Figure 4.40 The 4,1 and 5,1 modes for the unstiffened cylinder.

Figure 4.41 The 5,2 mode for the unstiffened and stiffened cylinder.
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Figure 4.42  The 6,2 mode for the unstiffened and stiffened cylinder.

Figure 4.43  The 7,3 mode for the unstiffened and stiffened cylinder.

Figure 4.44  The 5,4 mode for the unstiffened and stiffened cylinder.
4.5.2 Active vibration control

Figure 4.46 shows the theoretical and experimental acceleration distributions for each of the four lines where accelerometers were placed in the experiment. For both the uncontrolled case and the controlled case with control actuators driven by the same signal, the experimental results and theoretical curves are in close agreement. For the controlled case with independently driven control sources, the theoretical analysis predicts greater reduction in acceleration level than was achieved experimentally. An error analysis showed that a very small error (0.12%) in the control signal would produce a decrease in attenuation corresponding to the difference between the experimental and theoretical data.

Table 4.6 compares the acceleration levels of radial, axial and tangential acceleration measured at four points on the cylinder both without and with active vibration control. The measurements were made using tri-axial accelerometers. The table shows that axial and tangential acceleration
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Figure 4.46 Experimental acceleration distributions.

(1) Uncontrolled case; (2) Controlled case. Theoretical results and experimental data for the case with three control sources driven by the same signal; Theoretical results and experimental data for the case with independently driven control sources.

(a) $\theta = \pi/6$; (b) $\theta = 11\pi/24$; (c) $\theta = 7\pi/12$; (d) $\theta = 3\pi/4$.

$x_p =$ primary source location; $x_t =$ control source location; $x_e =$ error sensor location.
Table 4.6

<table>
<thead>
<tr>
<th>Measurement Location</th>
<th>Radial Acceleration Level (dB)</th>
<th>Axial Acceleration Level (dB)</th>
<th>Tangential Acceleration Level (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x (m)</td>
<td>θ (rad)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uncontrolled</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>π/6</td>
<td>-0.8</td>
<td>-1.2</td>
</tr>
<tr>
<td>0.2</td>
<td>π/3</td>
<td>1.1</td>
<td>0.4</td>
</tr>
<tr>
<td>1.8</td>
<td>π/6</td>
<td>-7.2</td>
<td>-7.7</td>
</tr>
<tr>
<td>1.8</td>
<td>π/3</td>
<td>-6.8</td>
<td>-7.4</td>
</tr>
<tr>
<td>Controlled</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>π/6</td>
<td>-0.9*</td>
<td>0.2*</td>
</tr>
<tr>
<td>0.2</td>
<td>π/3</td>
<td>-0.7*</td>
<td>-0.3*</td>
</tr>
<tr>
<td>1.8</td>
<td>π/6</td>
<td>-20.6</td>
<td>-20.6</td>
</tr>
<tr>
<td>1.8</td>
<td>π/3</td>
<td>-21.9</td>
<td>-22.4</td>
</tr>
</tbody>
</table>

*These values are measurements from upstream of the control location.

are significantly reduced by minimising radial acceleration at the ring of error sensors.

The measurements taken at $\theta = \pi/6$ radians support the theoretical finding that nodes in the tangential vibration occur at opposite circumferential locations to nodes in axial and radial vibration. At $\theta = \pi/6$, the tangential acceleration level is several dB less than the levels recorded for axial and radial acceleration, while the levels are similar at the other circumferential location recorded.
Chapter 4. Control of vibrations in a stiffened cylinder

4.6 SUMMARY

A theoretical model has been developed to describe the vibration response of a ring-stiffened cylinder to a range of excitation types, and in particular to describe the vibration response of ring-stiffened cylinders to point force primary excitation sources and angle stiffener and piezoceramic stack control sources. The numerical results indicate that flexural vibrations in cylinders can be actively controlled using piezoceramic stack actuators placed between the flange of an angle stiffener and the cylinder surface. Numerical results also indicate:

(1) Vibration in the axial, tangential and radial directions is coupled, and vibration amplitudes in each direction are of a similar order of magnitude for the cylinders considered.

(2) Circumferential modes in the axial and radial vibration standing waves occur at the same circumferential locations on the cylinder while modes of tangential vibration are located out-of-phase relative to the axial and radial modes.

(3) Optimally controlling radial vibration also significantly reduces axial and tangential vibration levels.

(4) The mean amplitude of the control forces required for optimal control is not greatly dependent on frequency, axial control source location or axial error sensor location. There are minor fluctuations, particularly for the finite cylinder, but no pattern or
Chapter 4. Control of vibrations in a stiffened cylinder

increasing or decreasing trends.

(5) The optimum control forces are either in phase or $180^\circ$ out of phase with the primary sources. This is true for the semi-infinite cylinder as well as the finite cylinder, because a standing wave is generated by the vibration reflections from the finite end and the angle stiffener.

(6) Increasing the separation between the primary and control sources does not greatly affect the mean attenuation of acceleration level downstream of the ring of error sensors.

(7) Increasing the separation between the error sensors and the control sources significantly improves the mean attenuation of acceleration level downstream of the ring of error sensors.

(8) Because of the distance between nodes in axial standing waves in the cylinders considered, there are no axial locations of control sources and error sensors that give maxima in control source amplitude or minima in attenuation.

(9) Little or no reduction is achieved with control sources driven by a common control signal, because higher order circumferential modes of vibration contribute significantly to the vibration response of the cylinder even at low frequencies.
Chapter 4. Control of vibrations in a stiffened cylinder

(10) The circumferential location of the control sources is significant. Generally, for every control source, there are two locations at which placement of an additional control source will require excessive control source amplitudes for optimal control.

(11) For the cylinder and frequency considered, optimal attenuation can be achieved with four control sources and four error sensors. Attenuation is only a little less with three control sources and three error sensors, but very little control is achieved with two or less control sources and error sensors.

The theoretical model outlined was verified experimentally for the cylinder with simply supported ends. A modal analysis of the cylinder indicated that the angle stiffener made a significant difference to the vibration response of the cylinder and that higher order circumferential modes contributed significantly to the overall response. Comparison between experimental results and theoretical predictions for the vibration of the cylinder with and without active vibration control showed that:

(1) The theoretical model accurately predicted the vibration response of the cylinder for the uncontrolled case and the case with control sources driven by the same signal.

(2) The theoretical model predicted more attenuation than could be achieved experimentally for the case with independently driven control sources. An error analysis indicated that an error in the control source signal of 0.12% would produce a decrease in attenuation
Chapter 4. Control of vibrations in a stiffened cylinder

corresponding to the difference between the theoretical prediction and the experimental result. Nevertheless, around 18 dB attenuation was achieved experimentally for the case with independently driven control sources.

(3) Experimental measurements of axial and tangential vibration were of similar order to measurements of radial vibration, as predicted by the theoretical model. Axial and tangential vibration was significantly reduced as well as radial vibration when radial vibration was used as the error function.
CHAPTER 5. SUMMARY AND CONCLUSIONS

5.1 SUMMARY OF NUMERICAL ANALYSIS

The application of piezoceramic stack actuators as actuators for active vibration control has been examined for three types of physical structure; beams, plates and cylinders. For each type of structure, a theoretical model has been developed to describe the vibration response of the structure to excitation by point forces and to active vibration control using piezoceramic stack control actuators and vibration sensors. The effect of stiffeners on the vibration response of the structures has been included where appropriate.

The analysis of vibration in a beam was a one dimensional problem. An analytical model was developed from the one dimensional equation of motion for beams to describe the vibration response of an arbitrarily terminated beam to a range of excitation types. The numerical results indicated that flexural vibrations in beams can be effectively controlled using a single piezoceramic stack actuator and angle stiffener control source and a single vibration error sensor.

The dependence of the control force amplitude and phase and the attenuation of vibration level achieved on a number of parameters was investigated. It was found that the magnitude of the control force required for optimal control generally decreased with increasing stiffener flange length and increasing frequency. The control force amplitude required for optimal control was less when the beam was excited at a resonance frequency.
Chapter 5. Conclusions

When there was reflection from the beam terminations, the optimum control force was either in phase or 180° out of phase with the primary source, but when there was no reflection from beam terminations (the infinite beam case), the control force phase cycled through 180° as the excitation frequency was increased.

Comparison between the results obtained for infinite beam and finite beam cases indicated that the standing wave generated when there were reflections from beam terminations had a significant effect on the effectiveness of active vibration control. Maxima occurred in the control force amplitude required for optimal control when the separation between control and primary sources was given by \( x = (c + n\lambda_p/2) \) where \( n \) is an integer and \( c \) is a constant dependent on frequency and the type of boundary condition. These maxima occurred when the control source was located at a node in the standing wave generated by reflection from the beam terminations. Minima in the mean attenuation of acceleration level downstream of the error sensor also occurred when the control source was located at a node in the standing wave.

When the error sensor was located at a node in the standing wave in a finite beam, the attenuation achieved was less than that achieved with the error sensor located away from a node. However, locating the error sensor at a node did not affect the control force amplitude required for optimal control.

Increasing the separation between the primary and control source did not improve the
Chapter 5. Conclusions

attenuation. The amount of attenuation achieved downstream of the error sensor increased
with increasing separation between the error sensor and the control source.

Numerical results also indicated that it is possible to achieve reductions in acceleration level
upstream of the primary source as well as the desired reduction downstream of the error
sensor. The maximum mean attenuation in acceleration level upstream of the primary source
was theoretically achieved with the separation between primary and effective control source
locations given by $x = (c + n\lambda_n/2)$ for $n = 1,2,3...$ For error sensor locations outside the
control source near field, the mean attenuation of acceleration level upstream of the primary
source did not depend on error sensor location.

An analytical model was developed from the two dimensional equation of motion for plates
to describe the vibration response of a plate to a range of excitation types. Two sets of
boundary conditions were considered, both with simply supported sides. One plate was
modelled as free at both ends, and the second was modelled as semi-infinite in length. The
effects of the mass and stiffness of an across-plate stiffener were included in the analysis.
The numerical results indicated that flexural vibrations in stiffened plates can be effectively
controlled using the piezoceramic stack actuators and a line of vibration error sensors.

The numerical model showed that control of plate vibration is a similar problem to control of
vibration in beams, but the added width dimension necessitated more than one control source
and error sensor across the plate width to achieve optimum control.
Chapter 5. Conclusions

Like the corresponding beam case, the mean amplitude of the control forces required for optimal control of plate vibration generally decreased with increasing frequency. The optimum control forces were either in phase or $180^\circ$ out of phase with the primary sources. This was true for the semi-infinite plate as well as the finite plate, because a standing wave was generated by the vibration reflections from the finite end and the angle stiffener. This was not so for the infinite beam case, as the beam was modelled as infinite in both directions and the mass loading of the angle stiffener was neglected.

Maxima occurred in the mean control force amplitude required for optimal control when the separation between control and primary forces was given by $x = (c + nx_s)$ where $n$ is an integer, $c$ is a constant dependent on frequency and the type of boundary condition, and $x_s$ is the separation between axial nodes in a standing wave. These maxima occurred when the control sources were located at a nodal line in the standing wave generated by reflection from the plate termination and the angle stiffener. Minima in the mean attenuation of acceleration level downstream of the error sensor occur when the control sources were located at a nodal line in the standing wave. Increasing the separation between the primary and control sources did not improve attenuation provided the control sources were located outside of the primary source near field.

The amount of attenuation achieved downstream of the error sensors increased with increasing separation between the error sensors and the control sources. When the line of error sensors was located at a nodal line in the standing wave that exists in finite plates, the
Chapter 5. Conclusions

attenuation achieved was less than that achieved with the error sensors located away from a node. Locating the error sensors at a node did not affect the amplitude of the control forces required for optimal control.

As more than one control source was used in the plate model, consideration was given to the effectiveness of active vibration control with the control sources driven by a common signal and with the control sources driven independently. At low frequencies, there was very little difference in the mean control effort required for optimal control and the mean attenuation downstream of the line of error sensors achieved, between control using independent control sources and control sources driven by a common signal. At higher frequencies when higher order across-plate modes became significant, very little attenuation was achieved with control sources driven by a common signal. Good reduction in acceleration level was achieved with independently driven control sources right across the frequency range considered. It was determined that use of three independently driven control sources and three error sensors was sufficient to give optimal attenuation.

Both the beam model and the plate model were extended to include a second angle stiffener and control source(s) downstream from the first, with the aim of controlling the vibration better when the first control source was located at a node in a standing wave. The magnitude of the first control force(s) was limited to a maximum value, and the second control source(s) used when the limit was reached. The maxima in control force amplitude and the minima in attenuation that occurred when the first stiffener and control source(s) were located at a
Chapter 5. Conclusions

standing wave node were eliminated in this way.

It was shown, for the beam case, that there was no simple practical method of using a second error sensor to eliminate the minima in attenuation that occur when the first error sensor is located at a standing wave node, because of the difficulty in determining when the first error sensor was located at a node without the introduction of several more error sensors.

The analysis of vibration in a cylinder was a significantly more complicated problem than the beam and plate cases. The Flügge equations for the vibration of a cylinder in the radial, axial and tangential directions were used to develop a model to describe the vibration response of a ring-stiffened cylinder to a range of excitation types, and in particular to point force primary excitation sources and angle stiffener and piezoceramic stack control sources. The numerical results indicated that flexural vibrations in cylinders can be actively controlled using piezoceramic stack actuators placed between the flange of an angle stiffener and the cylinder surface.

The cylinder vibration response was different to the simpler cases of the plate and beam in two main ways. In the cylinder, vibration in the axial, tangential and radial directions was significant, and the separation $x_s$ between axial nodes in standing waves was much longer than that in the plate and beam cases considered.

Vibration amplitudes in the radial, axial and tangential directions were of similar order.
Chapter 5. Conclusions

Modes of axial and radial vibration occurred at the same tangential locations while modes of tangential vibration were located out-of-phase relative to the axial and radial modes. Optimally controlling radial vibration also significantly reduced axial and tangential vibration levels.

Because the separation $x_s$ between axial standing wave nodes in the cylinder was so large, the mean amplitude of the control forces required for optimal control did not greatly dependent on frequency, axial control source location or axial error sensor location. There were minor fluctuations, particularly for the finite cylinder, but no pattern or increasing or decreasing trends. There were no axial locations of control sources and error sensors that gave maxima in control force amplitude or minima in attenuation.

Like both the plate cases and all but the infinite beam cases, the optimum control forces were either in phase or 180° out of phase with the primary sources. This was true for the semi-infinite cylinder as well as the finite cylinder, because a standing wave was generated by the vibration reflections from the finite end and the angle stiffener.

Increasing the separation between the primary and control sources did not greatly affect the attenuation achieved downstream of the ring of error sensors. Increasing the separation between the error sensors and the control sources improved the mean attenuation of acceleration level downstream of the ring of error sensors.
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Little or no reduction was achieved with control sources driven by a common control signal, because higher order circumferential modes of vibration contribute significantly to the vibration response of the cylinder even at low frequencies.

The circumferential location of the control sources was significant. Generally, for every control source, there were two locations at which placement of an additional control source required excessive control force amplitudes for optimal control. Four control sources and four error sensors were sufficient to achieve optimal attenuation of cylinder vibration.
Chapter 5. Conclusions

5.2 SUMMARY OF EXPERIMENTAL RESULTS

For at least one example of each of the three structure types considered, the theoretical model outlined was tested experimentally. In each experiment, the acceleration distribution resulting from measured force inputs was gauged and compared with the theoretical predictions, both with and without active vibration control.

Experiments were performed on a beam with four different sets of end conditions. The impedance corresponding to each termination was first calculated from experimental data. Comparison between experimental results and theoretical predictions showed that the accuracy of the theoretical model when compared to the experimental results was very high, both in predicting the control force amplitude and phase required relative to the primary force, and in determining the acceleration distribution occurring along the beam. The impedances calculated from experimental measurements gave more accurate results than the "classical" impedances corresponding to each termination. The theoretical model accurately predicted the amount of attenuation that could be achieved experimentally.

The theoretical model outlined for the plate was verified experimentally for a plate with simply supported sides, one end free and the other end anechoically terminated. A modal analysis of the plate indicated that the anechoic termination allowed some reflection and so did not exactly model the ideal infinite end, and that the angle stiffener made a significant difference to the vibration response of the plate. Comparison between experimental results and theoretical predictions for the vibration of the plate with and without active vibration
Chapter 5. Conclusions

control showed that the theoretical model accurately predicted the vibration response of the plate for the uncontrolled case and the case with control sources driven by the same signal. The angle stiffener reflected more of the vibration and transmitted less than the theoretical model predicted. The theoretical model predicted more attenuation than could be achieved experimentally for the case with independently driven control sources. An error analysis indicated that an error in the control source signal of 0.1% would have produced a decrease in attenuation corresponding to the difference between the theoretical prediction and the experimental result. Nevertheless, around 25 dB attenuation was achieved experimentally for the case with independently driven control sources.

Experiments were performed on a cylinder with simply supported ends. A modal analysis of the cylinder indicated that the angle stiffener made a significant difference to the vibration response of the cylinder and that higher order circumferential modes contributed significantly to the overall cylinder vibration response. Comparison between experimental results and theoretical predictions for the vibration of the cylinder with and without active vibration control showed that the theoretical model accurately predicted the vibration response of the cylinder for the uncontrolled case and the case with control sources driven by the same signal. The theoretical model predicted more attenuation than could be achieved experimentally for the case with independently driven control sources. Again, an error analysis indicated that an error in the control source signal of 0.1% would have produced a decrease in attenuation corresponding to the difference between the theoretical prediction and the experimental result. Around 18 dB attenuation was achieved experimentally for the case with
Chapter 5. Conclusions

independently driven control sources. Finally, experimental measurements indicated that the amplitudes of axial and tangential vibration were of similar order to the radial vibration amplitude, as predicted by the theoretical model, for both uncontrolled and controlled cases.
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5.3 Conclusions

It is possible to actively control vibration in stiffened structures using actuators placed between a stiffener flange and the structure surface. Actuators placed across the width of a plate-like structure or around the circumference of a cylindrical structure at a single axial location can be used in conjunction with a line or ring of error sensors to significantly reduce the vibration transmission along the structure.

The theoretical models outlined in this thesis can be used to determine the maximum amount of vibration reduction that can be achieved under ideal conditions. It may not be possible to achieve the predicted level of reduction in practice, but high levels of attenuation are achievable.

When the separation between nodes in the axial standing wave in the structure is less than the length of the structure, for any given frequency of excitation, there will be discrete axial locations of control sources that will not yield high levels of attenuation, or, for given locations of control sources, there will be certain frequencies at which vibration cannot be controlled effectively. This occurs when the control source location or error sensor location corresponds to a node in a standing wave generated by reflections from the structure terminations and stiffeners. It is possible to overcome this difficulty simply with a second set of control sources at another axial location. The level of attenuation is also reduced, but not as much, when the error sensor location corresponds to a standing wave node. Introduction of a second set of error sensors does not yield a simple solution to this problem.
Chapter 5. Conclusions

The separation between nodes in the axial standing wave in typical cylindrical structures is very long, so axial location of control sources and error sensors is not critical.

Increasing the separation between the error sensors and the control sources increases the maximum reduction in acceleration level that can be achieved in each type of structure considered. Increasing the separation between the control sources and the primary sources does not greatly affect the amount of reduction in acceleration level that can be achieved.

The effect of the circumferential location of control sources in cylindrical structures has been briefly investigated in this thesis, and is significant. Control sources are most effectively used when located at different circumferential spacings where each control source affects different circumferential modes of vibration. The circumferential location of the error sensors has not been investigated here, but it is suggested that the error sensors should be similarly unevenly spaced circumferentially.
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PAPERS PUBLISHED IN AND SUBMITTED TO REFEREED JOURNALS


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