Optimal spatially fixed and moving virtual sensing algorithms for local active noise control

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Abstract

Local active noise control systems aim to create zones of quiet at specific locations within a sound field. The created zones of quiet generally tend to be small, especially for higher frequencies, and are usually centred at the error sensors. For an observer to experience significant reductions in the noise, the error sensors therefore have to be placed relatively close to an observer’s ears, which is not always feasible or convenient. Virtual sensing methods have been proposed to overcome these problems that have limited the scope of successful local active noise control applications. These methods require non-intrusive sensors that are placed remotely from the desired locations of maximum attenuation. These non-intrusive sensors are used to provide an estimate of the sound pressures at these locations, which can then be minimised by a local active noise control system. This effectively moves the zones of quiet away from the physical locations of the transducers to the desired locations of maximum attenuation, such as a person’s ears.

A number of virtual sensing algorithms have been proposed previously. The difference between these algorithms is the structure that is assumed to compute an estimate of the virtual error signals. The question now arises as to whether there is an optimal structure that can be used to solve the virtual sensing problem, which amounts to a linear estimation problem. It is well-known that the Kalman filter provides an optimal structure for solving such problems. An optimal solution to the virtual sensing problem is therefore derived in this thesis using Kalman filtering theory. The proposed algorithm is implemented on an acoustic duct arrangement to demonstrate its effectiveness. The presented experimental results indicate that the zone of quiet was effectively moved away from the physical sensor towards the desired location of maximum attenuation.

The previously proposed virtual sensing algorithms have been developed with the aim to create zones of quiet at virtual locations that are assumed spatially fixed within the sound field. Because an observer is very likely to move their head, the desired locations of the zones of quiet are generally moving through the sound field rather than being spatially fixed. For effective control, a local active noise control system incorporating a virtual sensing method thus has to be able to create moving zones of quiet that track the observer’s ears. A moving virtual sensing method is therefore developed in this thesis that can be used to estimate the error signals at virtual locations that are moving through the sound field. It is shown that an optimal solution to the moving virtual sensing problem can be derived using Kalman filtering theory. A practical implementation of the developed algorithm is combined with an adaptive feedforward control algorithm and implemented on an acoustic duct arrangement. The presented experimental results illustrate that a narrowband moving zone of quiet that tracks the desired location of maximum attenuation has successfully been created.
Statement of originality

To the best of my knowledge, except where otherwise referenced and cited, all the material that is presented in this thesis is my own original work and has not been published previously for the award of any other degree or diploma in any university or other tertiary institution. If accepted for the award of the degree of Doctor of Philosophy in Mechanical Engineering, I consent that this thesis be made available for loan and photocopying.

Cornelis D. Petersen
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**Nomenclature**

**Signals**

- $d_p$: physical primary disturbance
- $d_v$: virtual primary disturbance
- $e_p$: physical error signal
- $e_v$: virtual error signal
- $r_p$: physical filtered-reference signal
- $r_v$: virtual filtered-reference signal
- $s$: disturbance source signal
- $u$: control signal
- $x$: feedforward reference signal
- $x_{fb}$: intrinsic feedback signal
- $y_p$: physical secondary disturbance
- $y_v$: virtual secondary disturbance
- $z$: state
- $\varepsilon_p$: physical innovation, physical output error
- $\varepsilon_v$: virtual innovation, virtual output error
- $\rho$: state estimation error

**Transfer paths and impedances, controllers and filters**

- $C$: feedback controller
- $G$: arbitrary plant or system
- $G_{ci}$: co-inner factor of $G$, $G_{ci}G_{ci}^* = I$
- $G_{co}$: co-outer factor of $G$, $G_{co}G_{co}^* = GG^*$
- $G_i$: inner factor of $G$, $G_i^*G_i = I$
- $G_o$: outer factor of $G$, $G_o^*G_o = G^*G$
- $G_{ci}^\perp$: co-inner factor perpendicular to $G_{ci}$, $G_{ci}^\perp G_{ci}^{\perp*} = I$ and $G_{ci}G_{ci}^{\perp*} = 0$
- $G_i^\perp$: inner factor perpendicular to $G_i$, $G_i^{\perp*}G_i^\perp = I$ and $G_i^{\perp*}G_i = 0$
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$G_{ps}$</td>
<td>physical primary transfer path</td>
</tr>
<tr>
<td>$G_{pu}$</td>
<td>physical secondary transfer path</td>
</tr>
<tr>
<td>$G_{xs}$</td>
<td>detector transfer path</td>
</tr>
<tr>
<td>$G_{xu}$</td>
<td>intrinsic feedback path</td>
</tr>
<tr>
<td>$G_{vs}$</td>
<td>virtual primary transfer path</td>
</tr>
<tr>
<td>$G_{vu}$</td>
<td>virtual secondary transfer path</td>
</tr>
<tr>
<td>$H$</td>
<td>filter</td>
</tr>
<tr>
<td>$W$</td>
<td>feedforward controller</td>
</tr>
<tr>
<td>$Z_{ps}$</td>
<td>physical primary transfer impedance</td>
</tr>
<tr>
<td>$Z_{pu}$</td>
<td>physical secondary transfer impedance</td>
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<td>intrinsic feedback impedance</td>
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<tr>
<td>$Z_{vs}$</td>
<td>virtual primary transfer impedance</td>
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<tr>
<td>$Z_{vu}$</td>
<td>virtual secondary transfer impedance</td>
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**Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>frequency (Hz)</td>
</tr>
<tr>
<td>$h$</td>
<td>physical sensor weight</td>
</tr>
<tr>
<td>$n$</td>
<td>time index</td>
</tr>
<tr>
<td>$x_p$</td>
<td>spatial location of physical sensor/primary source</td>
</tr>
<tr>
<td>$x_s$</td>
<td>spatial location of secondary source</td>
</tr>
<tr>
<td>$x_v$</td>
<td>spatial location of virtual sensor</td>
</tr>
<tr>
<td>$w$</td>
<td>control filter coefficient</td>
</tr>
<tr>
<td>$z$</td>
<td>complex variable in the $z$-transform, unit shift forward operator</td>
</tr>
<tr>
<td>$I$</td>
<td>number of control filter coefficients</td>
</tr>
<tr>
<td>$J$</td>
<td>cost function</td>
</tr>
<tr>
<td>$K$</td>
<td>number of feedforward reference sensors</td>
</tr>
<tr>
<td>$L$</td>
<td>number of control sources</td>
</tr>
<tr>
<td>$M_p$</td>
<td>number of physical sensors</td>
</tr>
<tr>
<td>$M_v$</td>
<td>number of virtual sensors</td>
</tr>
<tr>
<td>$N$</td>
<td>order of state-space system</td>
</tr>
<tr>
<td>$P$</td>
<td>covariance of state estimation error, inverse correlation matrix</td>
</tr>
<tr>
<td>$Q$</td>
<td>covariance of process noise</td>
</tr>
<tr>
<td>$R$</td>
<td>covariance of measurement noise</td>
</tr>
<tr>
<td>$S$</td>
<td>cross-covariance between process and measurement noise</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>convergence coefficient (normalised filtered-x LMS algorithm)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>regularisation parameter (filtered-x RLS algorithm)</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>regularisation parameter (normalised filtered-x LMS algorithm)</td>
</tr>
</tbody>
</table>
η attenuation (dB)
λ forgetting factor (filtered-x RLS algorithm)
µ convergence coefficient (filtered-x LMS algorithm)
ω angular frequency (rad/s)
σ variance
Π covariance of states

* discrete-time convolution
⊗ Kronecker matrix product
R field of real numbers
C field of complex numbers
∈ belong to
⊥ perpendicular to
∀ for all
□ end of proof
∼ defined by
≜ defined as
≈ approximately
≃ similar or equivalent

RH^{m\times n}_m set of all all rational m × n transfer function matrices with real coefficients and excluding singularities on the unit circle
RH^{m\times n}_∞ set of all asymptotically stable rational m × n transfer functions matrices with real coefficients
RH^{m\times n}_p set of all all rational proper m × n transfer function matrices with real coefficients

10 \log[\cdot] logarithm operator with base 10
var[\cdot] variance operator
E[\cdot] expectation operator
[\cdot]^* adjoint operator
[\cdot]^+ causality operator
[\cdot]^- non-causality operator
\| \cdot \|_2 H_2-norm operator
\cdot | magnitude operator
∠ phase operator
Matrix conventions

\( a \)  scalar
\( a \)  column vector
\( a^T \)  transpose of \( a \)
\( a^H \)  complex conjugate transpose of \( a \)
\( A \)  matrix
\( A^H \)  complex conjugate transpose of \( A \)
\( A^T \)  transpose of \( A \)
\( A^{-1} \)  inverse of \( A \)
\( A^{-H} \)  inverse of \( A^H \)
\( A^{-T} \)  inverse of \( A^T \)
\( A^\dagger \)  generalised or pseudo-inverse of \( A \)
\( I \)  identity matrix
\( \text{tr}(A) \)  trace of \( A \)
\( \kappa(A) \)  condition number of \( A \)

Abbreviations

AD  analogue to digital
ANC  active noise control
BPF  blade passage frequency
DA  digital to analogue
DARE  discrete-time algebraic Ricatti equation
FIR  finite impulse response
IIR  infinite impulse response
IMC  internal modal control
LMS  least mean-squares
LQG  linear quadratic gaussian
MIMO  multiple input multiple output
NMSV  normalised mean-square virtual output error
PEM  prediction error method
PO-MOESP  past output multi-variable output error state space
RLS  recursive least squares
SISO  single input single output
SMI  subspace model identification
SNR  signal to noise ratio
VAF  variance accounted for
Chapter 1

Introduction

1.1 Background

Occupational and community noise is now recognised as a serious contributor to environmental and occupational health issues. Prolonged exposure to excessive noise levels can cause hearing impairment, hypertension and ischaemic heart disease, adversely affect performance in reading, attentiveness, problem solving and memory, disturb sleep, and provoke annoyance responses and changes in social behaviour [1]. Consequently, commercial organisations are faced with more stringent noise regulations in order to ensure a safe working environment for their employees and to guarantee the safety of their customers. These customers also expect products to be more comfortable and quiet, which increases the demand for products and environments that are relatively silent and noise free. The traditional solution to occupational and community noise problems is to use passive control techniques [6]. These techniques are usually very effective at higher frequencies but generally require impractical bulky damping materials at lower frequencies. At low frequencies, active control techniques often provide a better and more practical solution to noise problems.

Active control techniques for noise problems

Since the 1980s, when fast and reliable digital signal processors became available, a substantial field of research into active control techniques for noise problems has emerged, which has been comprehensively detailed in a number of books [25, 48, 68, 88]. The active control techniques that have been developed since that decade can be divided into four main categories that are referred to as active vibration isolation control, active vibration control, active structural acoustic control, and active noise control techniques. The differences between these techniques can conveniently be discussed by considering the practical problem of engine-induced noise inside the cabin of a vehicle. In this
example of structure-borne noise, the vibrating engine, i.e. the disturbance source, is exciting the vehicle cabin, i.e. the receiving structure, which then radiates undesirable noise into the vehicle interior. If an active vibration isolation control approach is used to solve this noise problem, the aim is to isolate the engine vibrations from the vehicle cabin structure such that this structure is not vibrating and thus radiating noise into the vehicle interior. This is in contrast to an active vibration control approach, where instead of isolating the engine vibrations from the vehicle cabin structure, the aim is to reduce the vibrations of this structure. The desired reduction of these vibrations is generally achieved by appropriately exciting the structure using, for example, shakers or piezo-electric actuators. In active structural acoustic control, the aim is to control the vibrations of the vehicle cabin structure such that less noise is radiated into the vehicle interior. The difference with active vibration control is that instead of simply reducing the vibrations of the structure, knowledge about the interaction between the vibrating structure and the sound field inside the vehicle interior is taken into account to control the vibrations such that less noise is radiated into the vehicle interior. When an active vibration or structural acoustic control approach is adopted instead of an active vibration isolation control approach, the discussed structure-borne noise problem is thus solved further away from the disturbance source, i.e. the vibrating engine. When an active noise control approach is adopted, the aim is to reduce the noise inside the vehicle cabin by introducing anti-noise with loudspeakers. In comparison to the previously discussed techniques, the structure-borne noise problem is thus solved another step away from the disturbance source. The research presented in this thesis is mainly focussed on active noise control systems.

Active noise control

Active noise control systems aim to reduce unwanted or primary noise by actively adding anti- or secondary noise with loudspeakers. The principle is based on the fact that in most practical applications, the acoustic field of interest can be described by linear partial differential equations such that the principle of superposition holds [25]. This means that when the introduced anti-noise has equal amplitude and opposite phase to the unwanted noise, destructive interference between these noises occurs which results in the desired attenuation of the unwanted noise. The principle of active noise control was first suggested in a patent by Lueg [72], in which the three main components of an active noise control system were already identified, which are a sensor component, an actuator component, and a controller component. The sensor component generally consists of a number of microphones that measure the sound pressure at a number of locations in the acoustic field of interest. The actuator component generally consists of a number of loudspeakers that produce the desired anti-noise, which is generated to destructively interfere with the unwanted noise. These loudspeakers are also referred
to as the control or secondary sources and are driven by a control signal that is computed by the controller component. This component generally uses the measurements of the sound field provided by the sensor component to compute an appropriate control signal. The research presented in this thesis contributes to the development of sensing methods that aim to improve the performance of an active noise control system, and is therefore mainly focussed on the sensing component of an active noise control system.

Global and local active noise control

The field of active noise control is generally divided into global and local active noise control techniques [25]. In global active noise control, the aim is to reduce the unwanted noise throughout the entire acoustic domain of interest. When this approach is employed in the previously introduced example of engine-induced noise inside a vehicle cabin, the aim is to reduce the unwanted noise throughout the entire cabin. An immediate advantage of this approach is that when significant global control of the unwanted noise is obtained, the observer can freely move their head through the vehicle cabin while still experiencing a reduction in the unwanted noise. Unfortunately, global active noise control systems do not often attenuate the unwanted noise to an acceptable level in practical situations, especially in modally dense enclosures and in complex sound fields that have a large spatial variance [88]. In such cases, a local active noise control approach might be the only feasible solution. Instead of aiming to reduce the unwanted noise throughout the entire acoustic domain of interest, this approach aims to create local zones of quiet, for instance around the ears of a passenger inside a vehicle cabin. The research presented in this thesis is concentrated on local active noise control systems.

Limitations on local active noise control performance

An issue that has limited the scope of successful local active noise control applications is that the greatest noise reductions are usually achieved at the locations of the error microphones. Because these microphones cannot always be placed close enough to the observer’s ears, the created zones of quiet are generally not situated at the desired locations of maximum attenuation. This is further exacerbated by the fact that the size of the zones of quiet created around the error microphones generally tends to be small. The described problem is depicted in Fig. 1.1(a) where it is illustrated that although a significant reduction in the unwanted noise can usually be obtained at the error microphone, which is denoted by the physical microphone, the created zone of quiet is generally too small to extend to the observer’s ear.
Figure 1.1: Schematic representations [62] of the zone of quiet created by a local active noise control system using (a) a traditional sensing method; (b) a virtual sensing method.

Virtual sensing

A possible method to overcome the problem illustrated in Fig. 1.1(a) is to use a virtual sensing method [14, 27, 66, 104, 112], which has been illustrated in Fig. 1.1(b). In this figure, the measurement of the sound pressure provided by the physical microphone is used to compute an estimate of the sound pressure at a virtual microphone that is positioned at the desired location of maximum attenuation, i.e. the observer’s ear. This location is also referred to as the virtual location in the following. The estimate of the sound pressure at the virtual location is then minimised using a local active noise control system such that the zone of quiet is effectively moved away from the physical microphone to the observer’s ear, where maximum attenuation is required. The term virtual microphone refers to a fictitious microphone that is positioned at the desired location of maximum attenuation and which provides an estimate of the sound pressure at this location rather than a direct measurement. The term physical microphone denotes a microphone that provides a direct measurement of the sound pressure at a location remote from the virtual microphone, which is used to compute an estimate of the sound pressure at the virtual microphone.

Moving virtual sensing

The virtual sensing algorithms that have been proposed previously [14, 27, 66, 104, 112] aim to move the zone of quiet away from the physical microphone to a virtual location that is assumed spatially fixed within the sound field. This effectively means that these algorithms can be used when the observer illustrated in Fig. 1.1(b) is not moving their head, which is usually not the case in a practical situation. When the observer moves
their head away from the spatially fixed virtual location for which the virtual sensing algorithm has been designed in a preliminary identification stage, an increase in the unwanted noise is generally experienced by the observer, especially when the zone of quiet is small. The performance of a local active noise control system incorporating a virtual sensing method can thus further be improved by creating a moving zone of quiet that tracks the observer’s ear. A moving virtual sensing method therefore needs to be developed that can be used to estimate the sound pressure at a virtual location that is moving through the sound field rather than being spatially fixed. The developed moving virtual sensing method then needs to be incorporated into a local active noise control system. This system thus aims to create a moving zone of quiet at a virtual microphone that tracks the desired location of maximum attenuation.

General aim of research

The general aim of the research presented in this thesis is to develop spatially fixed and moving virtual sensing methods for local active noise control, and to combine these methods with active noise control algorithms with the aim to improve the performance of a local active noise control system. It is hoped that the developed methods will broaden the scope of successful local active noise control applications. An overview of the main contributions that are developed in this thesis to address this general aim is given after the literature review presented in the next section.

1.2 Literature review

In Sections 1.2.1 and 1.2.2, an overview of the field of research into global and local active noise control, respectively, is presented. For a more detailed discussion of these fields, the reader is referred to the textbooks on these topics [25, 48, 66, 88]. In Section 1.2.3, a brief discussion of the research into acoustic energy density sensing for active noise control is presented. This sensing method can be used to enlarge the zone of quiet created with a local active noise control system. In Section 1.2.4, a comprehensive overview of virtual sensing methods for local active noise control is presented. As discussed previously, these sensing methods can be used to move the zone of quiet away from the physical location of the transducers to the virtual location, where maximum attenuation is desired.

1.2.1 Global active noise control

Early work in the field of active noise control focussed on achieving global control where the objective was to reduce the noise throughout the entire acoustic domain of interest, for instance a vehicle cabin interior. In a series of publications [10, 26, 87],
an analysis of the active minimisation of harmonically excited enclosed sound fields was presented. The extent to which global control of the sound field could be achieved was analysed by minimising the total acoustic potential energy inside the enclosure using quadratic optimisation techniques. The analysis showed that global control of the sound field inside an enclosure can be achieved provided that the enclosure is lightly damped, and excited at frequencies close to the lower natural frequencies of the acoustic enclosure. For this specific case, the sound field inside the lightly damped enclosure is dominated by the modes belonging to these natural frequencies. These dominant modes can then generally be controlled without significantly affecting the other modes that contribute to the sound field, thereby achieving global control. However, if the control sources are located on nodes of the dominant modes, controllability problems arise as the dominant mode cannot be excited. Moreover, if the sound field is excited at frequencies between the resonance frequencies of the enclosure, many modes are contributing to the sound field. In this case, the control sources generally cannot control one of these modes without significantly exciting the other modes, which is usually referred to as the spill-over effect. Global control at anti-resonance frequencies of the considered enclosure is therefore generally not possible [88].

In the analysis presented in [10, 26, 87], the true total acoustic potential energy inside the considered acoustic enclosure was minimised. This is generally not possible in a practical application as this would require knowledge of the sound pressure at all points inside the enclosure. Real-time global active noise control is therefore usually attempted by minimising an estimate of the total acoustic potential energy given by the sum of the squared pressures at a number of microphones located throughout the enclosure [88]. This method can result in observability problems when the microphones are located on nodes of certain modes because these modes cannot be sensed in this instance. It is therefore important to know the shapes of the modes that contribute to the sound field inside the considered enclosure such that suitable locations for the microphones can be determined. These mode shapes are relatively easy to determine for enclosures that have a simple geometry, such as rectangular enclosures. However, for more complicated enclosures, such as vehicle interiors, the mode shapes are not as easily obtained. Determining suitable locations for the microphones is therefore not straightforward in this instance, which complicates the design of a global active noise control system.

Elliott et al. [31] experimentally investigated global active control of low frequency engine-induced noise inside the passenger cabin of a car. It was observed that the noise level inside the cabin is generally dominated by the engine firing frequency. A reference signal for feedforward control was therefore obtained from the ignition circuit of the engine. The implemented global active noise control system used six loudspeakers, which were shared with the in-car entertainment system, to minimise
two harmonics of the engine firing frequency at eight microphones. Overall reductions in the A-weighted sound pressure level of 4–5 dB(A) were obtained at higher engine speeds in front and rear seats, and 2–3 dB(A) at lower engine speeds in the rear seats. These real-time control results illustrate that it is generally not easy to obtain large global noise reductions inside acoustic enclosures in a practical situation.

Elliott et al. [30] investigated global active control of propeller-induced noise inside the passenger cabin of a twin turboprop aircraft. The spectrum of the sound field inside the cabin of such an aircraft contains strong tonal components that are related to the blade passage frequency (BPF) of the propellers, which are generally difficult to attenuate by passive absorption. The BPF and its second and third harmonics were targeted with the implemented global active noise control system, which employed three feedforward reference signals generated using a tachometer on one of the propeller engines. The global active noise control system consisted of 16 loudspeakers that were used to control the sum of the squared pressures measured at 32 microphones. In-flight overall reductions in the sum of the squared pressures at the 32 microphones of up to 7 dB(A) were measured while the active noise control system was targeting the BPF and its second and third harmonics. The best results were obtained at the BPF because the modal density at this frequency was low compared to the modal densities at the second and third harmonic of the BPF. This again illustrates that global active noise control is most effective at low frequencies where the modal overlap is small and where the sound field is mainly composed of one dominant mode.

An analysis of global active noise control for the free acoustic field case has also been undertaken [87, 88]. In this analysis, the power output of one monopole acoustic disturbance source was minimised by an array of monopole acoustic control sources each located at a distance \( d \) from the disturbance source. To achieve a given attenuation for this arrangement, it was observed that the required number of control sources increased as the square of the normalised separation distance \( d/\lambda \), with \( \lambda \) the acoustic wavelength. Also, to achieve 10 dB of attenuation in the total power output, it was shown that the maximum distance between the control sources may not exceed approximately half a wavelength. These results indicate that for shorter wavelengths, i.e. higher frequencies, a large number of closely spaced control sources is generally required to achieve global control of a primary point monopole source located in a free field. This illustrates that global active noise control in the free field case is usually difficult to obtain in practice because it is not straightforward to match the wavefronts of the disturbance and control sources in all three-dimensions [87, 88].

### 1.2.2 Local active noise control

As discussed in the previous section, global control strategies cannot always provide an effective solution for the active control of noise. A local active noise control approach is
often the only feasible solution, especially for the case of complex sound fields such as those found in acoustic enclosures excited at frequencies where the modal density is high. In contrast to global control strategies, local control strategies aim to reduce the noise at a number of points within the acoustic domain of interest rather than aiming to reduce the noise throughout this entire domain. As an example, global active noise control inside an airplane cabin interior would attempt to reduce the noise throughout the entire cabin, whereas local control would only attempt to reduce the noise at the passenger’s ears.

Figure 1.2: Electronic sound absorber used in an airplane or automobile to reduce the noise in the vicinity of the occupant’s head [91].

The principle of local active noise control was first suggested as early as 1953 by Olson and May [91]. The proposed system, which was called the electronic sound absorber, consisted of a microphone, an amplifier, and a loudspeaker. These components together operated as a feedback system that aimed to reduce the sound pressure at the microphone. Overall attenuations of 10–25 dB were achieved around the microphone in the frequency range below 200 Hz. Fig. 1.2 shows the illustration presented by Olson and May [91] as an example of a possible application of the electronic sound absorber. In this figure, the electronic sound absorber is installed on the back of a passenger’s seat in order to create a zone of quiet around the passenger’s head. This application of local active noise control is now known as an active headrest and has been investigated by many researchers [9, 40, 89, 94, 107, 109].

Elliott et al. [29] analysed the active cancellation of the sound pressure at a point in a pure tone diffuse primary sound field. The cancellation point was assumed to be in the far-field of the secondary source. The spatial extent of the zone of quiet was theoretically analysed. The analysis showed that the zone of quiet, in which the primary noise is reduced by 10 dB or more, typically has the shape of a sphere with a diameter of one-tenth of the acoustic wavelength. Furthermore, the analysis demonstrated that
large increases in the mean-square pressure away from the point of cancellation can occur as a result of an unfortunate combination of source and sensor locations.

Joseph et al. [58] analysed the active cancellation of the sound pressure at a point in the near-field of a secondary source for a pure tone diffuse primary sound field. In contrast to cancelling the sound pressure at a point in the far-field [29], the principal advantage of this arrangement is that the mean-square pressure well away from the cancellation point was generally found to be largely unaffected, with typical increases of only a small fraction of one decibel. This is due to the increased coupling between the source and the closely spaced microphone for this arrangement, which allows the sound pressure to be reduced with only a small secondary source strength. The theoretical analysis showed that the shape and size of the zone of quiet in the near-field of the secondary source was predominantly governed by geometric factors relating to the near-field radiation characteristics of the secondary source. In the near-field of a point monopole secondary source \((2\pi r_0 \ll \lambda)\), the diameter of the zone of quiet was found to be about \(0.63r_0\), with \(r_0\) the distance between the source and the cancellation point, and \(\lambda\) the acoustic wavelength. The size of the near-field zone of quiet thus increases as the cancellation point is moved away from the source, but is generally very small compared to the acoustic wavelength. In the far-field, the analysis resulted in a zone of quiet with a diameter of about \(\lambda/10\), which was also predicted by Elliott et al. [29]. Similar results were also found by Joseph et al. [58] in a theoretical analysis of more realistic secondary sources such as circular loudspeakers. Experimental work confirmed the general trends predicted by the developed theory.

Rafaely [106] extended the theoretical analysis presented in [29, 58] for the case of a pure tone diffuse primary sound field to the case of a broadband diffuse primary sound field. Both near-field and far-field zones of quiet were theoretically derived and investigated through a number of simulation examples. It was shown that for simple low-pass filtered noise, the spatial correlation characteristics and the created zone of quiet are comparable to those of a pure tone at the centre frequency of the considered frequency band. Previous experimental results obtained by Rafaely et al. [108] confirmed these theoretical findings.

**Active headrest**

A practical application of local active noise control known as the active headrest is depicted in Fig. 1.3 [108]. The aim of the active headrest illustrated in this figure is to create zones of quiet around the observer’s ears.

Rafaely et al. [109] and Rafaely and Elliott [107] studied the attenuation of broadband disturbances that can be obtained with an active headrest system. A feedback controller was designed using mixed \(H_2/H_\infty\) control methodologies and an internal model control (IMC) structure. Various cost functions and constraints were included in
the design method to obtain the required performance, robust stability against plant uncertainties, and limited disturbance enhancement. A single input single output (SISO) controller was designed, which attenuated the sound pressure at one microphone using one loudspeaker. The actual system depicted in Fig. 1.3 had two microphones and two loudspeakers, which would typically require the design of a multiple input multiple output (MIMO) controller for strongly coupled sources and sensors. Experimental results showed that an overall attenuation of 15.7 dB was obtained at the microphone location in the frequency band of 100–400 Hz. However, it was reported in subsequent numerical investigations [108] that the overall attenuation at the manikin’s ear was predicted to be only 3.7 dB due to the limited extent of the zone of quiet created at the microphone.

In a practical situation, a local active noise control system will generally be affected by the presence of a human head and a headrest, which causes reflection of the sound field. The performance of a local control system near a reflecting surface in a diffuse primary sound field was theoretically investigated by Garcia-Bonito and Elliott [38] and Garcia-Bonito et al. [40]. The reflecting surfaces considered were a sphere, a wall, a two wall edge and a corner. The simulation results showed that the presence of a reflecting surface slightly improved the size of the zone of quiet due to the imposed zero pressure gradient on the reflecting surfaces. These results suggest that the presence of a human head in a local control system could positively influence the size of the zone of quiet.

Nielsen et al. [89] developed an active headrest system for drivers of diesel locomotives. The predominantly low frequency characteristics of the engine-induced noise, which was shown to cause degraded work performance of the drivers, makes this noise problem a very suitable target for an active headrest system. A practical implementation of the developed system was tested and received positive feedback from the locomotive drivers. A quantitative analysis of the control performance was not presented.

Figure 1.3: Local active noise control system assembled on a headrest of a passenger’s seat [108]. This system is typically referred to as an active headrest.
1.2 Literature review

Brothánek and Jiříček [9] investigated the performance of an active headrest system in a free field. A double-input double-output broadband adaptive feedforward controller was designed, which minimised the sound pressure at two error microphones using two loudspeakers mounted on the headrest. The error microphones were located 5 cm away from the loudspeakers. Two performance microphones were placed at the mannequin’s ears 7 cm away from the error microphones. Overall attenuations of 20 dB were achieved at the error microphones in the frequency band of 100–500 Hz. However, the overall attenuation at the performance microphones was only 10 dB due to the limited extent of the zones of quiet.

Pawelczyk [94] examined a double-input quadruple-output adaptive feedback controller for an active headrest system. The controller was based on an IMC structure and was designed to minimise the sound pressure at two microphones using four loudspeakers. A comparison was made with a double-input double-output adaptive feedback controller. The error microphones were located about 15 cm away from the mannequin’s ears. Spatial distributions of the controlled sound field around the error microphones and mannequin’s ears were presented for a tonal frequency of 250 Hz. The double-input double-output controller obtained attenuations of 30 dB at the error microphones and 11 dB at the mannequin’s ears. The double-input quadruple-output controller provided an improved performance with attenuations of 32 dB at the error microphones and 18 dB at the mannequin’s ears. The doubling of the number of loudspeakers also achieved larger zones of quiet.

1.2.3 Energy density sensing

As discussed previously, the size of the zone of quiet that can be created with a local active noise control system is generally small and often does not extend from the error microphone to the observer’s ear. A possible method to overcome this problem is to enlarge the zone of quiet by minimising the acoustic energy density instead of the sound pressure. The acoustic energy density is defined as the sum of the acoustic potential energy density and the acoustic kinetic energy density [65]. Minimising the energy density at a point in space therefore effectively minimises the sound pressure and the particle velocity at that point. Since the particle velocity is directly proportional to the sound pressure gradient through Euler’s equation [65], minimising the energy density at a point is equivalent to minimising the sound pressure and sound pressure gradient at that point.

Elliott and Garcia-Bonito [28] investigated the active cancellation of both sound pressure and sound pressure gradient along a single axis in a pure tone diffuse primary sound field. The zone of quiet was expected to increase if sound pressure as well as sound pressure gradient were simultaneously minimised by two secondary sources. This expectation was confirmed by computer simulations as the zone of quiet was
found to have the shape of a cylinder with rounded ends, which had a length of about half an acoustic wavelength along the axis of the controlled sound pressure gradient, and a diameter of about one-tenth of an acoustic wavelength. If only sound pressure is minimised, the shape of the zone of quiet is expected to be a sphere with a diameter of about one-tenth of an acoustic wavelength [29]. Minimising an acoustic energy density based cost function instead of a traditional pressure squared cost function thus has the potential to increase the size of the zone of quiet created by a local active noise control system [43].

An acoustic energy density based cost function can also be used in global control applications to overcome the observability problems often encountered when using a pressure squared cost function. Cook and Schade [23] analysed the spatial variance of the acoustic energy density for several types of standing wave fields. The results showed that the acoustic energy density generally has a smaller spatial variance than the acoustic potential energy density in these types of sound fields. If an estimate of the acoustic energy density cost function is obtained for these types of sound fields by measuring the acoustic energy density at a number of points throughout the sound field, observability problems, which are often encountered when employing a traditional pressure squared cost function as an estimate of the acoustic potential energy, are less likely to occur. Note that this argument assumes that the particle velocity is directly measured and not estimated using an energy density sensor based on multiple sound pressure microphones.

Sommerfeldt and Nashif [115] investigated an acoustic energy density sensing approach both analytically and numerically for global active noise control inside a one-dimensional sound field in an acoustic duct. The numerical results demonstrated that minimising the energy density at discrete locations generally provided better global performance than minimising the squared pressure at these locations, which was confirmed through experiments in an acoustic duct [85, 86, 116]. Sommerfeldt and Nashif [117] also derived an adaptive algorithm based on the filtered-x LMS algorithm that can be implemented when attempting energy density control in a real-time situation. Experiments in an acoustic duct showed the effectiveness of the derived adaptive algorithm. A two-microphone method was used in these experiments to compute an estimate of the energy density. One disadvantage of this method is the increased signal processing that is required for the measurement of the acoustic energy density in comparison to a simple pressure measurement.

Sommerfeldt et al. [118] extended their previous work [85, 86, 115, 116] to three-dimensional enclosed sound fields. A rectangular enclosure with internal dimensions of $1.93 \times 1.22 \times 1.54$ m$^3$ was used in the experiments, and a numerical model of the enclosure was built for simulation purposes. One loudspeaker was used to create a tonal primary disturbance. The sound field was controlled with one loudspeaker,
either by minimising the acoustic energy density or the squared sound pressure at one location. The sound pressure sensor was placed at a poor location for the case of controlling the squared sound pressure, i.e. on a node in the sound field. The acoustic energy density at the considered location was measured using a six-microphone energy density sensor. The six microphones in the energy density sensor needed to be highly phase-matched in order to obtain an accurate estimate of the acoustic energy density. This highlights one of the disadvantages of using an acoustic energy density based cost function instead of a sound pressure squared cost function, because the cost of the employed sensors is generally higher. Numerical and experimental results again confirmed that minimising the acoustic energy density at discrete locations generally provided better global performance than minimising the squared pressure at these locations.

Faber and Sommerfeldt [32] investigated the global active control of low frequency engine-induced tonal noise inside a mock tractor cabin using an acoustic energy density based cost function. A laboratory model of a mock cabin was used to investigate the global control performance of the system. The results indicated that good global control performance was obtained inside the cabin for synthesised static and dynamic test signals as well as for noise recorded from an actual tractor cabin. Global attenuations of 20 dB and 10 dB were obtained for the first two static engine tones. Local attenuations at the ear locations of 20 dB were achieved at these static tones. The size of the resulting zones of quiet created around the ears was not investigated.

1.2.4 Virtual sensing

The discussions presented in Section 1.2.2 indicated that in the case of local control in the near-field of a secondary source, the error microphone may have to be placed at such a distance from this source that it restricts the movement of the observer’s head [27]. Local control in the far-field of a secondary source generally requires the error microphone to be placed relatively close to the observer’s ear because the produced zone of quiet tends to be small [29]. Unfortunately, locating the error microphone close to the observer’s ear is not always possible or at least a very inconvenient solution. This problem can be overcome by using a virtual sensing approach which is illustrated in Fig. 1.1(b). In this approach, a virtual microphone is located at the observer’s ear where the maximum noise reduction is required. An estimate of the sound pressure at this virtual microphone is then computed using the sound pressure measured with a physical microphone that is positioned at some suitable location remote from the observer’s head. The estimate of the sound pressure at the virtual microphone can then be minimised such that the zone of quiet is moved away from the physical microphone towards the observer’s ear.
The use of virtual sensing methods is not limited to local active noise control applications but can be employed in a wide variety of practical control and estimation problems, such as strip thickness control in rolling mills \cite{2, 45}, and the estimation of the response of a flexible structure during operation at locations at which physical sensors cannot be placed \cite{61}. In general, a virtual sensor provides an estimate of an output of a dynamic system without directly measuring this output with a suitable physical transducer during real-time control. Instead, an estimate of the virtual sensor output is computed using a model of the dynamic system under consideration and measurements of other outputs of this dynamic system provided by a number of physical sensors. A virtual sensor can thus be used when a physical transducer that directly measures the output of interest is not available or very expensive, when the environment in which the virtual sensor is located is too hostile, or when installing a physical sensor at the desired measurement location is not possible or very inconvenient. Note that the last reason relates to the application of virtual sensing to active noise control.

The first investigations into virtual sensing methods for local active noise control are ascribed to Elliott and David \cite{27}, although it appears that the principle of these methods was first suggested by Carme and Roure \cite{12} as claimed by these authors in their publication \cite{11}.ootnote{Unfortunately, it has not been possible to get access to the publication by Carme and Roure \cite{12} that was cited in Ref. \cite{11}.} Since the start of the 1990s, a number of virtual sensing methods have been proposed which are discussed here.

**Virtual microphone arrangement**

The *virtual microphone arrangement* suggested by Elliott and David \cite{27} requires knowledge of the transfer paths between the secondary sources and the physical and virtual microphones. Furthermore, it is assumed that the primary sound field changes relatively little between the physical and virtual microphones, which leads to the assumption of equal primary sound pressures at these microphones. This assumption, together with the knowledge of the secondary transfer paths, allows the sound pressure at the virtual microphone to be estimated. The estimated sound pressure can then be minimised by the secondary sources thereby creating a zone of quiet around the desired location of maximum attenuation, i.e. the virtual location.

Garcia-Bonito et al. \cite{41, 42} investigated the use of the virtual microphone arrangement for generating a local zone of quiet at a virtual microphone located in a diffuse primary sound field. Their results showed that for low frequencies, the virtual microphone arrangement produced very similar zones of quiet compared to cancelling the true sound pressure at the virtual location. However, for higher frequencies, the size of the zone of quiet drastically decreased compared to cancelling the true sound pressure.
The reason for this was that at higher frequencies the assumption of a spatially uniform primary sound field between the physical and virtual microphones is no longer valid.

Matuoka et al. [76] and Horihata et al. [53] investigated the virtual microphone arrangement for a three-dimensional rectangular acoustic enclosure with dimensions $1.45 \times 0.92 \times 0.65\text{m}^3$. One primary loudspeaker and one control loudspeaker were mounted in the walls at half the height of the enclosure. The primary loudspeaker was used to create a tonal disturbance of 120 Hz. For one virtual location, a comparison was made between minimising the estimated sound pressure at this location using the virtual microphone arrangement and minimising the true sound pressure at this location. The results showed that the controlled sound pressure distributions almost coincided for both cases, although cancelling the true sound pressure obtained 40 dB more reduction at the considered virtual location. Moreover, the locations of the physical and virtual microphones were chosen such that the assumption of equal primary sound pressures at these microphones, which is the fundamental assumption made in the virtual microphone arrangement, was satisfied.

The virtual microphone arrangement was experimentally tested by Garcia-Bonito et al. [41, 42] for the active headrest case illustrated in Fig. 1.3. A two-channel adaptive feedforward controller was employed to cancel the tonal primary sound field at two virtual microphones, which were located 2 cm away from a mannequin’s ears and 10 cm away from the physical microphones. The primary sound field was a progressive plane wave moving parallel to the physical and virtual microphones. This should result in almost equal primary sound pressures at the physical and virtual microphones, which is one of the assumptions made in the virtual microphone arrangement. For this type of primary sound field, the virtual microphone arrangement was shown to effectively create a zone of quiet at the virtual location. The 10 dB zone of quiet extended approximately 8 cm forwards and 10 cm sidewards from the virtual location up to an excitation frequency of 500 Hz.

In a practical situation, an active headrest system will be affected by the presence of a passenger’s head and the headrest itself, which causes reflections of the sound field. The performance of the virtual microphone arrangement near a reflecting sphere was theoretically investigated by Garcia-Bonito and Elliott [39], and Garcia-Bonito et al. [42, 43]. The results showed that the presence of a reflecting sphere near the virtual microphone considerably improved the size of the zone of quiet. The reason for this improvement is that the primary sound field becomes spatially more uniform near the reflecting sphere, which generally makes the assumption of equal primary sound pressures at the virtual and physical microphones more valid. These results suggest that the presence of a human head in a practical active headrest system employing the virtual microphone arrangement could positively influence the size of the created zone of quiet.
Rafaely et al. [108, 109] presented a performance analysis of an active headrest system using a feedback controller to attenuate broadband disturbances at a virtual microphone based on the assumptions made in the virtual microphone arrangement. The virtual microphone was located at a mannequin’s ear, approximately 10 cm away from the physical microphone. The controller was designed using a mixed $H_2/H_\infty$ method and an IMC structure as suggested by Rafaely et al. [109] and Rafaely and Elliott [107]. The performance was analysed off-line using estimates of the required frequency response functions, which were obtained from an experimental set-up. A comparison between two controllers was made in order to analyse the benefits of a virtual microphone approach. One controller was designed to minimise the sound pressure at the physical microphone, and another controller was designed to minimise the sound pressure at the virtual microphone. Both controllers were designed as single-input single-output systems that attenuated the sound pressure at one microphone using one loudspeaker. The results for the physical microphone controller indicated that an overall attenuation of 19.1 dB was obtained at the physical microphone in the frequency band of 100–400 Hz. However, the overall attenuation at the virtual microphone was predicted to be only 3.7 dB when using this approach due to the limited extent of the zone of quiet created at the physical microphone. The results of the virtual microphone controller indicated that an overall attenuation of 9.5 dB was obtained at the virtual microphone, which was an improvement of 5.8 dB over the physical microphone controller. The overall attenuation was less than the 19.1 dB attenuation achieved at the physical microphone by the physical controller. This difference was expected since the virtual microphone was about 10 cm further away from the loudspeaker than the physical microphone. This causes a longer delay in the secondary transfer path between the control source and the location of control, which generally has a negative effect on the achievable performance of a feedback control system. Tseng et al. [122] extended the previous work by including an additional constraint in the design method which guaranteed stability of the feedback controller. The reason for this extension was that, although the resulting feedback controller stabilises the closed-loop system, the IMC approach can result in an unstable feedback controller [114].

Holmberg et al. [52] constructed a simple one-channel active headrest system in order to analyse the performance of a feedback control approach based on the virtual microphone arrangement assumptions. A robust feedback controller was designed to minimise a 140 Hz tonal primary disturbance at the virtual microphone, which was located about 8 cm away from the physical microphone. Attenuations of 10 dB were obtained at the virtual microphone, while the noise was increased by 5 dB at the physical microphone. Since the designed feedback controller was not adaptive, the developed method cannot account for a change in frequency of the tonal primary sound pressure, which limits its use in practical applications.
Pawelczyk [96] developed a double-input double-output adaptive feedback controller for an active headrest system using the virtual microphone arrangement assumptions. The adaptive controller was based on an IMC structure, and was designed to minimise the tonal sound pressures at two virtual microphones using two loudspeakers. The virtual microphones were located about 15 cm away from the physical microphones. Spatial distributions of the controlled sound field around the physical and virtual microphones were presented for a tonal frequency of 250 Hz. The zone of quiet was successfully shifted from the physical microphone locations to the virtual microphone locations. Attenuations of 18 dB were achieved at the virtual microphones for the considered tonal frequency. Pawelczyk [95] extended his work on active headrests to a double-input quadruple-output adaptive feedback controller. With this controller, attenuations of 20 dB were obtained at the virtual microphones, and slightly larger zones of quiet could be created compared to the double-input double-output controller.

Pawelczyk [97] also investigated active noise control in a telephone based on the virtual microphone arrangement and feedback control. The aim was to reduce road noise in the frequency band of 60–600 Hz. Experiments showed that the overall reductions reached 4.8 dB in the frequency band of interest, and 3.5 dB over the entire frequency range.

Pawelczyk [98, 99] proposed another algorithm for active noise control at virtual microphones, in which the virtual microphone arrangement assumptions need not be made. The suggested algorithm was again based on a double-input double-output adaptive feedback controller, which aimed to minimise the noise at two virtual microphones. In a preliminary identification stage, physical microphones are positioned at the two virtual locations, and the feedback controller is adapted such that the error signals measured by these microphones are minimised. Furthermore, knowledge about the physical error signals that exist once the adaptive controller has converged, is embedded in an additional transfer function. During the control stage, this transfer function is employed to determine a set-point for the physical error signals, and the feedback controller is adapted such that the difference between this set-point and the measured physical error signals is minimised. In this way, the controller converges to the same solution as during the tuning stage, and the zones of quiet are thus shifted to the virtual microphones. The proposed algorithm was implemented in an active headrest system, with the virtual microphones located about 6.5 cm away from the physical microphones. Attenuations at the virtual microphones of 30 dB were obtained for tonal noise of 250 Hz, and 4 dB for broadband noise. Pawelczyk [100] also designed a fixed-gain feedback controller for the considered active headrest system using a polynomial approach. Overall attenuations at the virtual microphone of 8.6 dB were reported in the frequency range of 100–400 Hz.

2. See also discussion of Pawelczyk [94] on page 11 of this thesis.
Kuo et al. [67] and Kuo and Gan [66] investigated the application of virtual sensing to active noise control in an electronic muffler. In this application, the error microphone cannot be placed at the desired location of maximum attenuation due to a high temperature, fast air flow, and corrosive environment. A broadband adaptive feedforward controller that aimed to cancel the unwanted noise at a virtual microphone was investigated. Although the suggested virtual sensing algorithm was slightly different to the virtual microphone arrangement [27], these algorithms are both based on the assumption of equal primary sound pressures at the physical and virtual microphones and on knowledge of the secondary transfer paths between the control source and the physical and virtual microphones. The numerical control performance of the suggested algorithm was tested based on transfer paths that were measured on an experimental set-up. The numerical results showed that performance improvement was obtained when a virtual sensing method was employed instead of a traditional sensing method.

Diaz et al. [24] investigated the application of the virtual microphone arrangement to the local control of wheel-rail interaction induced noise inside a railway vehicle sleeping cabin. An enclosure consisting of wooden panels and measuring $1.80 \times 2.30 \times 1.80$ m$^3$ was built to serve as a prototype cabin for experimental work. A bed of width 80 cm and length 180 cm was located in one of the corners inside the enclosure. Two control loudspeakers were fitted in the panel at the bed head, 10 cm above the bed surface, and 29 cm from the middle of the bed. Two physical microphones were located between the two control loudspeakers at bed surface height, 10 cm to the right and left of the middle of the bed. Two virtual microphones were located at bed surface height close to the location of the passenger’s ears, 20 cm from the panel at the head end of the bed and 5.5 cm to the right and left of the middle of the bed. The primary broadband noise was generated using a shaker attached to the outside of the panel at the bottom end of the bed, which was excited in the frequency range between 90–290 Hz. An accelerometer was placed close to the point of excitation on the inside of the excited panel. The signal from this accelerometer was used as a feedforward reference signal in the filtered-x LMS algorithm during real-time control experiments. The zone of quiet obtained while controlling the noise at the two physical microphones was compared to the zone of quiet obtained while controlling the noise at both the two physical and virtual microphones. The control performance of the implemented algorithms was analysed by measuring the primary and controlled sound pressure distributions at bed surface height. The experimental results showed that the zones of quiet were moved towards the virtual microphones. When using the virtual microphone arrangement, higher attenuations were achieved at distances larger than 15 cm from the panel at the bed head. Attenuations at the virtual microphone locations of up to 15 dB(A) were reported.
Remote microphone technique

Popovich [104] and Roure and Albarrazin [112] independently suggested a virtual sensing method called the remote microphone technique. The name for this algorithm was suggested by Roure and Albarrazin [112], who investigated a practical implementation of the remote microphone technique. This technique requires the estimation of two secondary transfer paths and one filter, which are usually estimated in a preliminary system identification step. The two secondary transfer paths are the transfer paths between the control sources and the physical and virtual microphones, which are also needed in the virtual microphone arrangement [27]. However, an additional filter needs to be estimated in the remote microphone technique to calculate an estimate of the primary sound pressure at the virtual microphone given the primary sound pressure at the physical microphone. The virtual microphone arrangement assumes this filter to be unity, which follows from the assumptions of equal primary sound pressures at the physical and virtual microphones, and is therefore a simplified version of the remote microphone technique.

Roure and Albarrazin [112] tested the performance of the remote microphone technique in a room with metal walls simulating an airplane interior. Estimates of the required secondary transfer paths, which were modelled as FIR filters, were determined using an adaptive system identification technique based on the LMS algorithm [51]. Their experimental results were obtained using six physical microphones, twelve virtual microphones, and nine secondary sources. A tonal primary disturbance of 170 Hz was produced by one loudspeaker located in a corner of the room. An average attenuation of about 20 dB was obtained at the twelve virtual microphone locations when using the remote microphone technique. Minimising the true sound pressures at the virtual locations resulted in an average attenuation of about 29 dB at the twelve virtual microphones. This difference was ascribed to the sensitivity of the remote microphone technique to errors in the estimated secondary transfer path models and in the filter that was used to compute an estimate of the primary sound pressure at the virtual microphone.

Renault et al. [110] experimentally tested and compared the performance of the virtual microphone arrangement and the remote microphone technique using a feedforward control approach. A loudspeaker excited by white noise in a frequency range of 50–300 Hz was located about one metre outside an acoustic enclosure. Nine monitoring microphones were placed inside this enclosure to measure the performance of the considered algorithms. One of these microphones, which was located at the intended location of the zone of quiet, was used during a preliminary identification stage of the virtual sensing algorithms. The physical microphone was placed on the wall of the enclosure about 25 cm away from the virtual microphone. The results showed that the performance of the virtual microphone arrangement was inferior to the remote
microphone technique. This was because, in contrast to the assumption made in the virtual microphone arrangement, the primary sound pressures at the physical and virtual microphones were not equal in the conducted experiments, which is to be expected for a reactive sound field inside an acoustic enclosure.

Although Popovich [104] was the first to suggest the remote microphone technique and Roure and Albarrazin [112] were the first to investigate a practical implementation, Radcliffe and Gogate [105] had already performed computer simulations that analysed the theoretical performance limit of such a virtual sensing approach. The simulations were based on a three-dimensional acoustic finite element model of a car interior. The knowledge of the acoustical model of the car interior was used to modify a conventional feedforward control approach in order to cancel the noise at a number of virtual microphones located away from the physical microphones. This modification required models of the secondary transfer paths between the control sources and the physical and virtual microphones and a filter that computed an estimate of the primary sound pressure at the virtual microphones, which were also needed in the remote microphone technique. The results showed that for the case of tonal disturbances, for which the required filter can be modelled as a transfer impedance matrix, the attenuations that can in theory be achieved at the virtual locations when using either virtual or physical microphones at these locations are equivalent.

Hashimoto et al. [49] suggested a virtual sensing algorithm that was exactly the same as the remote microphone technique, which they called the remote error sensing method. This algorithm was experimentally tested in an acoustic enclosure. A single-channel broadband adaptive feedforward controller was used to attenuate broadband pink noise. Overall attenuations of about 15 dB in the frequency band of 100–1000 Hz were achieved at the virtual microphone, which was located 24 cm away from the physical microphone.

Friot et al. [37] introduced a simplified version of the remote microphone technique in order to reduce computational complexity. This algorithm requires only one filter that is designed to compute an estimate of the sound pressures at the virtual microphones given the sound pressures at the physical microphones. In contrast to the original formulation of the remote microphone technique [104, 112], it is thus no longer necessary to separate the sound pressures at the physical microphones into their primary and secondary components. However, it is important to realise that the proposed approach can only be employed when the frequency response functions between the secondary sound pressures at the physical and virtual microphones are equivalent, or at least very similar, to the frequency response functions between the primary sound pressures at the physical and virtual microphones. If this is not the case, an accurate estimate of the sound pressures at the virtual microphones given the sound pressures at the physical microphones cannot be computed using only one filter as suggested by Friot et al. [37].
Yuan [128] developed a slightly modified version of the remote microphone technique, which was experimentally tested in an acoustic duct using an adaptive broadband feedforward controller. The primary and secondary sources were located at opposite ends of the duct. The virtual microphone was placed in between two physical microphones, which were spaced 0.8 m apart. Similar to the remote microphone technique, estimates of the secondary transfer paths between the control source and the two physical microphones were computed. The difference with the remote microphone technique was the way in which the primary sound pressure at the virtual microphone was estimated. By judiciously placing the two physical microphones inside the acoustic duct, the sensitivity of the virtual sensing algorithm to transmission zeros between the primary sound pressures at the physical and virtual microphone was reduced. This allowed an accurate estimate of the sound pressure at the virtual microphone to be obtained over a frequency range of 100–600 Hz. Experimental results showed that overall attenuations of approximately 20 dB were achieved at the virtual microphone in this frequency range.

Berkhoff [7] investigated the application of the remote microphone technique to the active control of traffic noise using active noise barriers. These active barriers aim to reduce traffic induced noise at some distance away from the barriers, such as on the facades of houses. One of the main design objectives was to only use physical microphones located near the active noise barriers, and not near buildings. The signals at these near-field physical microphones can then be used to compute an estimate of the error signals at the far-field virtual microphones. Simulations were performed using five physical sensors spaced 0.3 m apart in the near-field of the secondary sources at a distance of 0.5 m, five virtual sensors spaced 1.5 m apart in the far-field of the secondary sources at a distance of 19.5 m, and five reference sensors spaced 0.3 m apart at a distance of 0.5 m from the secondary sources towards the primary sources. The primary sources were three independent broadband noise sources located 0.15 m apart at a distance of 5 m from the secondary sources. The simulation results showed that minimising the estimated far-field virtual error signals resulted in a far-field overall attenuation of 9.0 dB, while minimising the near-field physical error signals resulted in a far-field overall attenuation of only 1.9 dB.

Forward difference prediction techniques

Cazzolato [13] suggested an alternative approach to the virtual microphone arrangement [27] and the remote microphone technique [104, 112], based on forward difference prediction theory. In this approach, the sound pressure at the virtual location is estimated by summing the weighted sound pressures from a number of physical microphones in an array. The weights for each physical microphone element are determined using forward difference prediction techniques. The forward difference prediction approach was
investigated by Kestell [62] and Munn [80] for a long narrow acoustic duct and a free field. In these investigations, a linear prediction method using a two microphone array, and a quadratic prediction method using a three microphone array were evaluated both theoretically and experimentally. While the quadratic prediction method theoretically gave the highest attenuation at the desired location, experiments showed that the linear prediction method proved to be favourable in a practical situation [63, 64, 84]. This was attributed to the high sensitivity of the quadratic prediction method to short wavelength extraneous noise.

In an effort to overcome the problem of the high sensitivity to short wavelength extraneous noise, higher-order virtual microphone arrays were investigated [80]. This method uses a larger number of microphones in the array than the order of the forward difference prediction technique, resulting in an over-constrained problem which can be solved by a least squares approximation. The higher-order forward difference prediction algorithm then acts to spatially filter out the extraneous noise. The experimental results showed that the performance of these higher-order prediction algorithms was very much limited by the phase and sensitivity mismatches and relative position errors between the physical microphones in the array [82]. These mismatches and position errors are generally unavoidable, especially when a greater number of physical microphones are used.

Kestell [62] investigated the combination of virtual sensing and energy density sensing using forward difference prediction techniques for the acoustic duct and free field cases. Virtual energy density control combines the benefits of virtual sensing, i.e. providing a method for creating a zone of quiet remote from the physical transducers, and energy density control, i.e. enlarging the created zone of quiet. The results presented by Kestell et al. [64] were variable and no attempts were made to improve on the performance of the developed virtual energy density sensing methods. Moreover, it was shown by Cazzolato et al. [17] that the conclusions drawn from the virtual energy density sensing simulations and experiments presented by Kestell [62] were most likely flawed.

**Adaptive LMS virtual microphone technique**

In an effort to overcome the problem of phase and sensitivity mismatches and relative position errors between the elements of the physical microphone array that is employed in the forward difference prediction techniques, Cazzolato [14] explored the use of the adaptive LMS algorithm for determining optimal weights for the elements in the array. The *adaptive LMS virtual microphone technique* places a physical microphone at the virtual location in a preliminary identification stage in which the physical microphone weights are adapted by the LMS algorithm so as to optimally predict the sound pressure at this
location. After the weights have converged, the physical microphone is removed from the virtual location and the weights are fixed to their optimal values.

Munn [80] numerically investigated the use of the adaptive LMS virtual microphone technique for an acoustic duct. It was found that the adaptive LMS algorithm could completely compensate for relative position errors and sensitivity mismatches and partly compensate for phase mismatches that are present in the physical microphone array. Real-time control results for an acoustic duct showed that the adaptive LMS virtual microphone technique outperformed the forward difference prediction techniques [80, 81]. The performance of the adaptive LMS virtual microphone technique was also compared to the performance of the virtual microphone arrangement for both a free field and a reactive field inside an acoustic duct [80]. The performance of the virtual microphone arrangement was found to be poor in the reactive environment found inside an acoustic duct when compared to the adaptive LMS virtual microphone technique. The reason for this is that the assumption of equal pressures at the physical and virtual microphones is generally not satisfied in a reactive environment, since this environment usually has a high spatial rate of change in the primary sound field. In the free field, the virtual microphone arrangement performed much better than in an acoustic duct, since the spatial rate of change of the primary sound field is much lower for this case than in a reactive environment. However, the adaptive LMS virtual microphone technique consistently obtained the highest level of attenuation.

Gawron and Schaaf [44] suggested a virtual sensing method very similar to the adaptive LMS virtual microphone technique. The suggested method was applied to local active noise control of engine-induced noise inside a car cabin. Real-time control experimental results were presented in which attenuations of up to 15 dB on the second engine harmonic were reported. However, it was not made clear whether these attenuations were achieved at the physical or virtual microphone locations.

### Hybrid adaptive feedforward observer

Tran and Southward [119, 120, 121] suggested and investigated a virtual sensing algorithm based on a *hybrid adaptive feedforward observer*, which consisted of a conventional observer that was augmented with an adaptive feedforward component. This adaptive component was designed to track the mapping of the unknown disturbance source signals onto the observer states, with the adaptation obtained using an algorithm based on the least mean-squares gradient descent method. The estimation performance of the hybrid adaptive feedforward observer was investigated using a numerical model of an acoustic duct. The numerical results presented in Fig. 6 of their paper [119] indicated that a perfect estimate of the physical output was obtained. If the observer would work as suggested by Tran and Southward [119], this would indicate that a perfect estimate of the plant states was computed by the hybrid adaptive feedforward observer. However,
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This is not confirmed by the numerical results presented in Fig. 7 of their paper [119], where it can be seen that a perfect estimate of the virtual outputs has not been obtained. This indicates that, although a perfect estimate of the physical outputs is computed, perfect estimates of the plant states are not obtained after convergence of the adaptive algorithm. Since accurate state estimates are required to obtain accurate estimates of the virtual outputs, the suggested hybrid adaptive feedforward observer only appears suitable for rejecting non-stationary disturbances at the physical sensor outputs and not for virtual sensing purposes.3

Room transfer function extrapolation

Haneda [47] suggested a virtual sensing method for local active noise control inside an acoustic enclosure based on room transfer function extrapolation. In this method, an accurate modal model of the enclosure under consideration is required, which describes the transfer paths between the sources and the employed physical microphone array. An estimate of the transfer paths between the sources and the virtual microphone is then computed by extrapolating the mode shapes of the considered enclosure from the locations of the physical microphones to the location of the virtual microphone. This extrapolation effectively gives an estimate of the zeros of the transfer paths between the sources and the virtual microphone. The poles of these transfer paths are then determined by using the fact that all the transfer paths should have common poles. In the presented numerical results, a rectangular enclosure was used for which a modal model is easily determined. The disadvantage of the proposed method is that for more complex acoustic enclosures, which are often encountered in practical applications of local active noise control, it is generally not easy to determine an accurate modal model.

Beam-forming

Finn [33] patented a virtual sensing method based on phased-array sensors. In the suggested virtual sensing algorithm, beam-forming methods are used to provide the phased-arrays with the highest directional sensitivity in the direction of the virtual microphone. Multiple phased-arrays are then applied to create beams that intersect at the intended virtual location. By appropriately processing the signals from the multiple phased-arrays, it is claimed that the sound pressure at this virtual location can be estimated. This estimate is then minimised by an active noise control system, which effectively creates a zone of quiet at the virtual location away from the phased-array sensors. Numerical or experimental results that confirm the effectiveness or feasibility of the suggested beam forming method have not been presented so far to the author’s knowledge.

3 A detailed analysis of the hybrid adaptive feedforward observer is presented in Section 3.6 of this thesis.
1.3 Motivation and contributions

In Section 1.3.1, the literature review presented in the previous section is summarised to arrive at a motivation and general aim of the research that is described in this thesis. Section 1.3.2 gives an overview of the main contributions of the research presented in this thesis.

1.3.1 Motivation and aim of research

Early work in the field of active noise control focussed on achieving global control, where the objective was to reduce the noise throughout the entire acoustic enclosure, for instance a car interior. Global active noise control systems do not often attenuate the noise to an acceptable level in practical problems, especially in modally dense enclosures and in sound fields which have a large spatial variance. In this case, local active noise control is the only feasible solution. Unfortunately, local control generally achieves the greatest noise reductions at the error sensor locations, which might not always be the location where the maximum attenuation is required. This problem is further exacerbated by the fact that the size of the zone of quiet created around the error sensor tends to be very small.

One possible way to overcome the problems often encountered in local active noise control systems is to use a method that aims to enlarge the created zone of quiet. Minimising an acoustic energy density based cost function instead of a traditional pressure squared cost function has the potential to increase the size of the zone of quiet. However, there are several disadvantages of employing an acoustic energy density based cost function instead of a pressure squared cost function [117]. One disadvantage is the increased signal processing that is required for the measurement of the acoustic energy density compared to simply measuring the acoustic sound pressure. Another disadvantage is that typically a number of highly phase-matched microphones are required to obtain an accurate estimate of the acoustic energy density at a point, which can result in a relatively expensive sensor.

Another possible way to overcome the problems often encountered in local active noise control systems is to use a virtual sensing method. This method requires a non-intrusive sensor that is placed remote from the desired location of maximum attenuation, i.e. the virtual location. The non-intrusive sensor is used to compute an estimate of the sound pressure at this location. The estimated sound pressure is then minimised by a local active noise control system. This results in a zone of quiet that has been moved away from the physical location of the error sensor to the desired location of maximum attenuation. As discussed in Section 1.2.4, a number of virtual sensing methods for local active noise control systems have been suggested previously. These methods can be used when the movements of the observer’s head are centred around a spatially
fixed location and are small in comparison to the size of the created zone of quiet. However, this might not always be the case because the created zone of quiet generally tends to be small.

The general aim of the research presented in this thesis is to provide a comprehensive overview and analysis of the spatially fixed virtual sensing methods that have been proposed so far, and to develop new and improved spatially fixed and moving virtual sensing algorithms for local active noise control systems, which can be used to further improve the performance of local active noise control systems. An overview of the contributions that are developed in this thesis towards achieving this general aim is now presented.

1.3.2 Contributions

A number of algorithms have been proposed in previous research to solve the spatially fixed virtual sensing for active noise control problem, e.g. the virtual microphone arrangement [25], the remote microphone technique [104, 112], the adaptive LMS virtual microphone technique [14], the hybrid adaptive feedforward observer [119], and the secondary path equalisation method [66]. The main contributions to the field of spatially fixed virtual sensing for active noise control presented in this thesis are listed below.

- The common aim of the previously proposed virtual sensing algorithms is to compute an accurate estimate of the sound pressure at the virtual location without directly measuring this sound pressure during real-time control. The difference between these algorithms is the way in which the estimate is computed, i.e. the assumed structure is what makes the virtual sensing algorithms proposed so far different from one another. The question now arises as to whether there is an optimal structure that can be used to solve the virtual sensing for active noise control problem, which amounts to a linear estimation problem. It is well-known that the Kalman filter provides an optimal structure for solving linear estimation problems [60]. One of the main contributions of this thesis is the development of an optimal solution to the spatially fixed virtual sensing for active noise control problem using Kalman filtering theory. The developed algorithm is implemented on an acoustic duct arrangement to demonstrate its effectiveness in a practical implementation.

- A comprehensive discussion of the previously proposed virtual sensing algorithms is presented based on a general state-space model of the considered local active noise control system. For each of these algorithms, the assumed estimation structure is introduced and an optimal solution for the unknown parameters of the algorithm is derived by optimising the estimation performance. The presented analysis allows a comparison of the optimal estimation performance that can be obtained with the previously proposed virtual sensing algorithms. This
is for instance very useful when determining the most suitable virtual sensing algorithm for the local active noise control problem under consideration. The presented analysis also allows for a straightforward interpretation of the factors that determine the optimal estimation performance of the previously proposed virtual sensing algorithms. This can be used to determine the locations and the number of physical sensors that are required to obtain a certain estimation performance.

- The adaptive LMS virtual microphone technique [14] has only been introduced for the case of one virtual sensor. This technique is therefore extended to the case of multiple virtual sensors. Furthermore, an optimal solution for the unknown parameters of this technique is derived given a state-space model of the local active noise control system under consideration. This solution is especially useful in an initial numerical analysis of this technique because the unknown parameters no longer need to be determined in a computationally more intensive adaptive manner, which was the case in previous research [14, 80].

- It is shown that the hybrid adaptive feedforward observer [119] is only suitable for rejecting non-stationary disturbances at the physical sensors and not for virtual sensing purposes.

- Virtual sensing methods for local active noise control have predominantly been implemented in combination with an adaptive feedforward control approach. The question now arises whether this implementation results in a decrease in the optimal control performance that can be obtained at the virtual sensors after convergence of the adaptive algorithm in comparison to using physical sensors at the virtual locations. If this is the case, the control performance that is actually lost needs to be quantified. These questions are addressed for the case of using the various virtual sensing algorithms that have been suggested previously [14, 27, 104, 112].

- The optimal feedback control at virtual sensors problem is solved by recognising that this problem is equivalent to the general feedforward/feedback control problem, which is a standard problem often encountered in active noise control [25].

As discussed previously, the virtual sensing algorithms proposed so far [14, 27, 66, 104, 112] have all been developed with the aim to move the zone of quiet to a virtual location that is assumed spatially fixed within the sound field. Because an observer is very likely to move their head, the desired location of the zone of quiet is generally moving through the sound field rather than being spatially fixed. For effective control, a local active noise control system incorporating a virtual sensing method thus has to be able to create a moving zone of quiet that tracks the observer’s ears. One of the main
contributions of this thesis is the development of a *moving virtual sensing method*. This method can be used to estimate the sound pressure at a virtual location that is moving through the sound field rather than being spatially fixed. The following contributions relating to this problem are presented in this thesis.

- An *optimal solution* to the moving virtual sensing for local active noise control problem is derived by using Kalman filtering theory. The developed algorithm is implemented on an acoustic duct arrangement to demonstrate its effectiveness in a practical implementation.

- The previously proposed *spatially fixed* virtual sensing algorithms [14, 27, 104, 112] are modified to account for a virtual location that is moving through the sound field rather than being spatially fixed.

- When the virtual location is moving through the sound field, the primary disturbance that needs to be attenuated is a *non-stationary* process. To account for this, an adaptive control algorithm needs to be used that has the ability to *track* the statistical changes in the primary disturbance at the moving virtual location. In this thesis, a number of adaptive feedforward control algorithms are modified and combined with the developed moving virtual sensing algorithms, i.e. the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms. The ability of these algorithms to track the non-stationarities in the estimate of the primary disturbance at the moving virtual location is experimentally analysed for narrowband disturbances inside an acoustic duct arrangement.

In a practical application of the proposed moving virtual sensing methods, an important issue that would need to be addressed is how to determine the desired locations of maximum attenuation, i.e. the moving virtual locations. This could for instance be done using a 3D head tracking system based on *camera vision* or on *ultrasonic position sensing* such as the Logitech® head tracker. However, the issue of determining the desired locations of the zones of quiet is beyond the scope of this thesis. The aim here is therefore to develop methods that can be used to create zones of quiet at a number of virtual locations that move through the sound field, while assuming that an exact measurement of these locations is available.

### 1.4 Outline of thesis

This thesis consists of two parts. The first part is titled *Algorithms for active noise control at virtual sensors* and consists of Chapters 2–5. This part presents the active noise control algorithms that are used in this thesis, a detailed analysis of the previously proposed spatially fixed virtual sensing algorithms, and the developed optimal spatially fixed
and moving virtual sensing algorithms. The first part of this thesis is therefore relatively theoretical and mathematical in nature. The second part titled Experimental validation of algorithms is more practical and consists of Chapters 6–10. In this part, the algorithms that have been developed in part I are implemented on an acoustic duct arrangement in order to verify their performance in a practical implementation. A detailed outline of each chapter follows.

Part I: Algorithms for active noise control at virtual sensors

In Chapter 2, the considered active noise control system is introduced and a number of control strategies that are frequently used in local active noise control are presented. The signals and transfer paths that describe the input-output behaviour of the system are defined, and the mathematical notation and assumptions used throughout this thesis are introduced. The narrowband control performance that can be achieved at the virtual locations is derived by minimising the true sound pressure at the virtual locations using a quadratic optimisation technique \[88\]. A solution to the optimal feedforward control problem, in which the true broadband sound pressure at the virtual location is minimised without directly measuring it during real-time control, is presented using a factorisation approach \[34, 126\]. This solution is called the causal Wiener filter solution and is used frequently throughout this thesis to solve linear estimation and control problems. For instance, the optimal feedback control problem, in which the true broadband sound pressure at the virtual location is minimised using a feedback controller while assuming this sound pressure is directly measured during real-time control, is solved in Chapter 2 by reformulating it as an equivalent feedforward control problem using an internal model control approach \[25\]. This equivalent feedforward problem can then be solved using the causal Wiener filter solution. The general feedforward/feedback control problem, which can be implemented without directly measuring the true sound pressure at the virtual location during real-time control, is solved in a similar way. A number of adaptive feedforward control algorithms are also introduced that can be used when the sound pressures at the virtual locations are directly measured during real-time control.

The optimal feedforward and general feedforward/feedback control approaches introduced in Chapter 2 can be used without directly measuring the sound pressures at the virtual locations during real-time control. However, the presented optimal feedback and adaptive feedforward control approaches can only be used when these sound pressures are directly measured during real-time control. The reason is that these control approaches require the feedback information contained in these virtual sound pressures to compute an appropriate control signal. If a direct measurement of the sound pressures at the virtual locations is not available during real-time control, these control approaches can still be implemented by using an estimate of these sound pressures as a feedback signal. Chapter 3 introduces a number of previously proposed spatially fixed virtual sensing
algorithms that can be used to compute an estimate of the virtual sound pressures. The *adaptive LMS virtual microphone technique* [14] is extended to the case of multiple virtual sensors, the factors that limit the optimal estimation performance of the *remote microphone technique* [104, 112] and the *virtual microphone arrangement* [27] are derived, and it is shown that the *hybrid adaptive feedforward observer* [119] cannot be used as a virtual sensing algorithm. The main contribution in Chapter 3 is the development of an *optimal solution* to the virtual sensing for active noise control problem, which is derived using Kalman filtering theory.

In Chapter 4, the *adaptive feedforward control* algorithms introduced in Chapter 2 are modified to account for the situation when the sound pressures at the virtual locations are not directly measured during real-time control but estimated using one of the *spatially fixed* virtual sensing algorithms introduced in Chapter 3. The presented modified implementation of the adaptive feedforward algorithms has been used by a number of researchers that investigated spatially fixed virtual sensing algorithms for active noise control. The only difference between this implementation and the standard implementation of the adaptive feedforward algorithms introduced in Chapter 2 is that an *estimate* of the sound pressures at the virtual locations is adaptively minimised instead of the *true* sound pressures. It is shown in Chapter 4 that this results in a decrease in the optimal control performance that can be obtained at the virtual locations, with the degree of control performance lost depending on the estimation accuracy of the applied spatially fixed virtual sensing algorithm. Using the causal Wiener filter theory introduced in Chapter 2 and the analyses of the spatially fixed virtual sensing algorithms presented in Chapter 3, the degree of control performance that is lost is quantified and the factors that determine this amount are discussed.

Finally, the *optimal feedback control* algorithm introduced in Chapter 2 is modified to account for the situation when the virtual sound pressures are not directly measured during real-time control. A solution to the resulting *optimal feedback control at virtual sensors problem* is derived by showing that this problem is equivalent to the general feedforward/feedback control problem introduced in Chapter 2.

In Chapter 5, *moving virtual sensing* algorithms are developed that can be used to estimate the sound pressure at a virtual location that is *moving* through the sound field rather than being spatially fixed as assumed in Chapter 3. It is shown that an *optimal solution* to the moving virtual sensing problem can be derived by using Kalman filtering theory. Furthermore, the previously proposed spatially fixed virtual sensing algorithms [14, 104, 112] are modified to account for a moving virtual location. The developed moving virtual sensing algorithms are combined with the adaptive feedforward control algorithms introduced in Chapter 2. These adaptive algorithms enable the *tracking* of the non-stationarities in the estimate of the primary disturbance at the moving virtual location. The proposed method aims to create a moving zone of
quiet that tracks the desired location of maximum attenuation, and are implemented on an acoustic duct arrangement for the case of narrowband disturbances in the second part of this thesis.

**Part II: Experimental validation of algorithms**

Chapter 6 presents the acoustic duct arrangement on which the algorithms presented in Part I of this thesis are implemented during real-time experiments. Numerical models of the acoustic duct arrangement are introduced. The contribution presented in this chapter is the derivation of a more succinct expression for a previously proposed *travelling wave model* [129]. Furthermore, *subspace model identification techniques* are introduced [50, 124]. These system identification techniques are used in the presented real-time experiments to estimate an innovations model of the acoustic duct arrangement in a preliminary identification stage.

In Chapter 7, the *adaptive LMS virtual microphone technique* [14] which has been analysed in Chapter 3 is implemented on the acoustic duct arrangement and the *narrowband and broadband estimation performance* is analysed for a number of virtual locations and physical sensor configurations. This technique is then combined with the filtered-x LMS algorithm as described in Chapter 4 and the *narrowband and broadband adaptive feedforward control performance* is analysed. The adaptive LMS virtual microphone technique has been analysed previously for the acoustic duct case [80] and some additional insights are presented that explain the disagreement between numerical and experimental results reported in this previous research. Finally, *forward difference prediction techniques*, which have been proposed and investigated previously [13, 80], are compared to the adaptive LMS virtual microphone technique for the case of single tone disturbances inside the acoustic duct arrangement. The analytical analysis shows that the forward difference prediction weights approach the analytical optimal solutions for the weights for low frequencies and small distances between the physical and virtual microphones, which explains the experimental results observed in previous research [80].

In Chapter 8, the *Kalman filter based spatially fixed virtual sensing algorithm* developed in Chapter 3 is implemented on the acoustic duct arrangement and the *narrowband and broadband estimation performance* is analysed for a number of virtual locations. This virtual sensing algorithm is then combined with the filtered-x LMS algorithm as described in Chapter 4 such that the estimate of the sound pressure at the virtual location is minimised. The *narrowband and broadband adaptive feedforward control performance* that is obtained is analysed for a number of virtual locations.

In Chapter 9, the tracking performance of the *filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms* is investigated for the case of a physical sensor that moves through a narrowband sound field inside an acoustic duct arrangement. The physical
sensor tracks the desired location of maximum attenuation such that the sound pressure at this location is directly measured during real-time control. The adaptive feedforward control algorithms considered are implemented as described in Chapter 5, and the aim is to create a moving zone of quiet at a moving physical sensor. The narrowband control performance is compared for various speeds of movement and various spatial characteristics of the sound field.

In Chapter 10, the moving virtual sensing algorithms developed in Chapter 5 are implemented on the acoustic duct arrangement with the aim to create a moving zone of quiet at a virtual sensor that tracks the desired location of maximum attenuation. The difference with the experiments presented in Chapter 9 is that the sound pressure at the moving virtual location is now estimated instead of directly measured using a moving physical sensor. To minimise the estimate of the sound pressure at the moving virtual location, the filtered-x RLS algorithm is implemented as described in Chapter 5 because the experimental results presented in Chapter 9 show that this algorithm provides the best tracking of the non-stationary primary disturbance that needs to be attenuated. The narrowband control performance that is obtained at the moving virtual location is compared for various speeds of movement and various spatial characteristics of the sound field. It is also compared with the narrowband control performance that is obtained in Chapter 9, where a moving physical sensor that directly measures the sound pressure at the moving virtual location is used instead of a moving virtual sensor.

In summary, the main contributions of this thesis are the development of optimal spatially fixed and moving virtual sensing algorithms that can be used to improve the performance of local active noise control systems. The optimal virtual sensing algorithms are derived using Kalman filtering theory and are combined with local active noise control algorithms. The developed methods are implemented on an acoustic duct arrangement. The results show that the proposed spatially fixed virtual sensing algorithm can effectively be used to move the zone of quiet away from the physical sensor to a desired location of maximum attenuation that is spatially fixed. The developed moving virtual sensing algorithm can be used to create a moving zone of quiet that tracks the desired location of maximum attenuation. The experimental results show that this has been achieved for narrowband disturbances inside an acoustic duct arrangement. The presented research has led to further research questions, which are outlined in Chapter 11.

1.5 Publications

The journal and conference publications arising from the research presented in this thesis are listed in the following.
Journal publications


Conference publications


Part I

Algorithms for active noise control at virtual sensors
Chapter 2

Active noise control algorithms

2.1 Introduction

In this chapter, the considered local active noise control problem is introduced and the theory that is used throughout this thesis to solve linear estimation and control problems is presented. The signals and transfer paths that describe the input-output behaviour of the local active noise control system are defined, and the mathematical notation and assumptions used throughout this thesis are presented.

In Section 2.2, a block diagram of the considered local active noise control system which incorporates a virtual sensing method is presented. This block diagram is included to visualise the input-output behaviour of the active noise control system, and provides a consistent definition of the input signals, output signals, and transfer paths that frequently occur in the analyses presented throughout this thesis. A standard state-space model [60] of the active noise control system illustrated in the presented block diagram is introduced as well.

In Section 2.3, the optimal narrowband control performance that can be achieved at a number of spatially fixed virtual locations is derived by minimising the true virtual error signals using a quadratic optimisation technique. This technique can be used in an initial analysis of the active noise control system in which a theoretical limit on the narrowband control performance that can be obtained at the spatially fixed virtual locations can be computed given the chosen control source and virtual sensor configurations.

In Section 2.4, the optimal broadband control performance that can be obtained at the virtual locations is derived while assuming that the virtual locations are spatially fixed within the sound field. If a suitable feedforward reference signal is available, an optimal feedforward control approach can be adopted to minimise the virtual error signals without directly measuring them during real-time control. A solution to the resulting optimal feedforward control problem is presented in Section 2.4.1 using a factorisation
approach. This approach is adopted as it allows a convenient interpretation of the factors that limit the optimal feedforward control performance that can be obtained at the spatially fixed virtual locations. The presented optimal feedforward control solution is called the causal Wiener filter solution, and is used frequently throughout this thesis to solve linear estimation and control problems. If the virtual locations are spatially fixed but a suitable feedforward reference signal is not available, an optimal feedback control approach can be adopted. A solution to the resulting optimal feedback control problem is presented in Section 2.4.2 assuming that the virtual error signals are directly measured during real-time control. Although the presented feedback solution cannot be used here, since it is assumed that the virtual error signals are not directly measured during real-time control, it is still included because it is of relevance in Section 2.4.3 where it is used to derive a solution to the general feedforward/feedback control problem. The general feedforward/feedback control approach can be adopted if the virtual locations are spatially fixed and suitable feedforward reference signals are available but these signals are contaminated by intrinsic feedback from the control sources. The presented general feedforward/feedback control solution can be used to minimise the virtual error signals without directly measuring them during real-time control.

In Section 2.5, a number of adaptive feedforward control algorithms are introduced that can be used when the virtual error signals are directly measured during real-time control. Although this is not the case for the local active noise control problem considered here, these adaptive algorithms are presented because they will be combined in Chapter 4 with the spatially fixed virtual sensing algorithms that will be introduced in Chapter 3. The combination of these algorithms provides a method for adaptively minimising an estimate of the virtual error signals at a number of spatially fixed virtual locations. Furthermore, the adaptive algorithms that are presented in this chapter will be modified in Chapter 5 and combined with moving virtual sensing algorithms, which can be used to compute an estimate of the virtual error signals at virtual locations that are moving through the sound field. The combination of these algorithms provides a method for adaptively minimising an estimate of the virtual error signals at a number of moving virtual locations. The adaptive feedforward control algorithms to be introduced in this chapter, and the modified versions of these algorithms that will be presented in Chapters 4 and 5, will also used in the acoustic duct experiments presented in Chapters 7–10.

2.2 Problem definition and assumptions

In this section, a block diagram of the considered active noise control system incorporating a virtual sensing method is presented. This block diagram is included to visualise the input-output behaviour of the considered active noise control system,
and provides a consistent definition of the input signals, output signals, and transfer paths that frequently occur in the analyses presented throughout this thesis. A standard state-space model [60] of the active noise control system is presented, and the covariance properties of its stochastic input signals are defined. The block diagram is also used to introduce the various control approaches that are adopted in this thesis to solve the considered active noise control problem.

### 2.2.1 Block diagram of active noise control system

A block diagram of an active noise control system incorporating a virtual sensing method is shown in Fig. 2.1 for the multiple-reference, multiple-input, and multiple-output case. It is assumed in this figure that there are \( S \) disturbance source signals, \( L \) control sources, \( K \) reference sensors, \( M_p \) physical sensors, and \( M_v \) virtual sensors. In this thesis, the disturbance source signals \( s(n) \in \mathbb{R}^S \), with \( n \) a discrete-time index, are assumed to be either single tone signals of the same normalised frequency \( \omega T_s \), with \( T_s \) the sample time, or unknown white and stationary random processes with zero-mean and unit covariance, such that [5]

\[
E \left[ \begin{bmatrix} s(n) \\ 1 \end{bmatrix} s^T(k) \right] = \begin{bmatrix} I_0 \\ 0 \end{bmatrix} \delta_{nk}, \quad (2.1)
\]

where \( E[\cdot] \) denotes the expectation of the term inside brackets, \( I \in \mathbb{R}^{S \times S} \) is the identity matrix, and \( \delta_{nk} \) the Kronecker delta function defined as

\[
\delta_{nk} = \begin{cases} 
1, & \forall n = k \\
0, & n \neq k.
\end{cases} \quad (2.2)
\]

In Fig. 2.1, the disturbance source signals \( s(n) \) propagate through the discrete-time virtual primary transfer paths \( G_{vs}(z^{-1}, n) \in \mathcal{RH}_{m \times n}^{M_v \times S} \), where \( \mathcal{RH}_{m \times n}^{M_v \times n} \) is the set of all proper and asymptotically stable rational \( m \times n \) transfer function matrices with real coefficients\(^1\) [54], and where \( z^{-1} \) is the unit delay operator, such that [25]

\[
z^{-1}s(n) = s(n-1). \quad (2.3)
\]

Note that a transfer function is called rational if its numerator and denominator are polynomials, proper if the order of the numerator polynomial is lower than or equal to the order of the denominator polynomial, and asymptotically stable if all its poles are inside the unit circle. Also note that the dependence of \( G_{vs}(z^{-1}, n) \) on the time index \( n \) indicates that the virtual primary transfer path may be time-variant, which is the case

\(^1\) This restriction is put on the transfer function matrices since the systems to be controlled in active noise control applications are generally asymptotically stable [34].
Figure 2.1: Block diagram of an active noise control system incorporating a virtual sensing algorithm, with $S$ disturbance sources, $L$ control sources, $K$ reference sensors, $M_p$ physical sensors, and $M_v$ virtual sensors, with -- signals available during preliminary identification stage but not during real-time control -- feedforward reference signals that are potentially available during real-time control.
2.2 Problem definition and assumptions

when the virtual locations are moving through the sound field. If the virtual locations are spatially fixed, the virtual primary transfer path is assumed to be linear time-invariant, and the dependence on the time index \( n \) then disappears. In the following, the argument \((z^{-1}, n)\) will often be omitted such that \( G_{\text{vs}} \) will refer to the transfer function matrix \( G_{\text{vs}}(z^{-1}, n) \).

As shown in Fig. 2.1, the output of the virtual primary transfer paths are the virtual primary disturbances \( d_v(n) \in \mathbb{R}^{M_v} \) that need to be attenuated. This is achieved by generating appropriate control signals \( u(n) \in \mathbb{R}^L \) that actuate the control sources to produce the virtual secondary disturbances \( y_v(n) \in \mathbb{R}^{M_v} \) at the virtual sensors via the virtual secondary transfer paths \( G_{vu} \in \mathcal{RH}_\infty^{M_v \times L} \). Note again that this transfer path matrix is time-variant when the virtual locations are moving through the sound field. The virtual secondary disturbances interfere with the virtual primary disturbances \( d_v(n) \) at the virtual sensors to yield the virtual residual or error signals \( e_v(n) \in \mathbb{R}^{M_v} \) given by

\[
e_v(n) = d_v(n) + y_v(n) = G_{\text{vs}}s(n) + G_{vu}u(n).
\]

The aim of the active noise control system illustrated in Fig. 2.1 is to minimise the error signals \( e_v(n) \) at the virtual sensors assuming that these signals are not directly measured during real-time control. These virtual sensors are positioned at the desired locations of maximum attenuation, such as the ears of an observer. These locations are also referred to as the virtual locations in the following, and are assumed to be either spatially fixed within or moving through the sound field. Thus, the aim of the active noise control system considered here is to minimise the virtual error signals \( e_v(n) \) at the virtual locations without directly measuring these error signals during real-time control.

Optimal feedforward control

As indicated by the dash-dotted line in Fig. 2.1, a number of feedforward reference signals \( x(n) \in \mathbb{R}^K \) are potentially available that are strongly correlated to the virtual primary disturbances \( d_v(n) \) at the virtual locations. If this is the case and the virtual locations are spatially fixed, an optimal feedforward control approach can be adopted to achieve the minimisation of the virtual error signals \( e_v(n) \) without directly measuring these signals during real-time control. A fixed-gain feedforward controller \( W \in \mathcal{RH}_\infty^{L \times K} \) can then be designed that uses the feedforward reference signals \( x(n) \) to compute a control signal \( u(n) \) as

\[
u(n) = Wx(n).
\]

2. The variable \( z \) is also used in this thesis as a complex variable in the z-transform [51]. The virtual primary transfer path is then denoted by \( G_{\text{vs}}(z) \) instead of \( G_{\text{vs}}(z^{-1}) \), and defines the relationship between the z-transforms of the disturbance source signals and the virtual primary disturbances, see Appendix B.
In the optimal feedforward control problem, it is assumed that the feedforward reference signals \( x(n) \) are not contaminated by intrinsic feedback signals \( x_{fb}(n) \in \mathbb{R}^K \) that can potentially originate from the control sources as shown in Fig. 2.1. As an example, if the feedforward reference signal is generated by a tachometer on the engine of a vehicle, an intrinsic feedback path between the control loudspeakers and the feedforward reference signal does usually not exist, and the feedforward reference signal will not be contaminated by intrinsic feedback [30]. In the optimal feedforward control problem, the feedforward reference signals \( x(n) \) are thus assumed to be generated as

\[
x(n) = G_{xs}s(n),
\]

where \( G_{xs} \in \mathcal{RH}_{[0,\infty) \times [S]}^{K \times S} \) is usually referred to as the detector transfer path matrix. A solution to the optimal feedforward control problem is presented in Section 2.4.1 for the general case of broadband disturbances.

**General feedforward/feedback control**

If an intrinsic feedback path between the control sources and the reference sensors does exist, the feedforward reference signals \( x(n) \) in Fig. 2.1 are contaminated by the intrinsic feedback signals \( x_{fb}(n) \) given by

\[
x_{fb}(n) = G_{xu}u(n),
\]

with \( G_{xu} \in \mathcal{RH}_{[0,\infty) \times [L]}^{K \times L} \) the intrinsic feedback transfer path matrix. The contamination of the feedforward reference signals occurs, for instance, in the active control of noise inside ducts where there usually is an intrinsic feedback path between the control source and the reference sensor placed upstream of this source [68]. Assuming that the virtual locations are spatially fixed, the resulting control problem is referred to as the general feedforward/feedback control problem, and the feedforward reference signals \( x(n) \) are then given by

\[
x(n) = G_{xs}s(n) + G_{xu}u(n),
\]

with the control signals \( u(n) \) computed as

\[
u(n) = Cx(n),
\]

where \( C \) is the fixed-gain feedback controller with \( K \) inputs and \( L \) outputs. If contaminated feedforward reference signals \( x(n) \) are available, a general feedforward/feedback control approach can thus be adopted to achieve the minimisation of the virtual error signals \( e_v(n) \) without directly measuring these signals during real-time control. A solution to the general feedforward/feedback control problem is presented in Section 2.4.3 for the general case of broadband disturbances.
2.2 Problem definition and assumptions

Adaptive feedforward control

If suitable feedforward reference signals $x(n)$ are available, the most common approach in active noise control systems is to use an adaptive feedforward control approach instead of an optimal feedforward control approach [25, 68]. For the case considered here, adaptive control algorithms generally need the feedback information contained in the virtual error signals $e_v(n)$ to update the feedforward controller appropriately. In Section 2.5, a number of adaptive feedforward control algorithms are introduced while assuming that the virtual error signals are directly measured during real-time control and that the virtual locations are spatially fixed.

In Chapter 4, the adaptive feedforward control algorithms presented in Section 2.5 are combined with the spatially fixed virtual sensing algorithms introduced in Chapter 3 to account for the fact that it is assumed here that the virtual error signals are not directly measured during real-time control. As illustrated in Fig. 2.1, the virtual sensing algorithm computes an estimate $\hat{e}_v(n)$ of the virtual error signals, which can then be used as a feedback signal to the adaptive feedforward control algorithm. This implementation was originally suggested by Elliott and David [27], and the optimal control performance that can be achieved at the spatially fixed virtual locations when using this implementation is analysed in Chapter 4.

In Chapter 5, the adaptive feedforward control algorithms presented in Section 2.5 are further modified to also account for the situation when the virtual locations are moving through the sound field rather than being spatially fixed. Moving virtual sensing algorithms that can be used to compute an estimate of the virtual error signals at the moving virtual locations are therefore developed in Chapter 5. As illustrated in Fig. 2.1, this estimate can then be used as a feedback signal to the adaptive feedforward control algorithm. Note that when the virtual locations are moving through the sound field, an adaptive control approach needs to be adopted to account for the fact that the virtual primary disturbances that need to be attenuated are non-stationary signals. This will be discussed in more detail in Chapter 5.

Optimal feedback control

If suitable feedforward reference signals are not available to the controller in Fig. 2.1, an optimal feedback control approach can be adopted provided that the virtual locations are spatially fixed and the virtual error signals $e_v(n)$ are directly measured during real-time control. In this instance, the feedback control signal can be computed as

$$u(n) = Ce_v(n),$$

where $C$ is the fixed-gain feedback controller with $M_v$ inputs and $L$ outputs. A solution to the resulting optimal feedback control problem is presented in Section 2.4.2. However, it
Chapter 2  Active noise control algorithms

It is assumed in this thesis that the virtual error signals are not directly measured during real-time control, and the control signals can therefore not be computed as defined in Eq. (2.10). Instead, the feedback control signals can effectively be computed as

\[ u(n) = C\hat{e}_v(n), \]

where \( \hat{e}_v(n) \) is an estimate of the virtual error signals computed by a virtual sensing algorithm, as illustrated in Fig. 2.1. In Section 4.4, an \( H_2 \) optimal solution [18] to the optimal feedback control at spatially fixed virtual sensors problem is presented, assuming that the virtual error signals are not directly measured during real-time control. It will be shown that this solution can be computed by noting that this problem is equivalent to the general feedforward/feedback control problem solved in Section 2.4.3.

Virtual sensing algorithm

As discussed previously, when the virtual error signals are not directly measured during real-time control, a virtual sensing algorithm needs to be used when adopting either an adaptive feedforward or optimal feedback control approach. The control performance that is then obtained at the locations of the virtual sensors will depend on the estimation accuracy of the virtual sensing algorithm. As illustrated in Fig. 2.1, the estimation accuracy of the virtual sensing algorithm can be characterised by the virtual output errors \( \varepsilon_v(n) \in \mathbb{R}^{M_v} \), which are defined as

\[ \varepsilon_v(n) = e_v(n) - \hat{e}_v(n). \]

The aim of the virtual sensing algorithm in Fig. 2.1 is to minimise the virtual output errors in Eq. (2.12) such that accurate estimates of the virtual error signals are available to the controller. For this purpose, the virtual sensing algorithm can use information contained in the physical error signals \( e_p(n) \in \mathbb{R}^{M_p} \), which are measured by a number of physical sensors that are assumed to be spatially fixed. As illustrated in Fig. 2.1, the physical error signals are given by

\[ e_p(n) = d_p(n) + y_p(n), \]

with \( d_p(n) \in \mathbb{R}^{M_p} \) the physical primary disturbances, and \( y_p(n) \in \mathbb{R}^{M_p} \) the physical secondary disturbances. The physical primary disturbances originate from the disturbance source signals \( s(n) \) which propagate through the linear time-invariant physical primary transfer paths \( G_{ps} \in \mathbb{R}^{M_p \times S} \) to give

\[ d_p(n) = G_{ps}s(n). \]
The physical secondary disturbances result from exciting the control sources with the control signals $u(n)$, which propagate through the linear time-invariant physical secondary transfer paths $G_{pu} \in \mathcal{RH}_\infty^{M_p \times L}$ to give
\[ y_p(n) = G_{pu}u(n). \] (2.15)

In Chapter 3, the virtual sensing problem is analysed for the case of virtual locations that are *spatially fixed* within the sound field. A new virtual sensing algorithm based on Kalman filtering is developed, and a number of previously suggested virtual sensing algorithms [14, 27, 104, 112, 119] are analysed in detail. In Chapter 5, the virtual sensing problem is analysed for the case of virtual locations that are *moving* through the sound field rather than being spatially fixed. A *moving virtual sensing* algorithm based on Kalman filtering is developed and a number of previously proposed spatially fixed virtual sensing algorithms [14, 27, 104, 112] are modified to the case of a moving virtual location.

### 2.2.2 Standard state-space model

In this section, a *standard state-space model* [60] of the considered active noise control system is introduced that describes the input-output relationship between all of the input and output signals in the block diagram shown in Fig. 2.1. These input and output signals have been defined in the previous section, and the input-output relationship can be written in transfer path matrix form as
\[
\begin{bmatrix}
    e_p(n) \\
    e_v(n) \\
    x(n)
\end{bmatrix}
= \begin{bmatrix}
    G_{pu} & G_{ps} \\
    G_{vu} & G_{vs} \\
    G_{xu} & G_{xs}
\end{bmatrix}
\begin{bmatrix}
    u(n) \\
    s(n)
\end{bmatrix}, \tag{2.16}
\]

where $G \in \mathcal{RH}_\infty^{(M_p+M_v+K) \times (L+S)}$ denotes the *plant transfer path matrix*, which contains all the transfer paths that have been introduced in the previous section. Measurement noise on the reference, physical, and virtual sensors has not been included in Eq. (2.16) and Fig. 2.1. A standard state-space model that describes the considered plant, in which measurement noise on the sensors is included, can now be defined as [60]
\[
\begin{align*}
    z(n+1) &= Az(n) + B_u u(n) + B_s s(n) \\
    e_p(n) &= C_p z(n) + D_{pu} u(n) + D_{ps} s(n) + v_p(n) \\
    e_v(n) &= C_v z(n) + D_{vu} u(n) + D_{vs} s(n) + v_v(n) \\
    x(n) &= C_x z(n) + D_{xu} u(n) + D_{xs} s(n) + v_x(n),
\end{align*} \tag{2.17}
\]

with $z(n) \in \mathbb{R}^N$ the states of the system, where $N$ is the system order, $v_p(n) \in \mathbb{R}^{M_p}$ the physical measurement noise, $v_v(n) \in \mathbb{R}^{M_v}$ the virtual measurement noise, and
\( v_s(n) \in \mathbb{R}^K \) the reference measurement noise. The state-space matrices in Eq. (2.17) are real-valued and of appropriate dimensions. Note that the state-space matrices \( C_{v}(n) \), \( D_{vu}(n) \), and \( D_{vs}(n) \) are time-variant when the virtual locations are moving through the sound field. The dependence on the time index \( n \) disappears for these state-space matrices when the virtual locations are spatially fixed. The state-space model of the plant in Eq. (2.17), excluding the measurement noise on the sensors, will be denoted by

\[
G \sim \begin{bmatrix}
A & B_u & B_s \\
C_p & D_{pu} & D_{ps} \\
C_v & D_{vu} & D_{vs} \\
C_x & D_{xu} & D_{xs}
\end{bmatrix}.
\]

An overview of some of the calculus that can be performed on state-space models is given in Appendix B. The material presented in this appendix will be used frequently throughout this thesis.

### 2.2.3 Covariance properties

In this section, the covariance and cross-covariance properties of the measurement noise on the physical, virtual, and reference sensors are defined. It is assumed that the measurement noise signals in Eq. (2.17) are zero-mean white and stationary random processes, such that the following covariance matrices can be defined

\[
E \begin{bmatrix}
s(n) \\
v_p(n) \\
v_v(n) \\
v_x(n)
\end{bmatrix}^T = \begin{bmatrix}
1 & S_{ps}^T & S_{vs}^T & S_{xs}^T & 0 \\
S_{ps} & R_p & R_{pv} & R_{px} & 0 \\
S_{vs} & R_{pv} & R_v & R_{vx} & 0 \\
S_{xs} & R_{px} & R_{vx} & R_x & 0
\end{bmatrix} \delta_{nk}
\]

with \( R_p \in \mathbb{R}^{M_p \times M_p} \) the covariance matrix of the physical measurement noise, \( R_v \in \mathbb{R}^{M_v \times M_v} \) the covariance matrix of the virtual measurement noise, and \( R_x \in \mathbb{R}^{K \times K} \) the covariance matrix of the reference measurement noise. These measurement noise signals and the disturbance source signals \( s(n) \) are often uncorrelated [60], such that the cross-covariance matrices \( S_{ps} \in \mathbb{R}^{M_p \times S} \), \( S_{vs} \in \mathbb{R}^{M_v \times S} \), and \( S_{xs} \in \mathbb{R}^{K \times S} \) in Eq. (2.19) are often equal to zero. The measurement noise on the physical, virtual, and reference sensors are also usually uncorrelated, such that the cross-covariance matrices \( R_{pv} \in \mathbb{R}^{M_p \times M_v} \), \( R_{px} \in \mathbb{R}^{M_p \times K} \), and \( R_{vx} \in \mathbb{R}^{M_v \times K} \) in Eq. (2.19) are usually equal to zero as well. The covariance properties defined in Eq. (2.19) are used frequently in the analyses of the spatially virtual sensing algorithms presented in Chapter 3 and the moving virtual sensing algorithms presented in Chapter 5.
2.3 Optimal narrowband control

In this section, the optimal narrowband control performance that can be obtained at the virtual locations when minimising the true virtual error signals is derived for the case of virtual locations that are spatially fixed within the sound field. It is therefore assumed that the disturbance source signals $s(n)$ in Fig. 2.1 are single tone signals of normalised frequency $\omega T_s$, with $T_s$ the sample time. To analyse the considered active noise control problem at a single frequency, it is useful to define the relevant variables in Fig. 2.1 in a complex representation [25]. A technique called quadratic optimisation can then be used to compute the narrowband control performance that can be achieved when minimising the true virtual error signals. The presented technique can be used in an initial analysis of the active noise control system in which a theoretical limit on the narrowband control performance that can be obtained at the spatially fixed virtual locations can be computed, given the chosen control source and virtual sensor configuration.

**Quadratic optimisation**

The tonal disturbance source signals $s(n)$ propagate through the virtual primary transfer paths resulting in tonal virtual primary disturbances $d_v(n)$ of the same normalised frequency $\omega T_s$. These virtual primary disturbances can be attenuated by exciting the control sources with tonal control signals $u(n)$ of the same normalised frequency, whose magnitudes and phases need to be adjusted appropriately [25]. The optimal adjustment of the magnitudes and phases of the control signals can be achieved using a quadratic optimisation technique. A block diagram of the multi-channel quadratic optimisation problem is shown in Fig. 2.2.

![Figure 2.2: Block diagram of the multi-channel quadratic optimisation problem for a normalised frequency $\omega T_s$, with $L$ control sources and $M_v$ spatially fixed virtual sensors.](image)

In Fig. 2.2, the vectors of complex numbers $u(e^{j\omega T_s}) \in C^L$ and $d_v(e^{j\omega T_s}) \in C^{M_v}$ define the magnitudes and phases of the control signals and the virtual primary disturbances, respectively, with respect to some arbitrary reference. The virtual error signals $e_v(e^{j\omega T_s}) \in C^{M_v}$ can then be defined as

$$e_v(e^{j\omega T_s}) = d_v(e^{j\omega T_s}) + y_v(e^{j\omega T_s}) = d_v(e^{j\omega T_s}) + Z_{vu}u(e^{j\omega T_s}), \quad (2.20)$$
where $Z_{vu} \in \mathbb{C}^{M_v \times L}$ is the complex virtual secondary transfer impedance matrix. This matrix is determined by evaluating the virtual secondary transfer path matrix $G_{vu}$ at the specific normalised frequency $\omega T_s$ of interest, such that

$$Z_{vu} \triangleq G_{vu}(e^{j\omega T_s}).$$

The transfer impedance matrix $Z_{vu}$ relates the magnitudes and phases of the virtual secondary disturbances $y_v(e^{j\omega T_s})$ to the magnitudes and phases of the control signals $u(e^{j\omega T_s})$. In the following, the dependence on $\omega T_s$ will be omitted for notational convenience.

The virtual error signals are now minimised via a cost function $J$ that is defined as the sum of the mean-square values of the virtual error signals, such that

$$J = e_{v}^{H} e_{v},$$

where $(\cdot)^H$ denotes the complex conjugate transpose of the term inside parentheses. Using Eq. (2.20), this cost function can be written in Hermitian quadratic form as

$$J = u^{H} A u + u^{H} b + b^{H} u + c,$$

where the matrix $A \in \mathbb{C}^{L \times L}$ is a Hermitian matrix defined as

$$A = Z_{vu}^{H} Z_{vu},$$

the column vector $b \in \mathbb{C}^{L}$ is given by

$$b = Z_{vu}^{H} d_v,$$

and the real scalar $c$, which is equal to the sum of the mean-square values of the virtual primary disturbances, is defined as

$$c = d_v^{H} d_v.$$

The optimal control signal $u_o$ can now be found by minimising the cost function $J$ defined in Eq. (2.23). The optimal solution depends on the relative number of control sources $L$ and spatially fixed virtual sensors $M_v$. A general formulation for the optimal solution can be written as

$$u_o = -Z_{vu}^{+} d_v,$$

where $(\cdot)^{+}$ denotes the pseudo-inverse of the term inside parentheses. If there are more virtual sensors than control sources, such that $M_v > L$, the system is said to be over-determined. The pseudo-inverse is then defined as

$$Z_{vu}^{+} = \left[Z_{vu}^{H} Z_{vu}\right]^{-1} Z_{vu}^{H}.$$
and the minimum value of the cost function defined in Eq. (2.22) can be expressed for this case as

\[ J_{\text{min}} = c - b^H A^{-1} b. \]  

(2.29)

If the number of control sources and virtual sensors are the same, such that \( L = M_v \), the pseudo-inverse is given by

\[ Z_v^\dagger = Z_v^{-1}. \]  

(2.30)

The minimum value of the cost function in Eq. (2.23) is equal to zero in this case. In other words, all the virtual error signals \( e_v \) can be minimised to zero, such that the virtual primary disturbance can be completely cancelled in theory. If there are more control sources than virtual sensors, such that \( L > M_v \), the system is said to be under-determined, and the pseudo-inverse is then given by

\[ Z_v^\dagger = Z_v^H \left[ Z_v Z_v^H \right]^{-1}. \]  

(2.31)

This solution is found by minimising the control effort \( u^H u \) while constraining the cost function in Eq. (2.23) to be equal to zero [25]. This case rarely occurs in practice because the number of error sensors implemented in a narrowband active noise control system is generally larger than or equal to the number of control sources.

### 2.4 Optimal broadband control

In this section, the optimal broadband control performance that can be obtained at the virtual locations is derived assuming that these locations are spatially fixed within the sound field. It is therefore assumed that the disturbance source signals \( s(n) \) in Fig. 2.1 are white and stationary random processes with zero-mean and unit covariance as defined by Eq. (2.1). As discussed in Section 2.2, an optimal feedforward control approach can be adopted if a feedforward reference signal \( x(n) \) is available that is strongly correlated to the virtual primary disturbances and if the virtual locations are spatially fixed. In Section 2.4.1, a solution to the resulting optimal feedforward control problem is presented. The presented solution can be used to minimise the broadband virtual error signals at the spatially fixed virtual locations without directly measuring these signals during real-time control. If the virtual locations are spatially fixed but a suitable feedforward reference signal is not available, an optimal feedback control approach can be adopted. In Section 2.4.2, a solution to the optimal feedback control problem is presented assuming that the virtual error signals are directly measured during real-time control. Although the optimal feedback solution presented in Section 2.4.2 cannot be used here, since the virtual error signals are not directly measured during real-time control, this solution is still presented because it is used in Section 2.4.3 to derive a solution to the general feedforward/feedback control problem. As discussed in Section 2.2,
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This general feedforward/feedback control approach can be adopted if the virtual locations are spatially fixed and if suitable feedforward reference signals are available even if these signals have been contaminated by intrinsic feedback from the control sources. The presented general feedforward/feedback control solution can be used to minimise the broadband virtual error signals at the spatially fixed virtual locations without directly measuring these signals during real-time control.

2.4.1 Optimal feedforward control

A block diagram of the optimal feedforward control problem considered in this section is shown in Fig. 2.3. The signals and transfer paths used in this block diagram have been defined in Section 2.2.

![Block diagram of the optimal feedforward control problem](image)

Figure 2.3: Block diagram of the optimal feedforward control problem, with $S$ disturbance sources, $L$ control sources, $K$ reference sensors, and $M_v$ spatially fixed virtual sensors.

The idea of the feedforward control approach illustrated in Fig. 2.3 is that the feedforward reference signals $x(n)$ are strongly correlated to the virtual primary disturbances $d_v(n)$. This strong correlation allows the virtual primary disturbances to be predicted from the feedforward reference signals. A feedforward controller $W$ can then be designed that computes a control signal as

$$ u(n) = Wx(n), \quad (2.32) $$

such that the virtual error signals $e_v(n)$ are minimised without directly measuring these signals during real-time control. It can be seen from Fig. 2.3 that the virtual error signals can be set to zero if the controller is given by

$$ G_{ys} = -G_{vy} W G_{xs}. \quad (2.33) $$

Provided that the virtual secondary transfer path matrix $G_{vy}$ and the detector transfer path matrix $G_{xs}$ are square and invertible, the controller $W$ should thus ideally be given by

$$ W = -G_{vy}^{-1} G_{ys} G_{xs}^{-1}. \quad (2.34) $$
To physically implement the controller \( W \) in a real-time control application, however, it is constrained to be \textit{causal} and \textit{stable}, which may prevent Eq. (2.33) being satisfied. As an example, the strong correlation between the feedforward reference signals \( x(n) \) and the virtual primary disturbances \( d_v(n) \) might be the result of a linear relationship that is mainly \textit{non-causal}. This means that the current value of the virtual primary disturbance \( d_v(n) \) mainly depends on future values of the feedforward reference signals. These non-causally related parts of the virtual primary disturbances can thus not be predicted from the current and previous values of the feedforward reference signals. To implement the controller \( W \) in a real-time control situation, it thus needs to be \textit{causally constrained} during the design stage. In choosing suitable locations for the reference sensors, it is therefore important that the feedforward reference signals contain \textit{time-advanced} information on the virtual primary disturbances, such that these disturbances can be causally predicted.

Another situation that prevents Eq. (2.33) being satisfied is the presence of non-minimum-phase zeros in the virtual secondary transfer path, which result in unstable poles in its inverse \( G_{vu}^{-1} \). This prevents the calculation of a causal and stable controller that satisfies Eq. (2.33). Furthermore, in the multi-channel case, the number of control sources \( L \) could be smaller than the number of virtual sensors \( M_v \), such that the true inverse \( G_{vu}^{-1} \) in Eq. (2.34) does not exist. This case corresponds to an \textit{under-actuated} system in which the virtual error signals generally cannot all be set to zero [25]. This again prevents the design of a controller that satisfies Eq. (2.33). Likewise, the number of reference sensors \( K \) could be smaller than the number of disturbance sources \( S \), such that the true inverse \( G_{xs}^{-1} \) in Eq. (2.34) does not exist. In this instance, the reference signals generally do not contain enough information to perfectly predict the virtual primary disturbances. The design of a controller that satisfies Eq. (2.33) is then not possible. In conclusion, it is not always possible to design a \textit{causal} and \textit{stable} controller \( W \) that satisfies Eq. (2.33), and perfect cancellation of the virtual primary disturbances \( d_v(n) \) is therefore not always possible.

\section*{Choice of cost function}

Instead of trying to design a controller that satisfies Eq. (2.33), the controller is generally calculated by minimising a cost function given by

\[
J = \text{tr}\left( E\left[ e_v(n)e_v(n)^T \right]\right),
\]

where \( \text{tr}(\cdot) \) denotes the trace of the matrix inside parentheses. The cost function \( J \) is thus defined as the sum of the mean-square values of the virtual error signals \( e_v(n) \), and is therefore a measure of the power of the residual signals that remain at the virtual sensors. This type of cost function is widely used in the design of optimal controllers.
for active noise and vibration control applications [25]. Using Parseval’s Theorem [51], and the fact that the signals $s(n)$ have been defined in Eq. (2.1) as white noise sequences with unit covariance, an equivalent frequency domain expression of the time-domain cost function in Eq. (2.35) is given by

$$J = \|G_{vs} + G_{vu}W G_{xs}\|_2^2$$

(2.36)

with $\| \cdot \|_2$ indicating the $H_2$-norm of the term inside the delimiters, which is defined as follows.

**Definition 2.1 (H$_2$-norm of a transfer function matrix [130]).**

The $H_2$-norm of a transfer function matrix $G(z) \in \mathcal{RH}^{m \times n}_\infty$ is defined as

$$\|G(z)\|_2 \triangleq \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}(G(e^{j\omega})G(e^{j\omega})^*) \, d\omega},$$

(2.37)

with $(\cdot)^*$ the complex conjugate transpose of the term inside parentheses.

Using the cost function in Eq. (2.36), the $H_2$ optimal feedforward controller design problem is formulated as follows.

**Problem 2.1 (H$_2$ optimal feedforward control problem).**

Given the transfer function matrices $G_{vs} \in \mathcal{RH}^{M_v \times S}_\infty$, $G_{vu} \in \mathcal{RH}^{M_v \times L}_\infty$, and $G_{xs} \in \mathcal{RH}^{K \times S}_\infty$, determine the optimal controller $W_o \in \mathcal{RH}^{L \times K}_\infty$ such that

$$W_o = \text{arg} \min_{W \in \mathcal{RH}^{L \times K}_\infty} J(W),$$

(2.38)

with the cost function $J(W)$ defined by

$$J(W) = \|G_{vs} + G_{vu} W G_{xs}\|_2^2.$$  

(2.39)

Problem 2.1 will be solved here using a factorisation approach [126]. This approach is adopted as it allows a straightforward interpretation of the minimum value of the cost function $J$ in Eq. (2.39) that is obtained after the optimisation [34]. This minimum value can also conveniently be compared with the minimum value that is obtained when minimising an estimate of the virtual error signals instead of the true virtual error signals, with the estimate computed using one of the virtual sensing algorithms that are introduced in Chapter 3. This comparison will be presented in Chapter 4.

In the factorisation approach, an inner-outer factorisation [130] of the virtual secondary transfer path matrix $G_{vu}$, and an outer-inner factorisation [130] of the detector transfer path matrix $G_{xs}$ are computed. The idea behind the inner-outer and outer-inner factorisations is to factorise a dynamic system into a part that has a stable and causal inverse, i.e. the outer and co-outer factors, and a remaining part that only introduces...
a phase shift and does not affect the energy of the input signals, i.e. the inner and co-inner factors. The presented optimal solution to Problem 2.1 is called the *causal Wiener solution* [25]. Before presenting this solution, the adjoint operator, inner-outer factorisation, outer-inner factorisation, and causality and non-causality operators are formally defined. These factorisations and operators are frequently used throughout this thesis.

**Adjoint operator**

**Definition 2.2 (Adjoint operator [54]).**

The adjoint $G^*(z)$ of a transfer function matrix $G(z)$ is defined as

$$G^*(z) \triangleq G(z^{-1})^T.$$  \hfill (2.40)

In Appendix B, it is shown how a state-space realisation of the adjoint $G^*(z)$ can be computed given a state-space realisation of the transfer function matrix $G(z)$.

**Inner-outer and outer-inner factorisations**

To compute the causal Wiener solution to Problem 2.1, an inner-outer factorisation of the virtual secondary transfer path matrix $G_{vu}$ in Fig. 2.3 needs to be calculated. Therefore, let us first formally define when a transfer function matrix is called *inner*, and when it is called *outer*.

**Definition 2.3 (Inner and outer functions [130]).**

A transfer function matrix $G_i(z) \in \mathcal{RH}_\infty$ is called inner if

$$G_i^*(z)G_i(z) = I, \quad \forall z = e^{j\theta},$$  \hfill (2.41)

with $\theta \in [0, 2\pi]$. A square inner transfer function matrix is called all-pass. Furthermore, a transfer function matrix $G_o(z) \in \mathcal{RH}_\infty$ is called outer if all its transmission zeros are stable, i.e. inside the unit circle.

Using Definition 2.3, the inner-outer factorisation of a transfer function matrix $G(z) \in \mathcal{RH}_\infty^{m \times n}$ is now defined by the following Lemma, where the dependence on $z$ has again been omitted for notational convenience.

**Lemma 2.1 (Inner-outer factorisation [54]).**

An inner-outer factorisation of a transfer function matrix $G \in \mathcal{RH}_\infty^{m \times n}$ is given by

$$G = G_i G_o,$$  \hfill (2.42)

where $G_i \in \mathcal{RH}_\infty^{l \times l}$, with $l \leq \min(m, n)$, is the isometric inner factor, such that

$$G_i^* G_i = I_l,$$  \hfill (2.43)
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and where $G_o \in \mathcal{RH}_{\infty}^{l \times n}$ is the minimum-phase outer factor, which has a stable right-inverse $G_o^\dagger$, such that

$$G_i = GG_o^\dagger. \quad (2.44)$$

If $G$ has no zeros on the unit circle, the right-inverse of the outer factor is asymptotically stable. Furthermore, there exists a complementary inner transfer function matrix $G_i^\perp \in \mathcal{RH}_{\infty}^{m \times (m-l)}$, such that $[G_i G_i^\perp]$ is unitary

$$[G_i G_i^\perp]^*[G_i G_i^\perp] = [G_i G_i^\perp][G_i G_i^\perp]^* = I_m. \quad (2.45)$$

Also, due to the isometric properties of $G_i$, the following expression can be derived

$$G^*G = G_o^*G_o, \quad (2.46)$$

and the outer factor $G_o$ is therefore called a minimum-phase spectral factor of $G^*G$.

Here, an inner-outer factorisation of the virtual secondary transfer path $G_{vu}$ is computed as

$$G_{vu} = G_{vu,i}G_{vu,o}, \quad (2.47)$$

with $G_{vu,i}$ the inner factor and $G_{vu,o}$ the outer factor of the virtual secondary transfer path matrix. In Appendix B, it is shown how the inner-outer factorisation of a dynamic system described by a transfer path matrix $G$ can be calculated given a state-space realisation of $G$.

Next, let us formally define when a transfer function matrix is called co-inner and when it is called co-outer. These definitions are presented because an outer-inner factorisation of the detector transfer path matrix $G_{xs}$ needs to be computed to calculate the causal Wiener solution to Problem 2.1.

**Definition 2.4** (Co-inner and co-outer functions [130]).

A transfer function matrix $G_{ci}(z) \in \mathcal{RH}_{\infty}$ is called co-inner if

$$G_{ci}(z)G_{ci}^*(z) = I, \quad \forall z = e^{i\theta}, \quad (2.48)$$

with $\theta \in [0, 2\pi]$. A square co-inner transfer function matrix is called all-pass. Furthermore, a transfer function matrix $G_{co}(z) \in \mathcal{RH}_{\infty}$ is called co-outer if all its transmission zeros are stable, i.e. inside the unit circle.

Using Definition 2.4, the outer-inner factorisation of a transfer function matrix $G(z) \in \mathcal{RH}_{\infty}^{m \times n}$ is now defined by the following Lemma, where the dependence on $z$ has again been omitted for notational convenience.
2.4 Optimal broadband control

Lemma 2.2 (Outer-inner factorisation [54]).
An outer-inner factorisation of a transfer function matrix \( G \in \mathcal{RH}_\infty^{m \times n} \) is given by
\[
G = G_{ci} G_{co},
\]
(2.49)
where \( G_{ci} \in \mathcal{RH}_\infty^{l \times n} \), with \( l \leq \min(m, n) \), is the co-isometric co-inner factor, such that
\[
G_{ci} G_{ci}^* = I_l,
\]
(2.50)
and where \( G_{co} \in \mathcal{RH}_\infty^{m \times l} \) is the minimum-phase co-outer factor, which has a stable left-inverse \( G_{co}^+ \), such that
\[
G_{ci} = G_{co}^+ G_{co}.
\]
(2.51)
If \( G \) has no zeros on the unit circle, the left-inverse of the co-outer factor is asymptotically stable.

Furthermore, there exists a complementary co-inner transfer function matrix \( G_{ci}^\perp \in \mathcal{RH}_\infty^{(n-l) \times n} \), such that \([G_{ci} G_{ci}^\perp]^T\) is unitary
\[
[G_{ci} G_{ci}^\perp]^* [G_{ci} G_{ci}^\perp]^* = \begin{bmatrix} G_{ci}^\perp \end{bmatrix}^* [G_{ci} G_{ci}^\perp]^* = I_n.
\]
(2.52)
Also, due to the co-isometric properties of \( G_{ci} \), the following expression can be derived
\[
GG^* = G_{ci} G_{ci}^*,
\]
(2.53)
and the co-outer factor \( G_{co} \) is therefore called a minimum-phase spectral factor of \( GG^* \).

Here, an outer-inner factorisation of the detector transfer path matrix \( G_{xs} \) is computed as
\[
G_{xs} = G_{xs,ci} G_{xs,co},
\]
(2.54)
with \( G_{xs,ci} \) the co-inner factor and \( G_{xs,co} \) the co-outer factor of the detector transfer path matrix. Note that the co-outer factor \( G_{xs,co} \) is a minimum-phase spectral factor that defines the spectral density matrix of the feedforward reference signals \( x(n) = G_{xs}s(n) \) in Fig. 2.3. This spectral density matrix is thus given by
\[
G_{xs} G_{xs}^* = G_{xs,co} G_{xs,co}^*.
\]
(2.55)
since the disturbance source signal \( s(n) \) has been defined in Eq. (2.1) as a zero mean white and stationary random process with unit covariance. The minimum-phase co-outer factor has a stable left-inverse \( G_{xs,co}^+ \) and filtering the feedforward reference signals \( x(n) \) in Fig. 2.3 with this left-inverse gives the signals
\[
s'(n) = G_{xs,co}^+ x(n).
\]
(2.56)
The spectral density matrix of these signals is now given by

\[ G_{x_0^*, x_0}^* G_{x_0^*, x_0} = G_{x_0, x_0}^* G_{x_0, x_0} = I, \]

which shows that the signals \( s'(n) \) are white noise processes of unit covariance. The left-inverse \( G_{x_0, x_0}^* \) is therefore called a pre-whitening filter \([25]\) for the feedforward reference signals, and the filtering action in Eq. (2.56) is referred to as a pre-whitening of the feedforward reference signals. In Appendix B, it is shown how the outer-inner factorisation of a dynamic system described by a transfer path matrix \( G \) can be calculated given a state-space realisation of \( G \).

**Casuality operator**

Before presenting the causal Wiener solution to Problem 2.1, the casuality operator needs to be defined. For this purpose, let us first introduce a more general set of transfer function matrices than the set \( \mathcal{RH}^{m \times n}_{\infty} \) of all proper and asymptotically stable \( m \times n \) rational transfer function matrices with real coefficients introduced previously. This more general set, of which \( \mathcal{RH}^{m \times n}_{\infty} \) is a subset, is denoted by \( \mathcal{RH}^{m \times n} \), and includes all \( m \times n \) rational transfer function matrices with real coefficients and excluding singularities on the unit circle. A singularity on the unit circle occurs when the system has a pole on the unit circle \([130]\). It can be proven that for any transfer function matrix \( G(z^{-1}) \in \mathcal{RH}^{m \times n} \), there exists a (convergent) Laurent series given by \([51, 113]\)

\[ G(z^{-1}) = \sum_{i=-\infty}^{\infty} G_i z^{-i}, \quad G_i \in \mathbb{R}^{m \times n}. \] (2.58)

In more familiar terms, a transfer function matrix \( G(z^{-1}) \in \mathcal{RH}^{m \times n} \) can be represented by a matrix of finite impulse response filters that have an infinite number of filter coefficients. Note that filter coefficients for \( i < 0 \) are included in Eq. (2.58) as well. If these filter coefficients are not all equal to zero, the output signal that results from filtering an input signal with the transfer function matrix \( G(z^{-1}) \in \mathcal{RH}^{m \times n} \) will depend on future values of the input signal. For this case, the transfer function matrix thus defines a non-causal relationship between the input and output signals. The causality and non-causality operators are now defined to be able to separate the part of the transfer function matrix \( G \) that defines a causal input-output relationship from the part that defines a non-causal input-output relationship.

**Definition 2.5 (Causality and non-causality operator).**

*Given the Laurent series of \( G(z^{-1}) \in \mathcal{RH}^{m \times n} \) defined in Eq. (2.58), the causal part and*
non-causal part of $G$ are defined as
\[ [G]_+ \triangleq \sum_{i=0}^{\infty} G_i z^{-i}, \quad \text{causal part,} \quad (2.59) \]
\[ [G]_- \triangleq \sum_{i=-\infty}^{-1} G_i z^{-i}, \quad \text{non-causal part,} \quad (2.60) \]
respectively, with $G_i \in \mathbb{R}^{m \times n}$.

Now the adjoint operator, inner-outer factorisation, outer-inner factorisation, and causality and non-causality operators have been formally defined, the casual Wiener solution to Problem 2.1 can be presented.

**Causal Wiener solution**

Using the inner-outer factorisation defined in Lemma 2.1, the outer-inner factorisation defined in Lemma 2.2, the adjoint operator introduced in Definition 2.2, and the causality operator introduced in Definition 2.5, a solution to Problem 2.1 is given by the following theorem.

**Theorem 2.1** (Causal Wiener filter).

*Given the transfer function matrices $G_{vs} \in \mathcal{RH}_{\infty}^{M_v \times S}$, $G_{vu} \in \mathcal{RH}_{\infty}^{M_v \times L}$, and $G_{xs} \in \mathcal{RH}_{\infty}^{K \times S}$, and assuming that $G_{vu}$ and $G_{xs}$ do not have any zeros on the unit circle, the following inner-outer and outer-inner factorisations can be defined

\[ G_{vu} = G_{vu,i} G_{vu,o}, \quad (2.61) \]
\[ G_{xs} = G_{xs,ci} G_{xs,co}, \quad (2.62) \]

where $G_{vu,o}$ has a stable right-inverse $G_{vu,o}^+$, and $G_{xs,co}$ has a stable left-inverse $G_{xs,co}^\dagger$. Furthermore, let $G_{vu,i}^*$ and $G_{xs,ci}^\dagger$ be such that $[G_{vu,i}^* G_{vu,j}]$ and $[G_{xs,ci}^\dagger G_{xs,ci}^\dagger]^*$ are unitary. Then

\[ W_o = -G_{vu,o}^+ \begin{bmatrix} G_{vu,i}^\dagger G_{vu,j} & G_{vs}^\dagger G_{xs,ci} \end{bmatrix} + G_{xs,co}^\dagger \]

(2.63)

minimises

\[ J = \| G_{vs} + G_{vu}WG_{xs} \|_2^2, \quad \text{subject to } W \in \mathcal{RH}_{\infty}^{L \times K}, \quad (2.64) \]

and its minimum value is given by

\[ J_{\min} = \| G_{vs} G_{xs,ci}^\dagger \|_2^2 + \| G_{vu,i}^* G_{vs} G_{xs,ci}^\dagger \|_2^2 + \| [G_{vu,i}^* G_{vu,j} G_{vs}^* G_{xs,ci}]^\dagger \|_2^2. \quad (2.65) \]

**Proof.** A proof can be found in Vidyasagar [126], and an alternative proof by Fraanje [34] has been included in Appendix A.
Pre-whitening and post-conditioning filters

Theorem 2.1 shows that the optimal solution \( u_o(n) \) for the control signal in Fig. 2.3 is calculated from Eq. (2.63) as

\[
 u_o(n) = -G_{vu,o}^\dagger \left[ G_{vu,o}^* G_{vs} G_{xs,cl}^* \right] + G_{xs,co}^\dagger x(n),
\]

\[
 = -G_{vu,o}^\dagger \left[ G_{vu,o}^* G_{vs} G_{xs,cl}^* \right] + s'(n),
\]

(2.66)

with \( s'(n) \) the pre-whitened feedforward reference signal defined in Eq. (2.56). The pre-whitening filter \( G_{xs,co}^\dagger \) for the feedforward reference signals generally appears in optimal solutions for feedforward active noise control problems [25]. This can also be said for the right-inverse \( G_{vu,o}^\dagger \) of the outer factor of the virtual secondary transfer path. When writing the control signal \( u_o(n) \) in Eq. (2.66) as

\[
 u_o(n) = -G_{vu,o}^\dagger u_o'(n),
\]

(2.67)

the control signal \( u_o(n) \) can be viewed as a post-conditioned control signal computed by filtering the control signal \( u_o'(n) \) with the post-conditioning filter \( G_{vu,o}^\dagger \). Again, the post-conditioning filter \( G_{vu,o}^\dagger \) generally appears in optimal solutions for feedforward active noise control problems [25]. This post-conditioning filter is included to account for the influence of the virtual secondary transfer path on the power spectral density matrix of the virtual secondary disturbances \( y_v(n) \). These signals destructively interfere with the virtual primary disturbances to obtain the desired attenuation.

Comparison to the ideal solution

If the virtual secondary transfer path matrix \( G_{vu} \) and the detector transfer path matrix \( G_{xs} \) do not have any non-minimum-phase zeros, such that \( G_{vu,j} = I \) and \( G_{xs} = I \), the causal Wiener filter in Eq. (2.63) reduces to the unconstrained Wiener filter [25] given by

\[
 W_o = -G_{vu,o}^\dagger G_{vs} G_{xs}^\dagger.
\]

(2.68)

The unconstrained Wiener filter further reduces to the ideal solution defined in Eq. (2.34) when the virtual secondary transfer path matrix \( G_{vu} \) and the detector transfer path matrix \( G_{xs} \) are square. These transfer path matrices are square when the number of virtual sensors \( M_v \) is equal to the number of control sources \( L \), and the number of reference sensors \( K \) is equal to the number of disturbance sources \( S \).

Interpretation of minimum value of cost function [34]

If the number of reference sensors \( K \) is smaller than the number of disturbance source signals \( S \), the term \( G_{xs,ci}^\perp \) may not be equal to zero. The first term on the right-hand side in Eq. (2.65) then contributes to the minimum value of the cost function, provided that
2.4 Optimal broadband control

\(G_{\text{PS}}G_{\text{x}_{\text{S},\text{ci}}}^*\) is not equal to zero [34]. In this case, there are disturbances that contribute to the virtual primary disturbances at the virtual sensors, but these disturbances are not measured, or observed at the reference sensors. These parts of the virtual primary disturbances can thus not be predicted from the feedforward reference signals \(x(n)\), and can therefore not be controlled. This is related to the concept of unobservable modes of the active noise control system [34, 130].

If the number of control sources \(L\) is smaller than the number of virtual sensors \(M_v\), the term \(G_{\text{vu},\text{i}}^*\) may not be equal to zero. The second term on the right-hand side in Eq. (2.65) then contributes to the minimum value of the cost function, provided that \(G_{\text{vu},\text{i}}^*G_{\text{PS}}G_{\text{x}_{\text{S},\text{ci}}}^*\) is not equal to zero [34]. In this case, there are disturbances that contribute to the virtual primary disturbances at the virtual sensors, but these disturbances cannot be controlled with the chosen actuator configuration. This is related to the concept of uncontrollable modes of the active noise control system [34, 130].

The first and second term that contribute to the minimum value of the cost function defined in Eq. (2.65) are related to the actuator and virtual sensor configurations. The third term is related to the fact that the feedforward controller is constrained to be causal [34]. This term is determined by delays and non-minimum-phase zeros in the virtual secondary transfer path matrix and the detector transfer path matrix, which result in non-causal terms in \(G_{\text{vu},\text{i}}^*\) and \(G_{\text{x}_{\text{S},\text{ci}}}^*\). These non-causal terms contribute to the third term of the minimum value of the cost function defined in Eq. (2.65).

2.4.2 Optimal feedback control

If the virtual locations are spatially fixed but suitable feedforward reference signals are not available, an optimal feedback control approach can be adopted. A block diagram of the optimal feedback control problem is depicted in Fig. 2.4, where it is assumed that the virtual error signals \(e_{\text{v}}(n)\) are directly measured during real-time control.

![Block diagram of the optimal feedback control problem](image)

Figure 2.4: Block diagram of the optimal feedback control problem, with \(S\) disturbance sources, \(L\) control sources, and \(M_v\) spatially fixed virtual sensors.
Although the feedback solution shown in Fig. 2.4 cannot be used here since the virtual error signals are not directly measured during real-time control, this solution is still presented because it is used in Section 2.4.3 to derive a solution to the general feedforward/feedback control problem. The virtual error signals $e_v(n)$ in Fig. 2.4 are now given by

$$e_v(n) = (I - G_{vu}C)^{-1}G_{vs}s(n), \quad (2.69)$$

where $C$ is the feedback controller with $M_v$ inputs and $L$ outputs. The feedback control problem shown in Fig. 2.4 can be converted into an equivalent feedforward control problem using the internal model control (IMC) parameterisation [114]. This parameterisation of the feedback controller $C$ is shown in Fig. 2.5.

![Figure 2.5: Block diagram of the internal model control parameterisation for optimal feedback control, with $S$ disturbance sources, $L$ control sources, and $M_v$ spatially fixed virtual sensors.](image)

Fig. 2.5 illustrates that an internal model of the virtual secondary transfer path $G_{vu}$ is used to remove the influence of the control signals on the virtual error signals. The resulting virtual primary disturbance $d_v(n) = e_v(n) - G_{vu}u(n)$ acts as a feedforward reference signal to the feedforward controller $W$. The feedback controller $C$, which is contained within the grey box in Fig. 2.5, is now parameterised as

$$C = (I + WG_{vu})^{-1}W, \quad \forall W \in \mathcal{RH}_L^{L \times M_v}. \quad (2.70)$$

The parameterisation in Eq. (2.70) is also known as the Youla parameterisation, with $W \in \mathcal{RH}_L^{L \times M_v}$ the Youla parameter [114]. Given the stable transfer path matrix $G_{vu}$, the Youla parameterisation provides a parameterisation of all stabilising controllers $C \in \mathcal{RH}_p^{L \times M_v}$ that yield internal stability of the closed-loop system [114], with $\mathcal{RH}_p^{n \times m}$ the set of all $n \times m$ proper rational transfer function matrices with real coefficients [130]. From Fig. 2.5, the virtual error signals can now be written in terms of the feedforward controller $W$ as

$$e_v(n) = (G_{vs} + G_{vu}WG_{vs})s(n). \quad (2.71)$$
This equation shows that the feedback control problem is reformulated into an equivalent feedforward control problem when using the IMC parameterisation. The advantage of this approach is that the feedforward controller $W$ can now be designed using the causal Wiener filter solution defined in Theorem 2.1. The feedback controller $C$ can then be found by substituting the resulting feedforward controller into Eq. (2.70). The resulting optimisation problem is thus to minimise

$$J = \text{tr}\left( E\left[ e_v(n)e_v(n)^T \right] \right) = \| G_{vs} + G_{vu}WG_{vs} \|^2,$$

subject to $W \in \mathcal{H}^{L \times M_v}_{\infty}$. From Theorem 2.1, a solution to this problem is defined as

$$W_o = -G_{vu,\rho}^\dagger \left[ G_{vu,\rho}^* G_{vs} G_{vs,ci}^* \right] + G_{vs,co}^\dagger$$

and the minimum value of the cost function in Eq. (2.72) is given by

$$J_{\text{min}} = \| G_{vs} G_{vs,ci}^\perp \|^2 + \| G_{vu,\rho}^* G_{vs} G_{vs,ci}^* \|^2 + \| G_{vu,\rho}^* G_{vs} G_{vs,ci}^* \|^2 - \| G_{vu,\rho}^* G_{vs} G_{vs,ci}^* \|^2,$$

since

$$G_{vs} G_{vs,ci}^\perp = G_{vs,co} G_{vs,ci} G_{vs,ci}^\perp = 0.$$  (2.75)

Note that both the feedforward controller $W_o$ in Eq. (2.73), and the minimum value of the cost function in Eq. (2.74), depend on the co-outer factor, or minimum-phase spectral factor, $G_{vs,co}$ and not $G_{vs}$ itself. This indicates that the performance is not affected by delays and non-minimum-phase zeros in the virtual primary transfer path $G_{vs}$ [34]. The first term of the minimum value of the cost function in Eq. (2.74) is again related to the concept of uncontrollable modes of the active noise control system, and thus to the actuator configuration. The second term is again related to the fact that the controller is constrained to be causal.

### 2.4.3 General feedforward/feedback control

In this section, the optimal feedforward control problem illustrated in Fig. 2.3 is revisited, and it is now assumed that the feedforward reference signals $x(n)$ are contaminated by intrinsic feedback from the control sources. This contamination typically occurs when microphones are used as reference sensors. A block diagram of the resulting general feedforward/feedback control problem is shown in Fig. 2.6. The general feedforward/feedback control approach illustrated in this figure can be used to minimise the broadband virtual error signals at the spatially fixed virtual locations without directly measuring
these signals during real-time control. The virtual error signals in Fig. 2.6 can now be written in terms of the feedback controller \( C \) as

\[
e_v(n) = (G_{vs} + G_{vu}(I - CG_{xu})^{-1}CG_{xs})s(n).
\]

As in the previous section, the Youla parameterisation can be used to parameterise all internally stabilising controllers \( C \in \mathcal{RH}_p^{L \times K} \) as

\[
C = (I + WG_{xu})^{-1}W, \quad \forall W \in \mathcal{RH}_\infty^{L \times K}.
\]

With the controller \( C \) parameterised as in Eq. (2.77), the virtual error signals in Eq. (2.76) can also be written as

\[
e_v(n) = (G_{vs} + G_{vu}WG_{xs})s(n).
\]

The virtual error signals are now independent of the intrinsic feedback transfer path \( G_{xu} \), indicating that the contamination of the feedforward reference signals by intrinsic feedback from the control sources has been cancelled. Again, the control problem has been reformulated into an equivalent feedforward control problem, which can be solved using Theorem 2.1. The optimisation problem for the general feedforward/feedback problem is now to minimise

\[
J = \text{tr} \left( \mathbf{E} \left[ e_v(n)e_v(n)^T \right] \right) = \|G_{vs} + G_{vu}WG_{xs}\|_2^2,
\]

subject to \( W \in \mathcal{RH}_\infty^{L \times K} \). From Theorem 2.1, a solution to this problem is defined as

\[
W_o = -G_{vu,0}^* \left[ G_{vu,0}^* G_{vs}^* G_{xs,ci}^* \right] + G_{xs,0}^* \,
\]

and the minimum value of the cost function in Eq. (2.79) is given by

\[
J_{\text{min}} = \|G_{vs}G_{xs,cl}^*\|_2^2 + \|G_{vu,0}^* G_{vs}^* G_{xs,cl}^*\|_2^2 + \|G_{vu,0}^* G_{vs}^* G_{xs,cl}^*\|_2^2.
\]

Note that the minimum value of the cost function in Eq. (2.81) does not depend on the intrinsic feedback transfer path \( G_{xu} \).
2.5 Adaptive feedforward control

The most common approach in real-time active noise control applications is to use an adaptive feedforward control approach instead of an optimal feedforward control approach, provided that a suitable feedforward reference signal $x(n)$ is available [25, 68]. For the case considered here, adaptive control algorithms generally need the feedback information contained in the virtual error signals $e_v(n)$ to update the feedforward controller appropriately. In this section, a number of adaptive feedforward control algorithms are introduced, assuming that the virtual error signals are directly measured during real-time control. The general aim of this thesis is, however, to minimise the error signals at the virtual locations while not directly measuring them during real-time control. The adaptive feedforward control problem will therefore be revisited in Chapter 4, where it is assumed that an estimate of the virtual error signals is computed using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3. As illustrated in Fig. 2.1, this estimate is then used instead of the true virtual error signals to update the feedforward controller appropriately. Furthermore, the adaptive feedforward algorithms introduced in this section are modified in Chapter 5 and combined with moving virtual sensing algorithms, which can be used to compute an estimate of the virtual error signals at virtual locations that are moving through the sound field. The combination of these algorithms provides a method for adaptively minimising an estimate of the virtual error signals at a number of moving virtual locations. In this section, the multi-channel filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms are introduced, assuming that the virtual error signals are directly measured during real-time control.

2.5.1 Filtered-x LMS algorithm

The filtered-x LMS algorithm is the most commonly used adaptive control algorithm in real-time active noise control implementations. Its application to active noise control has been investigated extensively, and the derivation and properties of the algorithm are detailed comprehensively in a number of well-known publications [25, 48, 68]. In this section, a description of the multi-channel filtered-x LMS algorithm is presented without giving a full derivation. A block diagram of the multi-channel filtered-x LMS algorithm is shown in Fig. 2.7. Similar to the feedforward control problem depicted in Fig. 2.1, it is assumed that there are $L$ control sources, $K$ reference sensors, and $M_v^p$ spatially fixed virtual sensors. Furthermore, it is assumed that the virtual error signals $e_v(n)$ are directly measured during real-time control. The multi-channel filtered-x LMS algorithm will be presented assuming that the virtual secondary transfer paths $G_{vu}$ are
Figure 2.7: Block diagram of the multi-channel filtered-x LMS algorithm, using $L$ control sources, $K$ reference sensors, and $M_v$ spatially fixed virtual sensors.

represented by an impulse response matrix defined as

$$G_{vu} = \begin{bmatrix}
G_{vu,11} & G_{vu,12} & \cdots & G_{vu,1L} \\
G_{vu,21} & G_{vu,22} & \cdots & G_{vu,2L} \\
\vdots & \vdots & \ddots & \vdots \\
G_{vu,M_v1} & G_{vu,M_v2} & \cdots & G_{vu,M_vL}
\end{bmatrix}, \quad (2.82)$$

with $g_{vu,m,l}$ an FIR filter that models the impulse response between the $l$th control source and the $m$th spatially fixed virtual sensor. The controller $W$ is represented by an $L \times K$ matrix where each element is an adaptive FIR filter of order $I$ with filter coefficients $w_{lk}(n) \in \mathbb{R}^I$ defined by

$$w_{lk}(n) = \begin{bmatrix} w_{lk,0}(n) & w_{lk,1}(n) & \cdots & w_{lk,I-1}(n) \end{bmatrix}^T. \quad (2.83)$$

The $l$th control signal $u_l(n)$ is now calculated as

$$u_l(n) = \sum_{k=1}^{K} w_{lk}^T(n)x_k(n), \quad (2.84)$$

where $x_k(n) \in \mathbb{R}^I$ is defined as

$$x_k(n) = \begin{bmatrix} x_k(n) & x_k(n-1) & \cdots & x_k(n-I+1) \end{bmatrix}^T, \quad (2.85)$$

with $x_k(n)$ the $k$th feedforward reference signal. The multi-channel filtered-x LMS algorithm can now be used to adjust the $KLI$ control filter coefficients $w_{kl}(n)$ such that
2.5 Adaptive feedforward control

the virtual error signals $e_v(n)$ are minimised. The control signals $u_i(n)$ in Eq. (2.84) can now be contained in a column vector $u(n)$ that can be written as

$$ u(n) = X(n)^T w(n), \quad (2.86) $$

with the matrix $X(n) \in \mathbb{R}^{KLI \times L}$ given by

$$ X(n) = \begin{bmatrix} x(n) & 0 & \ldots & 0 \\ 0 & x(n) & \ldots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & x(n) \end{bmatrix}. \quad (2.87) $$

The column vector $x(n) \in \mathbb{R}^{KI}$ in Eq. (2.87) is called the generalised feedforward reference signal vector, and is defined as

$$ x(n) = \begin{bmatrix} x_1^T(n) \\ x_2^T(n) \\ \vdots \\ x_K^T(n) \end{bmatrix}^T, \quad (2.88) $$

with $x_k(n)$ defined in Eq. (2.85). The column vector $w(n) \in \mathbb{R}^{KLI}$ in Eq. (2.86) contains the $KLI$ control filter coefficients $w_{kl}(n)$, and is defined as

$$ w(n) = \begin{bmatrix} w_1^T(n) \\ w_2^T(n) \\ \vdots \\ w_L^T(n) \end{bmatrix}^T, \quad (2.89) $$

where $w_i(n) \in \mathbb{R}^{KI}$ is given by

$$ w_i(n) = \begin{bmatrix} w_{i1}^T(n) \\ w_{i2}^T(n) \\ \vdots \\ w_{iK}^T(n) \end{bmatrix}^T, \quad (2.90) $$

with $w_{ik}(n)$ defined in Eq. (2.83). The multi-channel filtered-x LMS algorithm can now be written as [68]

$$ w(n + 1) = w(n) - \mu R_v(n)e_v(n), \quad (2.91) $$

where the scalar $\mu \in \mathbb{R}^+$ is the positive scalar convergence coefficient. In Eq. (2.91), the matrix $R_v(n) \in \mathbb{R}^{KLI \times M_v}$ is a matrix of virtual filtered-reference signals that is generated as

$$ R_v(n) = G_{vu}^T \otimes x(n), \quad (2.92) $$

with the symbol $\otimes$ denoting the Kronecker matrix product [130]. For an $M_v \times L$ finite impulse response matrix $G_{vu}$ as defined in Eq. (2.82), the Kronecker matrix product
$G^T_{vu} \otimes x(n)$ defines a filtering procedure that results in a matrix $R_v(n)$ that is given by

$$R_v(n) = \begin{bmatrix}
  g_{vu,11} * x_1(n) & g_{vu,21} * x_1(n) & \ldots & g_{vu,M_1} * x_1(n) \\
  g_{vu,11} * x_2(n) & g_{vu,21} * x_2(n) & \ldots & g_{vu,M_1} * x_2(n) \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{vu,1L} * x_k(n) & g_{vu,2L} * x_k(n) & \ldots & g_{vu,M_L} * x_k(n)
\end{bmatrix}, \quad (2.93)$$

where $\ast$ denotes discrete-time convolution [68]. The matrix $R_v(n)$ in Eq. (2.93) can also be written as

$$R_v(n) = \begin{bmatrix}
  r_{111}(n) & r_{112}(n) & \ldots & r_{11M_L}(n) \\
  r_{211}(n) & r_{212}(n) & \ldots & r_{21M_L}(n) \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{K11}(n) & r_{K12}(n) & \ldots & r_{K1M_L}(n) \\
  r_{121}(n) & r_{122}(n) & \ldots & r_{12M_L}(n) \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{KL1}(n) & r_{KL2}(n) & \ldots & r_{KLM_L}(n)
\end{bmatrix}. \quad (2.94)$$

with the elements $r_{klm_v}(n) \in \mathbb{R}^l$ defined as

$$r_{klm_v}(n) = \begin{bmatrix} r_{klm_v}(n) & r_{klm_v}(n-1) & \ldots & r_{klm_v}(n-I+1) \end{bmatrix}^T, \quad (2.95)$$

where $r_{klm_v}(n)$ are the $KLM_v$ virtual filtered-reference signals shown in Fig. 2.7. These signals are generated as

$$r_{klm_v}(n) = g_{vu,lm_v} * x_k(n), \quad (2.96)$$

such that the virtual-filtered reference signal $r_{klm_v}(n)$ is generated by filtering the feedforward reference signal $x_k(n)$ with the virtual secondary transfer path between the $l$th control source and the $m_v$th spatially fixed virtual sensor. The update equations for the components $w_{lk}(n)$ of the vector of filter coefficients $w(n)$ in Eq. (2.91) can now also be written as

$$w_{lk}(n+1) = w_{lk}(n) - \mu \sum_{m_v=1}^{M_v} r_{klm_v}(n) e_{vm_v}(n). \quad (2.97)$$
2.5 Adaptive feedforward control

2.5.2 Normalised filtered-x LMS algorithm

The stability and convergence speed of the filtered-x LMS algorithm are controlled by the convergence coefficient \( \mu \). It has been shown that the maximum convergence coefficient that still yields stability is inversely proportional to the power of the virtual filtered-reference signals, which can be approximated by

\[
\text{tr}(R_v(n)R_v(n)^T).
\] (2.98)

One method to optimise the convergence speed of the filtered-x LMS algorithm is therefore to normalise the convergence coefficient \( \mu \) by the power of the virtual filtered-reference signals [68]. The resulting algorithm is called the normalised filtered-x LMS algorithm, and a multi-channel version of this algorithm is given by [56, 57]

\[
w(n+1) = w(n) - \mu(n)R_v(n)e_v(n),
\] (2.99)

with \( R_v(n) \) the matrix of virtual filtered-reference signals defined in Eq. (2.94), and the normalised convergence coefficient \( \mu(n) \in \mathbb{R}^{+} \) given by

\[
\mu(n) = \frac{\alpha}{\text{tr}(R_v(n)R_v(n)^T) + \epsilon},
\] (2.100)

with \( \alpha \in \mathbb{R}^{+} \) the positive scalar convergence coefficient, and \( \epsilon \in \mathbb{R}^{+} \) a small positive regularisation term. The normalised filtered-x LMS algorithm can thus be used to optimise the convergence speed when the virtual filtered-reference signals are non-stationary processes.

2.5.3 Filtered-x RLS algorithm

Another adaptive feedforward control algorithm that can be used to update the control filter coefficients \( w(n) \) defined in Eq. (2.89) is the filtered-x RLS algorithm. The cost function that is adaptively minimised by this algorithm is given by

\[
J = \sum_{i=1}^{n} \lambda^{n-i} \text{tr}(e_v(i)e_v(n)^T),
\] (2.101)

with \( \lambda \in \mathbb{R}^{+} \) the positive scalar forgetting factor. The cost function that is minimised is thus an exponentially weighted sum of the current and past squared virtual error signals \( e_v(i) \), with \( i = 1 \ldots n \). The forgetting factor is generally chosen in the range \( 0.9 \leq \lambda \leq 1 \), such that recent values of the squared virtual error signals are penalised more heavily if \( \lambda \neq 1 \). Tracking of non-stationarities in the statistics of the virtual filtered-reference signals and virtual primary disturbances can therefore be achieved
by setting $\lambda \neq 1$. Bouchard and Quednau [8] presented the update equations for the multi-channel filtered-x RLS algorithm, which are given by

$$K(n) = \lambda^{-1} P(n) R_v(n) \left( I + \lambda^{-1} R_v(n)^T P(n) R_v(n) \right)^{-1},$$  \hspace{1cm} (2.102)

$$P(n+1) = \lambda^{-1} P(n) - \lambda^{-1} K(n) R_v(n)^T P(n),$$  \hspace{1cm} (2.103)

$$w(n+1) = w(n) - K(n) e_v(n),$$  \hspace{1cm} (2.104)

with $R_v(n)$ the matrix of virtual filtered-reference signals defined in Eq. (2.94), $K(n) \in \mathbb{R}^{KL \times M_v}$ the gain matrix, and $P(n) \in \mathbb{R}^{KL \times KL}$ the inverse correlation matrix [51]. The multi-channel filtered-x RLS algorithm defined by Eqs (2.102)–(2.104) is usually initialised by setting

$$w(0) = 0,$$  \hspace{1cm} (2.105)

$$P(0) = \delta I,$$  \hspace{1cm} (2.106)

with $\delta \in \mathbb{R}^+$ a regularisation parameter [51]. The multi-channel filtered-x RLS algorithm realises a decorrelation of the virtual filtered-reference signals [8]. This reduces the eigenvalue spread in the auto-correlation matrix of the virtual filtered-reference signals. The convergence speed of the multi-channel filtered-x LMS algorithm depends on this eigenvalue spread, with a large eigenvalue spread generally resulting in slow convergence. By decorrelating the virtual filtered-reference signals, the multi-channel filtered-x RLS algorithm thus generally provides improved convergence speed over the multi-channel filtered-x LMS algorithm [8].

### 2.5.4 Generating the virtual filtered-reference signals

In the presented adaptive feedforward control algorithms, the virtual filtered-reference signals are assumed to be generated using an impulse response matrix representation of the virtual secondary transfer paths $G_{vu}$. In the real-time control experiments presented in Chapters 7–10, a state-space model of the virtual secondary transfer paths will be used to generate the virtual filtered-reference signals. In this instance, the virtual filtered-reference signals are usually generated by filtering each of the $K$ feedforward reference signals with $L$ single-input $M_v$ output state-space models [34, 90]. The $l$th state-space model is denoted by

$$G_{vu,l} \sim \begin{bmatrix} A_{vu,l} & B_{vu,l} \\ C_v & D_{vu,l} \end{bmatrix},$$  \hspace{1cm} (2.107)

where $B_{vu,l}$ and $D_{vu,l}$ are the $l$th column of the state-space matrices $B_u$ and $D_{vu}$, respectively. The state-space model $G_{vu,l}$ thus describes the virtual secondary transfer paths
between the $l^{th}$ control source and the $M_v$ virtual sensors. The virtual filtered-reference signals $r_{kl}(n) \in \mathbb{R}^{M_v}$, with

$$r_{kl}(n) = \begin{bmatrix} r_{k1l}(n) & r_{k2l}(n) & \ldots & r_{kMvl}(n) \end{bmatrix}^T,$$

(2.108)

are now generated by filtering the feedforward reference signal $x_k(n)$ with the state-space model $G_{vu,l}$, such that

$$z_{kl}(n+1) = Az_{kl}(n) + Bu,l x_k(n),$$

$$r_{kl}(n) = Cv_{kl}z_{kl}(n) + Du,l x_k(n).$$

(2.109)

The presented adaptive feedforward control algorithms can thus easily be implemented using a state-space representation of the virtual secondary transfer paths instead of an impulse response matrix representation.

### 2.6 Conclusion

In this chapter, the theory and mathematical notation that is used throughout this thesis to analyse the considered local active noise control problem have been introduced. A block diagram of the local active noise control system has been presented which enabled a consistent definition of the input signals, output signals, and transfer paths that frequently occur throughout this thesis.

The optimal narrowband control performance that can be obtained at a number of spatially fixed virtual locations has been derived by minimising the true virtual error signals using a quadratic optimisation technique [25]. This technique can be used in an initial analysis of the active noise control system in which a theoretical limit on the narrowband control performance that can be obtained at the spatially fixed virtual locations can be computed given the chosen control source and virtual sensor configurations.

The optimal broadband control performance that can be obtained at the virtual locations has been derived assuming that these locations are spatially fixed. If a suitable feedforward reference signal is available, an optimal feedforward control approach can be adopted to minimise the virtual error signals without directly measuring them during real-time control. The causal Wiener filter solution to the resulting optimal feedforward control problem has been presented using a factorisation approach [34, 126]. The presented causal Wiener filter solution is used frequently throughout this thesis to solve linear estimation and control problems.

If the virtual locations are spatially fixed but a suitable feedforward reference signal is not available, a feedback control approach can be adopted. A solution to the optimal feedback control problem has been presented using an internal model control approach [25]
and assuming that the virtual error signals are directly measured during real-time control. Although the presented feedback solution cannot be used in the considered local active noise control problem because the virtual error signals are not directly measured during real-time control, it has still been presented because it has been used to derive a solution to the general feedforward/feedback control problem. The general feedforward/feedback control approach can be adopted if the virtual locations are spatially fixed, and if suitable feedforward reference signals are available even if these signals are contaminated by intrinsic feedback from the control sources. The presented general feedforward/feedback control solution can be used to minimise the virtual error signals without directly measuring these signals during real-time control.

A number of adaptive feedforward control algorithms have been introduced that can be used when the virtual error signals are directly measured during real-time control. Although this is not the case in the local active noise control problem considered here, these adaptive algorithms have been presented because they are combined in Chapter 4 with the spatially fixed virtual sensing algorithms that will be introduced in the next chapter. The combination of these algorithms provides a method for adaptively minimising an estimate of the virtual error signals at a number of spatially fixed virtual locations. Also, these adaptive algorithms are modified in Chapter 5 and combined with moving virtual sensing algorithms, which can be used to compute an estimate of the virtual error signals at virtual locations that are moving through the sound field rather than being spatially fixed. The combination of these algorithms provides a method for adaptively minimising an estimate of the virtual error signals at a number of moving virtual locations with the aim to create a moving zone of quiet. The adaptive feedforward control algorithms that have been introduced in this chapter, and the modified versions of these adaptive algorithms that will be presented in Chapters 4 and 5, are also used in the acoustic duct experiments presented in Chapters 7–10.
Chapter 3

Spatially fixed virtual sensing algorithms

3.1 Introduction

In Chapter 2, active noise control algorithms have been introduced that can be used to create local zones of quiet at a number of virtual locations that are spatially fixed within the sound field. Provided that a suitable feedforward reference signal is available, either the optimal feedforward control approach presented in Section 2.4.1 or the general feedforward/feedback control approach presented in Section 2.4.3 can be adopted to minimise the virtual error signals without directly measuring them during real-time control. However, the optimal feedback control approach presented in Section 2.4.2 and the adaptive feedforward control approach presented in Section 2.5 cannot be implemented without directly measuring the virtual error signals during real-time control. This is because these control approaches require the feedback information contained in the virtual error signals to compute an appropriate control signal. To implement these control approaches without directly measuring the virtual error signals during real-time control, an estimate of the virtual error signals can instead be used as a feedback signal. This estimate can be computed using a virtual sensing algorithm. In this chapter, the virtual sensing component of the active noise control system illustrated in the block diagram of Fig. 2.1 is analysed. It is assumed in this chapter that the virtual locations are spatially fixed within the sound field. The case of virtual locations that are moving through the sound field will be addressed in Chapter 5, where moving virtual sensing algorithms will be developed.

In Section 3.2, the spatially fixed virtual sensing problem is introduced, and the assumptions made to analyse this linear estimation problem are presented. The analysis is based on the standard state-space model of the considered active noise control system.
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introduced in Chapter 2. The covariance properties of the stochastic input signals of this model are again defined as they are used frequently throughout this chapter.

In Section 3.3, the adaptive LMS virtual microphone technique \[14\] is analysed. This spatially fixed virtual sensing technique uses an array of physical sensors to compute an estimate of the virtual error signal. This estimate is computed as a weighted summation of the physical error signals measured by the physical sensor array. The weights are determined in a preliminary identification stage in which a physical sensor is temporarily located at the virtual location. The LMS algorithm \[127\] is then used to adapt the physical sensor weights such that the difference between the true virtual error signal directly measured at the virtual location and its estimate of the virtual error signal is minimised. After convergence of the weights, the physical sensor temporarily located at the virtual location is removed and the weights are fixed to their converged values, such that a spatially fixed virtual sensor is effectively created. The adaptive LMS virtual microphone technique has not been extended beyond the case of one virtual sensor, and the optimal Wiener solution \[51\] for the physical sensor weights has not been analysed previously. The optimal Wiener solution for the physical sensor weights is therefore derived, first for the case of one virtual sensor. The discussion is then extended to the case of multiple virtual sensors. An optimal solution for the physical sensor weights is also derived given the standard state-space model of the active noise control system introduced in Chapter 2.

In Section 3.4, the remote microphone technique proposed by both Popovich \[104\] and Roure and Albarrazin \[112\] is analysed. In this spatially fixed virtual sensing technique, transfer function models of the physical and virtual secondary transfer paths between the control sources and the physical and virtual sensors are estimated in a preliminary identification stage in which physical sensors are temporarily located at the virtual locations. Furthermore, a filter is estimated that computes an estimate of the virtual primary disturbances at the virtual locations given the physical primary disturbances at the physical sensors. This filter plus the physical and virtual secondary transfer path models are then used to compute an estimate of the virtual error signals, given the physical error signals and the control signals. Here, the factors that determine the optimal estimation performance of the remote microphone technique are analysed. The optimal estimation performance is derived using the factorisation approach that has been introduced in Chapter 2, where it has been used to derive the causal Wiener filter solution defined in Theorem 2.1. The factorisation approach allows an elegant interpretation of the factors that limit the theoretical optimal estimation performance of the remote microphone technique. The effect of measurement noise on the physical sensors, including the ones located at the virtual locations during the preliminary identification stage, is analysed as well.
In Section 3.5, the *virtual microphone arrangement* introduced by Elliott and David [27] is analysed. Similarly to the remote microphone technique [104, 112], transfer function models of the physical and virtual secondary transfer paths between the control sources and the physical and virtual sensors are estimated in a preliminary identification stage in which physical sensors are temporarily located at the spatially fixed virtual locations. The difference with the remote microphone technique [104, 112] is that the primary disturbances at the physical and virtual sensors are assumed to be equal. The additional filter that is used in the remote microphone technique to compute an estimate of the virtual primary disturbances given the physical primary disturbances is thus assumed to be unity in the virtual microphone arrangement. This algorithm is therefore a simplified version of the remote microphone technique. Here, the factors that limit the estimation performance of the virtual microphone arrangement are analysed. As expected, these factors are determined by the validity of the assumption that the primary disturbances at the physical and virtual sensors are equal.

In Section 3.6, the *hybrid adaptive feedforward observer* proposed by Tran and Southward [119] is analysed. The hybrid adaptive feedforward observer uses a state-space model of the active noise control system to compute an estimate of the virtual error signals at the spatially fixed virtual locations. It is shown here that if the number of physical sensors is less than the order of this state-space model, which is usually the case in practice, the proposed hybrid adaptive feedforward observer method is only suitable for rejecting non-stationary primary disturbances at the physical sensors, and not for virtual sensing purposes.

The common aim of the previously proposed spatially fixed virtual sensing algorithms analysed in Sections 3.3–3.6 is to compute an accurate estimate of the virtual error signals given the directly measured physical error signals. The difference between these algorithms is the way in which the estimate is computed, i.e. the assumed structure is what makes the virtual sensing algorithms proposed so far different from one another. Once the structure has been chosen, optimal solutions for the unknown parameters of the algorithm can be computed by optimising the estimation performance as described in Sections 3.3–3.6. The question now arises as to whether there is an *optimal structure* that can be used to solve the virtual sensing for active noise control problem. It is well-known that the *Kalman filter* provides an optimal structure for solving linear estimation problems [60]. Since virtual sensing for active noise control is a linear estimation problem, an *optimal solution* to this problem is derived in Section 3.7 using Kalman filter theory [36, 60]. The practical implementation of the proposed virtual sensing algorithm is also discussed.
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3.2 Problem description

The spatially fixed virtual sensing problem is analysed here assuming that information about the sound field is measured by a number of physical sensors. The physical error signals measured by these sensors can then be used to compute an estimate of the virtual error signals at the spatially fixed virtual locations within the sound field. From the standard state-space model defined in Eq. (2.17), the active noise control system that is considered here is described by the following standard state-space model [60]

\[
\begin{align*}
\mathbf{z}(n+1) &= A\mathbf{z}(n) + B_u\mathbf{u}(n) + B_v\mathbf{v}(n) \\
\mathbf{e}_p(n) &= C_P\mathbf{z}(n) + D_Pu\mathbf{u}(n) + D_Pv\mathbf{v}(n) + \mathbf{v}_p(n) & n \geq 0 \\
\mathbf{e}_v(n) &= C_v\mathbf{z}(n) + D_vu\mathbf{u}(n) + D_vs\mathbf{v}(n) + \mathbf{v}_v(n),
\end{align*}
\]

with \(\mathbf{z}(n) \in \mathbb{R}^N\) the states of the system, \(\mathbf{e}_p(n) \in \mathbb{R}^{M_p}\) the physical error signals, \(\mathbf{e}_v(n) \in \mathbb{R}^{M_v}\) the virtual error signals, and \(\mathbf{u}(n) \in \mathbb{R}^L\) the control signals. The disturbance source signals \(\mathbf{s}(n) \in \mathbb{R}^S\), the physical measurement noise signals \(\mathbf{v}_p(n) \in \mathbb{R}^{M_p}\), and the virtual measurement noise signals \(\mathbf{v}_v(n) \in \mathbb{R}^{M_v}\) are assumed to be zero mean white and stationary random processes, with covariance properties defined in Eq. (2.19) as

\[
E\left[ \begin{bmatrix} \mathbf{s}(n) \\
\mathbf{v}_p(n) \\
\mathbf{v}_v(n) \end{bmatrix} \mathbf{s}^T(k) \right] = \begin{bmatrix} \mathbf{I} & \mathbf{S}_{ps}^T & \mathbf{S}_{ps}^T \\
\mathbf{S}_{ps} & \mathbf{R}_p & \mathbf{R}_{pv} \\
\mathbf{S}_{ps}^T & \mathbf{R}_{pv}^T & \mathbf{R}_v \end{bmatrix} \delta_{nk}.
\]

Note that the state-space matrices \(\mathbf{C_v}, \mathbf{D_vu},\) and \(\mathbf{D_vs}\) in Eq. (3.1) are not dependent on the time-index \(n\) because it is assumed in this chapter that the virtual locations are spatially fixed. The aim of a spatially fixed virtual sensing algorithm is to compute an accurate estimate \(\hat{\mathbf{e}}_v(n) \in \mathbb{R}^{M_v}\) of the virtual error signals in Eq. (3.1) without directly measuring these signals during real-time control. The estimation performance of a virtual sensing algorithm can be analysed by defining virtual output errors \(\varepsilon_v(n) \in \mathbb{R}^{M_v}\) as

\[
\varepsilon_v(n) = \mathbf{e}_v(n) - \hat{\mathbf{e}}_v(n).
\]

In Sections 3.3–3.6, a number of spatially fixed virtual sensing algorithms suggested by other researchers [14, 104, 112, 119] are analysed. As discussed previously, the difference between these algorithms is the structure that is assumed to compute an estimate of the virtual error signals. Once the structure has been chosen, optimal solutions for the unknown parameters of the algorithms can be computed as described in Sections 3.3–3.6 by minimising a cost function \(I_\varepsilon\) defined as the mean-square value of the virtual output errors in Eq. (3.3), such that

\[
I_\varepsilon = \text{tr}\left( E\left[ \varepsilon_v(n)\varepsilon_v(n)^T \right] \right).
\]
3.3 Adaptive LMS virtual microphone technique

This section presents an analysis of the adaptive LMS virtual microphone technique introduced by Cazzolato [14]. This technique has not been extended beyond the case of one spatially fixed virtual sensor so far and the optimal Wiener solution [51] for the physical sensor weights that are used in this technique has not been analysed previously. The optimal Wiener solution for the physical sensor weights is therefore derived in Section 3.3.2, first for the case of one spatially fixed virtual sensor. The presented discussion is extended in Section 3.3.3 to the case of multiple spatially fixed virtual sensors. In Section 3.3.4, an optimal solution for the physical sensor weights is derived given the state-space description of the physical and virtual error signals defined in Eq. (3.1).

3.3.1 Algorithm structure

For the case of one spatially fixed virtual sensor, a block diagram of the adaptive LMS virtual microphone technique as implemented by previous researchers [80] is shown in Fig. 3.1.

![Block diagram](image)

Figure 3.1: Block diagram of the original implementation of the adaptive LMS virtual microphone technique, with $M_p$ physical sensors, and $M_v = 1$ spatially fixed virtual sensor.

An estimate $\hat{e}_v(n)$ of the virtual error signal is computed in Fig. 3.1 by summing the weighted physical error signals $e_p(n)$, which can be expressed as

$$\hat{e}_v(n) = \sum_{i=1}^{M_p} h_i e_{pi}(n) = h^T e_p(n), \quad (3.5)$$

where $h \in \mathbb{R}^{M_p}$ is a column vector containing the physical sensor weights defined as

$$h = [h_1 \ h_2 \ \cdots \ h_{M_p}]^T, \quad (3.6)$$

and where $e_p(n) \in \mathbb{R}^{M_p}$ is a column vector consisting of the physical error signals given by

$$e_p(n) = [e_{p1}(n) \ e_{p2}(n) \ \cdots \ e_{pM_p}(n)]^T. \quad (3.7)$$
The physical sensor weights are determined in a preliminary identification stage in which a physical sensor is placed at the virtual location. The LMS algorithm [51] is then used to adjust the physical sensor weights. After convergence of the weights, the physical sensor is removed from the virtual location, and the weights are fixed to their converged values.

**LMS algorithm**

The application of the LMS algorithm to finding an optimal solution for the physical sensor weights in an adaptive manner is now presented. Fig. 3.2 shows a block diagram of the LMS algorithm as applied in previous research [14, 80] into the adaptive LMS virtual microphone technique.

\[
\begin{align*}
 y_p(n) & \quad M_p \\
 h_u & \rightarrow \hat{y}_v(n) - \epsilon_v(n) \\
 LMS & \rightarrow y_v(n) \\
 \end{align*}
\]

Figure 3.2: Block diagram of the LMS algorithm as applied in the adaptive LMS virtual microphone technique, with \( M_p \) physical sensors, and \( M_v = 1 \) spatially fixed virtual sensor.

In this previous research [14, 80], an optimal solution for the weights was determined with the control source excited by band-limited white noise, and with the primary source switched off, such that \( s(n) = 0 \) in Eq. (3.1). For the single virtual sensor case, the virtual output error defined in Eq. (3.3) that needs to be minimised is then given by

\[
\epsilon_v(n) = y_v(n) - \hat{y}_v(n) = y_v(n) - h_u^T y_p(n),
\]

with \( y_v(n) \) the virtual secondary disturbance, \( \hat{y}_v(n) \) the estimated virtual secondary disturbance, \( y_p(n) \) the physical secondary disturbances, and \( h_u \in \mathbb{R}^{M_p} \) the physical sensor weights derived with only the control sources switched on. The LMS algorithm shown in Fig. 3.2 was then used to adapt the weights \( h_u \) such that the cost function defined in Eq. (3.4) was minimised, with \( \epsilon_v(n) \) defined in Eq. (3.8). The LMS algorithm used for updating the weights is then given by [51]

\[
h_u(n + 1) = h_u(n) + \mu y_p(n) \epsilon_v(n),
\]

with \( \mu \in \mathbb{R}^+ \) the positive scalar convergence coefficient. After convergence, the weights were fixed to their optimal value \( h_{uo} \). It is now important to note that the described
method results in physical sensor weights that are optimal for the estimation of the virtual secondary disturbance given the physical secondary disturbances.

Separation of estimation problem for the primary and secondary sound fields

In previous research [14, 80], an estimate of the virtual error signal was computed as illustrated in Fig. 3.1 as

\[ \hat{e}_v(n) = h_{uo}^T e_p(n), \tag{3.10} \]

with \( h_{uo} \) the optimal weights for the secondary sound field determined with the LMS algorithm shown in Fig. 3.2. Since the physical error signals are given by \( e_p(n) = d_p(n) + y(n) \), Eq. (3.10) can also be written as

\[ \hat{e}_v(n) = h_{uo}^T d_p(n) + h_{uo}^T y_p(n) = \hat{d}_v(n) + \hat{y}_v(n), \tag{3.11} \]

with \( \hat{d}_v(n) \) the estimated virtual primary disturbance, and \( \hat{y}_v(n) \) the estimated virtual secondary disturbance. Eq. (3.11) illustrates that the optimal weights \( h_{uo} \) for the secondary sound field are applied to both the primary and secondary physical disturbances. The underlying assumption made by previous researchers [80] is therefore that the weights \( h_{uo} \) for the secondary sound field are an optimal solution for the estimation of both \( d_v(n) \) and \( y_v(n) \), which might not always be true. As an example, for active noise control in the near-field of the secondary source, the spatial characteristics of the primary and secondary sound fields can be very different [27]. This property is for instance employed in the virtual microphone arrangement introduced by Elliott and David [27], where it is assumed that the primary sound field changes relatively little between the physical and virtual sensors, such that the primary disturbances at these sensors are assumed to be equal. The secondary disturbances at the physical and virtual sensors are, however, not assumed to be equal due to the near-field properties of the secondary source. For the adaptive LMS virtual microphone technique, the described situation will result in different optimal physical sensor weights for the primary and secondary sound fields. In conclusion, it is thus important to find optimal weights for the estimation of both the primary and secondary disturbances at the virtual sensor. If these weights are equal, the adaptive LMS virtual microphone technique can be implemented as illustrated in Fig. 3.1. If these weights are not equal, the virtual sensing algorithm needs to be able to separate the physical error signals \( e_p(n) \) into its primary and secondary components \( d_p(n) \) and \( y_p(n) \), respectively. This can be achieved as illustrated in Fig. 3.3, where a block diagram of the modified implementation of the adaptive LMS virtual microphone technique is shown.

The separation of the physical error signals into their primary and secondary components can be obtained as illustrated in Fig. 3.3 by using the physical secondary
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Figure 3.3: Block diagram of the modified implementation of the adaptive LMS virtual microphone technique, with $L$ control sources, $M_p$ physical sensors, and $M_v = 1$ spatially fixed virtual sensor.

### transfer path matrix $G_{pu}$, such that
\[
d_p(n) = e_p(n) - G_{pu}u(n) = e_p(n) - y_p(n).
\] (3.12)

The optimal weights for the primary and secondary sound fields can then be applied separately to the physical primary and secondary disturbances in order to get optimal estimates $\hat{d}_v(n)$ and $\hat{y}_v(n)$ of the virtual primary and secondary disturbances, respectively. These estimates can then be superposed to get an optimal estimate $\hat{e}_v(n)$ of the virtual error signal, such that
\[
\hat{e}_v(n) = \hat{d}_v(n) + \hat{y}_v(n) = h_{so}^T d_p(n) + h_{uo}^T y_p(n),
\] (3.13)

with $h_{so}$ the optimal physical sensor weights for the primary sound field, and $h_{uo}$ the optimal physical sensor weights for the secondary sound field. Using Eq. (3.12), Eq. (3.13) can also be written as
\[
\hat{e}_v(n) = h_{so}^T e_p(n) + (h_{uo} - h_{so})^T G_{pu}u(n).
\] (3.14)

Eq. (3.14) shows that if the optimal weights for the primary and secondary sound fields are equal, such that $h_{so} = h_{uo} = h_o$, the adaptive LMS virtual microphone technique can be implemented as illustrated in Fig. 3.1, which is the implementation used by previous researchers [14, 80].

### 3.3.2 Optimal Wiener solution for physical sensor weights

The LMS algorithm illustrated in Fig. 3.2 can be used to determine an optimal solution for the physical sensor weights for the secondary sound field in an adaptive manner. This optimal solution is obtained after convergence of the LMS algorithm and approximates a solution known as the optimal Wiener solution [51], which will be denoted by $h_{uo}$ in the following. The optimal Wiener solution is derived in this section by minimising the virtual output error $\hat{e}_v(n)$ in Fig. 3.2. The derived solution is thus optimal for the estimation of the virtual secondary disturbance $y_v(n)$ at one spatially fixed virtual
sensor given the physical secondary disturbances \( y_p(n) \). The optimal Wiener solution \( h_{uo} \) for the primary sound field can be derived in a similar way. Fig. 3.1 has been redrawn in Fig. 3.4 to illustrate the filter problem discussed here.

![Block diagram of the filter problem](image)

**Figure 3.4:** Block diagram of the filter problem that is solved to derive the optimal Wiener solution for the physical sensor weights \( h_u \) for the secondary sound field.

The aim of the filter problem illustrated in Fig. 3.4 is to find a set of optimal weights \( h_{uo} \) that minimise the virtual output error \( \varepsilon_v(n) \). The virtual output error in Fig. 3.4 is given by

\[
\varepsilon_v(n) = y_v(n) - \hat{y}_v(n) = y_v(n) - h_u^T y_p(n),
\]

where \( y_p(n) \in \mathbb{R}^{M_p} \) is a column vector containing the physical secondary disturbances defined as

\[
y_p(n) = \begin{bmatrix}
    y_{p1}(n) \\
y_{p2}(n) \\
    \vdots \\
y_{pM_p}(n)
\end{bmatrix}^T.
\]

The column vector \( h_u \in \mathbb{R}^{M_p} \) in Eq. (3.15) contains the physical sensor weights for the secondary sound field, and is defined as

\[
h_u = \begin{bmatrix}
h_{u1} \\
h_{u2} \\
\vdots \\
h_{uM_p}
\end{bmatrix}^T.
\]

The optimal weights are now defined as the weights that minimise the cost function \( J_\varepsilon \) given by the mean-square virtual output error

\[
J_\varepsilon = E[\varepsilon_v(n)^2].
\]

After substituting Eq. (3.15) into Eq. (3.18), this cost function can be written as

\[
J_\varepsilon = h_u^T R_u h_u - 2h_u^T y_p + \sigma_{vu},
\]

with \( \sigma_{vu} \) a scalar constant equal to the variance of the virtual secondary disturbance, such that

\[
\sigma_{vu} = E[y_v(n)^2].
\]

The matrix \( R_u \in \mathbb{R}^{M_u \times M_p} \) in Eq. (3.19) is given by

\[
R_u = E[y_p(n)y_p(n)^T],
\]
with $y_p(n)$ the vector of physical secondary disturbances defined in Eq. (3.16). The matrix $R_u$ is thus the covariance matrix of the physical secondary disturbances, and can also be written as

$$R_u = \begin{bmatrix} \sigma^2_1 & \sigma^2_2 & \cdots & \sigma^2_{M_p} \\ \sigma^2_2 & \sigma^2_3 & \cdots & \sigma^2_{M_p1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_{M_p} & \sigma^2_{M_p1} & \cdots & \sigma^2_{M_p2} \end{bmatrix},$$ (3.22)

where $\sigma^2_i$ is the variance of the physical secondary disturbance $y_{pi}(n)$ in Eq. (3.16), and $\sigma^2_{ij}$ the covariance between the physical secondary disturbances $y_{pi}(n)$ and $y_{pj}(n)$, with $\sigma^2_{ij} = \sigma^2_{ji}$. The column vector $p_u \in \mathbb{R}^{M_p}$ in Eq. (3.19) is defined as

$$p_u = E[y_p(n)y_v(n)],$$ (3.23)

which can be written as

$$p_u = \begin{bmatrix} \sigma^2_1v & \sigma^2_2v & \cdots & \sigma^2_{M_pv} \end{bmatrix}^T,$$ (3.24)

with $\sigma^2_{iv}$ the covariance between the physical secondary disturbance $y_{pi}(n)$ and the virtual secondary disturbance $y_v(n)$. The optimal weights $h_{uo}$ can now be found by differentiating the cost function $J_\epsilon$ in Eq. (3.19) with respect to each weight $h_{ui}$, with $i = 1 \ldots M_p$, and setting all of the resulting derivatives to zero [127]

$$\frac{\partial J_\epsilon}{\partial h_{ui}} = \begin{bmatrix} \frac{\partial J_\epsilon}{\partial h_{u1}} & \frac{\partial J_\epsilon}{\partial h_{u2}} & \cdots & \frac{\partial J_\epsilon}{\partial h_{uM_p}} \end{bmatrix}^T = 0.$$ (3.25)

Using the definition of $J_\epsilon$ from Eq. (3.19), this can be expressed as [127]

$$\frac{\partial J_\epsilon}{\partial h_{ui}} = 2[R_u h_{ui} - p_u] = 0.$$ (3.26)

The optimal solution $h_{uo} \in \mathbb{R}^{M_p}$ is thus given from Eq. (3.26) by

$$h_{uo} = R_u^{-1}p_u.$$ (3.27)

The optimal estimation performance can now be computed by substituting the optimal solution in Eq. (3.27) into the cost function in Eq. (3.19) to give

$$J_{\epsilon,\text{min}} = \sigma_{vu} - p_u^T h_{uo}.$$ (3.28)

The optimal solution and the residual mean-square virtual output error can thus be calculated directly from the statistical properties of the secondary disturbances at the physical and virtual sensors.
Principle of orthogonality

The vector of covariances between the physical secondary disturbances \( y_p(n) \) and the virtual output error \( \varepsilon_v(n) \) can be written, using Eq. (3.15), as

\[
E[y_p(n)\varepsilon_v(n)] = E[y_p(n)(y_v(n) - h^T_u y_p(n))] = p_u - R_u h_u. \quad (3.29)
\]

When substituting the optimal solution for the weights defined in Eq. (3.27) into Eq. (3.29), it is clear that all the elements of this vector are zero when the weights are adjusted to minimise the mean-square virtual output error. This means that the physical secondary disturbances \( y_{pi}(n) \) and the virtual output error \( \varepsilon_v(n) \) become uncorrelated if the weights are adjusted to the optimal solution given in Eq. (3.27). This observation is referred to as the principle of orthogonality [51, 60].

Optimal weights for narrowband disturbances

Suppose the physical secondary disturbances \( y_p(n) \) and the virtual secondary disturbance \( y_v(n) \) are tonal signals of normalised angular frequency \( \omega T_s \) given by

\[
\begin{bmatrix} y_p(n) \\ y_v(n) \end{bmatrix} = \begin{bmatrix} Y_p \cos(\omega T_s n + \theta_p) \\ Y_v \cos(\omega T_s n + \theta_v) \end{bmatrix}, \quad (3.30)
\]

where \( Y_v \in \mathbb{R} \) is the magnitude of the virtual secondary disturbance, and \( \theta_v \in \mathbb{R} \) the phase with respect to some arbitrary reference. The vectors \( Y_p \in \mathbb{R}^{M_p} \) and \( \theta_p \in \mathbb{R}^{M_p} \) in Eq. (3.30) are defined as

\[
Y_p = \begin{bmatrix} Y_{p1} & Y_{p2} & \ldots & Y_{pM_p} \end{bmatrix}^T, \quad (3.31)
\]

\[
\theta_p = \begin{bmatrix} \theta_{p1} & \theta_{p2} & \ldots & \theta_{pM_p} \end{bmatrix}^T, \quad (3.32)
\]

with \( Y_{pi} \in \mathbb{R} \) the magnitude of the physical secondary disturbance \( y_{pi}(n) \) in Eq. (3.16), and \( \theta_{pi} \in \mathbb{R} \) the phase with respect to some arbitrary reference. For this case, it can be shown that only two physical sensors are required to set the virtual output error \( \varepsilon_v(n) \) in Eq. (3.15) equal to zero. For the two physical sensor case, and with the physical and virtual secondary disturbances defined as in Eq. (3.30), the covariance matrix \( R_u \) defined in Eq. (3.22) is given by

\[
R_u = \frac{1}{2} \begin{bmatrix} Y_{p1}^2 & Y_{p1}Y_{p2} \cos(\theta_{p1} - \theta_{p2}) \\ Y_{p2}Y_{p1} \cos(\theta_{p2} - \theta_{p1}) & Y_{p2}^2 \end{bmatrix}, \quad (3.33)
\]
and the vector of cross-covariances $p_u$ defined in Eq. (3.24) by

$$p_u = \frac{1}{2} \begin{bmatrix} Y_{p1} Y_v \cos(\theta_{p1} - \theta_v) \\ Y_{p2} Y_v \cos(\theta_{p2} - \theta_v) \end{bmatrix}. \quad (3.34)$$

The optimal tonal weights $h_{uo} \in \mathbb{R}^2$ are now found by substituting Eqs (3.33) and (3.34) into Eq. (3.27), which results in

$$h_{uo} = \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} -\frac{Y_v}{Y_{p1}} \sin(\theta_{p2} - \theta_v) \\ \frac{Y_v}{Y_{p2}} \sin(\theta_{p1} - \theta_v) \end{bmatrix}. \quad (3.35)$$

Substituting this solution into Eq. (3.15), with $y_p(n)$ and $y_v(n)$ as defined in Eq. (3.30), results in

$$\epsilon_v(n) = y_v(n) - h_{uo}^T y_p(n) = 0. \quad (3.36)$$

This confirms that for single tone disturbances, a perfect estimate of the virtual secondary disturbance can be obtained by using two physical sensors.

**Conditioning of estimation problem for narrowband disturbances**

Eq. (3.35) indicates that the optimal weights can become very large when the phases of the two physical secondary disturbances are such that

$$\sin(\theta_{p1} - \theta_{p2}) \approx 0. \quad (3.37)$$

This means that the estimation problem becomes *ill-conditioned* if this is the case. The conditioning of the estimation problem can be analysed by computing the condition number $\kappa(R_u)$ of the covariance matrix $R_u$ in Eq. (3.33), which results in

$$\kappa(R_u) = \frac{1 + Y_{12}^2 + \sqrt{Y_{12}^4 + 2Y_{12}^2 \cos 2\Delta_\theta + 1}}{1 + Y_{12}^2 - \sqrt{Y_{12}^4 + 2Y_{12}^2 \cos 2\Delta_\theta + 1}}, \quad (3.38)$$

with $Y_{12} \triangleq Y_{p1} Y_{p2}^{-1}$ and $\Delta_\theta \triangleq \theta_{p1} - \theta_{p2}$. Eq. (3.38) shows that the condition number $\kappa(R_u)$ will become infinite when the phase difference $\Delta_\theta$ between the two physical secondary disturbances is such that $\cos 2\Delta_\theta = 1$. This occurs when

$$\Delta_\theta = k\pi, \quad (3.39)$$

where $k$ is an integer, which is precisely when the optimal weights in Eq. (3.35) become infinitely large. In other words, if the two physical secondary disturbances are approximately in phase or out of phase by $180^\circ$, the estimation problem becomes ill-conditioned. This observation can also be derived from the vector diagram shown in Fig. 3.5, which
3.3 Adaptive LMS virtual microphone technique

Figure 3.5: Vector diagram illustrating the tonal estimation problem using $M_p = 2$ physical sensors and $M_v = 1$ virtual sensor.

illustrates how the two weights are calculated given the magnitudes and phases of the virtual and physical secondary disturbances. It can be seen from this figure that the conditioning of the estimation problem will be good when the two vectors that represent the physical secondary disturbances are orthogonal, i.e. in quadrature. An ill-conditioned problem occurs when these two vectors are almost aligned, i.e. in phase or out of phase by 180°. These observations can be used to choose suitable locations for the physical sensors for the case of tonal disturbances.

3.3.3 Extension to multiple spatially fixed virtual sensors

As discussed earlier, the adaptive LMS virtual microphone technique has not been extended beyond the case of one spatially fixed virtual sensor. In this section, this technique is therefore extended to the case of multiple spatially fixed virtual sensors. Again, optimal weights for the secondary sound field are derived assuming that the primary sources are switched off. Optimal weights for the primary sound field can be derived in a similar fashion assuming that the control sources are switched off. For the multiple virtual sensors case, and with the primary sources switched off, the virtual output errors are given by

$$e_v(n) = y_v(n) - H^T_p y_p(n),$$  \hspace{1cm} (3.40)

with $y_v(n) \in \mathbb{R}^{M_v}$ a column vector containing the virtual secondary disturbances at the spatially fixed virtual locations defined as

$$y_v(n) = \left[ \begin{array}{c} y_{v1}(n) \\ y_{v2}(n) \\ \vdots \\ y_{vM_v}(n) \end{array} \right]^T.$$  \hspace{1cm} (3.41)
The matrix $H_u \in \mathbb{R}^{M_p \times M_v}$ in Eq. (3.40) is a matrix of physical sensor weights for the secondary sound field, which is defined as

$$H_u = \begin{bmatrix}
h_{u11} & h_{u12} & \cdots & h_{u1M_v} \\
h_{u21} & h_{u22} & \cdots & h_{u2M_v} \\
\vdots & \vdots & \ddots & \vdots \\
h_{uM_p1} & h_{uM_p2} & \cdots & h_{uM_pM_v}
\end{bmatrix} = \begin{bmatrix}
h_{u1} & h_{u2} & \cdots & h_{uM_v}
\end{bmatrix}, \quad (3.42)$$

with $h_{umv} \in \mathbb{R}^{M_p}$ the column vector of weights used for estimating the virtual secondary disturbance $y_{vm}(n)$ in Eq. (3.41) given the physical secondary disturbances $y_p(n)$. The optimal weights $H_{uo} \in \mathbb{R}^{M_p \times M_v}$ are now defined as the weights that minimise the cost function

$$J_\varepsilon = \text{tr}\left(\mathbb{E}[(\varepsilon_{p}(n)\varepsilon_{v}(n))^T]\right). \quad (3.43)$$

After substituting Eq. (3.40) into Eq. (3.43), this cost function can be written as

$$J_\varepsilon = \text{tr}\left(H_u^T R_u H_u - 2H_u^T P_u + \sigma_{uv}\right), \quad (3.44)$$

with $\sigma_{uv} \in \mathbb{R}^{M_v \times M_v}$ the covariance matrix of the virtual secondary disturbances given by

$$\sigma_{uv} = \mathbb{E}[y_v(n)y_v(n)^T], \quad (3.45)$$

and with $R_u \in \mathbb{R}^{M_p \times M_p}$ the covariance matrix of the physical secondary disturbances defined in Eq. (3.22). The cross-covariance matrix $P_u \in \mathbb{R}^{M_p \times M_v}$ in Eq. (3.44) is defined as

$$P_u = \mathbb{E}[y_p(n)y_v(n)^T], \quad (3.46)$$

and can be written as

$$P_u = \mathbb{E}\left[\begin{bmatrix}y_p(n)y_{v1}(n) & y_p(n)y_{v2}(n) & \cdots & y_p(n)y_{vM_v}(n)\end{bmatrix}\right] = \begin{bmatrix}
p_{u1} & p_{u2} & \cdots & p_{uM_v}
\end{bmatrix}. \quad (3.47)$$

The column vectors $p_{umv} \in \mathbb{R}^{M_p}$ in Eq. (3.48) are defined, similarly to Eq. (3.24), as

$$p_{umv} = \left[\sigma_{1vmv}^2 \sigma_{2vmv}^2 \cdots \sigma_{Mvmv}^2\right]^T, \quad (3.49)$$

with $\sigma_{vmv}^2$ the covariance between the physical secondary disturbance $y_{pl}(n)$ and the virtual secondary disturbance $y_{vm}(n)$. The optimal weights $H_{uo}$ can now be found by differentiating the cost function $J_\varepsilon$ in Eq. (3.44) with respect to the weights $H_u$, and setting all of the resulting derivatives to zero [25], which results in

$$\frac{\partial J_\varepsilon}{\partial H_u} = 2[R_u H_u - P_u] = 0. \quad (3.50)$$
The optimal weights are found by solving Eq. (3.50) with respect to the weights $H_u$. The optimal solution will be denoted by the matrix $H_{uo} \in \mathbb{R}^{M_v \times M_o}$, and is given by
\[
H_{uo} = R_u^{-1}P_u. \tag{3.51}
\]

Using Eqs (3.42) and (3.48), this can also be written as
\[
\begin{bmatrix}
  h_{u1o} & h_{u2o} & \cdots & h_{uM_vo}
\end{bmatrix} = R_u^{-1}
\begin{bmatrix}
  p_{u1} & p_{u2} & \cdots & p_{uM_v}
\end{bmatrix}. \tag{3.52}
\]

Eq. (3.52) indicates that each column $h_{umvo} \in \mathbb{R}^{M_p}$ of the matrix $H_{uo}$ in Eq. (3.51) can be calculated as defined previously in Eq. (3.27) for the case of one spatially fixed virtual sensor.

### 3.3.4 Optimal solution from state-space model

In this section, an optimal solution for the physical sensor weights is derived given the standard state-space model of the active noise control system defined in Eq. (3.1). One of the reasons this derivation is presented is that in the numerical acoustic duct analysis presented in Chapter 7, a state-space model of the acoustic duct is computed using a modal modelling technique. Using the derivations presented in this section, a numerical optimal solution for the physical sensor weights can be directly calculated from this state-space model. Furthermore, measurement noise on the physical and virtual sensors can easily be included in the presented derivations.

By setting the disturbance source signal $s(n) = 0$ in Eq. (3.1), optimal weights for the secondary sound field can be derived. Similarly, optimal weights for the primary sound field can be derived by setting the control signal $u(n) = 0$. Here, the disturbance source signal is set to zero, and the standard state-space model defined in Eq. (3.1) can then be written as
\[
\begin{align*}
  z(n+1) &= Az(n) + Bu(n) \\
  y_p(n) &= C_pz(n) + D_pu(n) + v_p(n) \quad n \geq 0 \tag{3.53}
\end{align*}
\]

An estimate of the virtual secondary disturbances $y_v(n)$ is again computed as
\[
\hat{y}_v(n) = H_u^T y_p(n), \tag{3.54}
\]

with $H_u$ the matrix of weights for the secondary sound field defined in Eq. (3.42). From Eq. (3.53), a state-space system that models the estimated virtual secondary disturbances $\hat{y}_v(n)$ is now given by
\[
\begin{align*}
  z(n+1) &= Az(n) + Bu(n) \\
  \hat{y}_v(n) &= H_u^T C_pz(n) + H_u^T D_pu(n) + H_u^T v_p(n). \tag{3.55}
\end{align*}
\]
The virtual output errors $\varepsilon_v(n)$ are again defined, assuming that $s(n) = 0$, as
$$
\varepsilon_v(n) = y_v(n) - \hat{y}_v(n).
$$
(3.56)

Using Eqs (3.53) and (3.55), a state-space system that models the virtual output errors $\varepsilon_v(n)$ is now given by
$$
z(n + 1) = Az(n) + Bu(n)$$
$$
\varepsilon_v(n) = C_vz(n) + D_{uv}u(n) + v_v(n) - H^T_s v_p(n),
$$
(3.57)
where the matrices $C_v \in \mathbb{R}^{M_v \times N}$ and $D_{uv} \in \mathbb{R}^{M_v \times L}$ are defined as
$$
C_v = C_v - H^T_s C_p$$
$$
D_{uv} = D_{vu} - H^T_s D_{pu}.
$$
(3.58)

The optimal weights $H_{uo} \in \mathbb{R}^{M_p \times M_v}$ are again defined as the weights that minimise the cost function
$$
J_\varepsilon = \text{tr} \left( E \left[ \varepsilon_v(n) \varepsilon_v(n)^T \right] \right).
$$
(3.59)
It is assumed that the control signal $u(n)$ is a zero-mean white and stationary random process during the identification of the weights. Furthermore, the measurement noise signals $v_p(n)$ and $v_v(n)$ are assumed to be zero-mean white and stationary random processes that are uncorrelated to $u(n)$, such that the following covariance matrices can be defined
$$
E \left[ \begin{bmatrix}
u(n) \\
v_p(n) \\
v_v(n) \\
z(0)
\end{bmatrix} \begin{bmatrix}u(k) \\
v_p(k) \\
v_v(k) \\
z(0)
\end{bmatrix}^T \right] = \begin{bmatrix}
Q_u & 0 & 0 & 0 \\
0 & R_p & R_{pv} & 0 \\
0 & R_{pv} & R_p & 0 \\
0 & 0 & 0 & \Pi_0
\end{bmatrix} \delta_{nk},
$$
(3.60)
with $z(0) \in \mathbb{R}^N$ the initial state estimate, and $\Pi_0 \in \mathbb{R}^{N \times N}$ the covariance matrix of the initial state estimate. It can then be shown that the cost function in Eq. (3.59) can be written as
$$
J_\varepsilon = \text{tr} \left( C_v \Pi_u C_v^T + D_{uv} Q_u D_{uv}^T + H^T_s R_p H_s - 2H^T_s R_{pv} + R_v \right),
$$
(3.61)
where $\Pi_u > 0$ is the solution to the discrete-time Lyapunov equation [130]
$$
\Pi_u = A \Pi_u A^T + B_u Q_u B_u^T.
$$
(3.62)
Eqs (3.61) and (3.62) can be derived as follows. First, the state covariance matrix $\Pi_u(n)$ is defined as
$$
\Pi_u(n) = E [z(n)z(n)^T].
$$
(3.63)
From Eq. (3.53), the state covariance matrix satisfies the recursion

$$\Pi_u(n+1) = A \Pi_u(n) A^T + AE[z(n)u(n)^T]B_u^T + B_u E[u(n)z(n)^T]A^T + B_u Q_u B_u^T. \tag{3.64}$$

It can be shown, using the state-space model in Eq. (3.53) and the covariance properties of its stochastic input signals defined in Eq. (3.60), that the current state $z(n)$ is uncorrelated to the current input $u(n)$ [60], such that

$$E[z(n)u(n)^T] = 0. \tag{3.65}$$

This can be seen by deriving the following expression from Eq. (3.53)

$$z(n) = A^n z(0) + \sum_{m=1}^{n} A^{n-m} B_u u(m-1). \tag{3.66}$$

The state $z(n)$ is thus a linear combination of the initial state $z(0)$ and the past inputs $\{u(m), m = 1, \ldots, n - 1\}$. From the covariance properties defined in Eq. (3.60), the input $u(n)$ is uncorrelated to all of these variables during identification of the weights, thereby arriving at Eq. (3.65). Eq. (3.64) therefore reduces to

$$\Pi_u(n+1) = A \Pi_u(n) A^T + B_u Q_u B_u^T. \tag{3.67}$$

When the state $z(n)$ reaches its mean steady state value, the state covariance matrix $\Pi_u(n+1) = \Pi_u(n) = \Pi_u$ in Eq. (3.67), and solving the discrete-time Lyapunov equation in Eq. (3.62) gives the steady state solution $\Pi_u$.

Using a similar reasoning to that used to derive Eq. (3.65), it can be shown that the current state $z(n)$ is uncorrelated to the current measurement noise signals $v_p(n)$ and $v_v(n)$ [60], such that

$$E[z(n)v_p(n)^T] = 0, \quad E[z(n)v_v(n)^T] = 0. \tag{3.68}$$

Using the covariance properties defined in Eqs (3.60), (3.65) and (3.68), the cost function in Eq. (3.59) can now be written as defined in Eq. (3.61).

Next, an optimal solution $H_{uo}$ for the physical sensor weights is derived that minimises the cost function defined in Eq. (3.61). Substituting the matrices defined in Eq. (3.58) into the cost function defined in Eq. (3.61) gives

$$J_e = tr \left( H_u^T R_u H_u - 2 H_u^T P_u + \sigma_{vu} \right), \tag{3.69}$$

with

$$R_u = C_p \Pi_u C_p^T + D_p u Q_u D_p^T + R_p, \quad P_u = C_v \Pi_u C_v^T + D_p u Q_u D_v^T + R_v, \tag{3.70}$$

$$\sigma_{vu} = C_v \Pi_u C_v^T + D_v u Q_u D_v^T + R_v.$$
where $R_u \in \mathbb{R}^{M_p \times M_p}$ is again the covariance matrix of the physical secondary disturbances defined in Eq. (3.22), $P_u \in \mathbb{R}^{M_p \times M_v}$ the cross-covariance matrix between the physical and virtual secondary disturbances defined in Eq. (3.48), and $\sigma_{vu} \in \mathbb{R}^{M_v \times M_v}$ the covariance matrix of the virtual secondary disturbances defined in Eq. (3.45). The optimal weights can now be found by differentiating the cost function $J_\epsilon$ in Eq. (3.69) with respect to the weights $H_u$, and setting all of the resulting derivatives to zero:

$$\frac{\partial J_\epsilon}{\partial H_u} = 2[R_u H_u - P_u] = 0. \quad (3.71)$$

Solving Eq. (3.71) with respect to the weights $H_u$ results in

$$H_{uo} = R_u^{-1} P_u. \quad (3.72)$$

Similarly, optimal weights $H_{so} = R_s^{-1} P_s$ for the primary sound field can be computed, with

$$R_s = C_p \Pi_s C_p^T + D_{ps} Q_s D_{ps}^T + R_p,$$
$$P_s = C_p \Pi_s C_v^T + D_{ps} Q_s D_{ps}^T + R_{pv}. \quad (3.73)$$

where $R_s \in \mathbb{R}^{M_p \times M_p}$ is the covariance matrix of the physical primary disturbances, $P_s \in \mathbb{R}^{M_p \times M_v}$ the cross-covariance matrix between the physical and virtual primary disturbances, and $\Pi_s > 0$ the solution to the discrete-time Lyapunov equation

$$\Pi_s = A \Pi_s A^T + B_s Q_s B_s^T. \quad (3.74)$$

The above discussions are now summarised in the following theorem.

**Theorem 3.1** (Optimal physical sensor weights from state-space model).

*Given the state-space model in Eq. (3.1), the optimal weights $H_{uo} \in \mathbb{R}^{M_p \times M_v}$ for the secondary sound field are given by

$$H_{uo} = R_u^{-1} P_u, \quad (3.75)$$

with $R_u$ and $P_u$ the covariance matrices defined in Eq. (3.70). These weights minimise the cost function

$$J_\epsilon = \text{tr} \left( E \left[ \epsilon_v(n) \epsilon_v(n)^T \right] \right), \quad (3.76)$$

where the virtual output error is defined as

$$\epsilon_v(n) = y_v(n) - H_u^T y_p(n), \quad (3.77)$$

with $y_p(n)$ and $y_v(n)$ the outputs of the state-space model in Eq. (3.1) while setting the disturbance source signal $s(n) = 0$. Similarly, the optimal weights $H_{so} \in \mathbb{R}^{M_p \times M_v}$ for the primary sound field are given by

$$H_{so} = R_s^{-1} P_s, \quad (3.78)$$

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with $\mathbf{R}_s$ and $\mathbf{P}_s$ the covariance matrices defined in Eq. (3.73). These weights minimise the cost function in Eq. (3.76), where the virtual output error is now given by

$$\varepsilon_v(n) = d_v(n) - \mathbf{H}_s^T d_p(n),$$  \hspace{1cm} (3.79)

with $d_p(n)$ and $d_v(n)$ the outputs of the state-space model in Eq. (3.1) while setting the control signal $u(n) = 0$.

Theorem 3.1 will be used in Chapter 7 to analyse the estimation performance of the adaptive LMS virtual microphone technique when implemented on an acoustic duct arrangement. The numerical analysis presented in that chapter is based on a state-space model of the acoustic duct, which is computed using a modal modelling technique. Using the derivations presented in this section and summarised in Theorem 3.1, a numerical optimal solution for the physical sensor weights can be directly calculated from this state-space model, such that the numerical estimation performance of the adaptive LMS virtual microphone technique can be conveniently analysed. The main advantage of this is that the physical sensor weights no longer need to be calculated using the computationally more intensive adaptive method illustrated in Fig. 3.2, which was the case in previous research [13, 80].

3.4 Remote microphone technique

In this section, the remote microphone technique proposed by Popovich [104] and Roure and Albarrazin [112] is analysed. In this spatially fixed virtual sensing technique, transfer function models of the physical and virtual secondary transfer paths between the control sources and the physical and virtual sensors are estimated in a preliminary identification stage in which physical sensors are temporarily located at the virtual locations. Furthermore, a filter is calculated that computes an estimate of the virtual primary disturbances at the virtual locations given the physical primary disturbances. This filter plus the physical and virtual secondary transfer path models are then used to compute an estimate of the virtual error signals given the physical error signals and the control signals. Here, the factors that limit the optimal estimation performance of the remote microphone technique are analysed. The optimal estimation performance is derived using the factorisation approach that has been introduced in Chapter 2 to derive the causal Wiener filter solution in Theorem 2.1. The factorisation approach allows a convenient interpretation of the factors that limit the optimal estimation performance of the remote microphone technique. The effect of measurement noise on the physical sensors, including the ones located at the virtual locations during the preliminary identification stage, is analysed as well.
3.4.1 Algorithm structure

A block diagram of the remote microphone technique is shown in Fig. 3.6. The signals and transfer paths in this block diagram have been defined in Chapter 2.

![Block diagram of the remote microphone technique](image)

Figure 3.6: Block diagram of the remote microphone technique without measurement noise, and with \( L \) control sources, \( M_p \) physical sensors, and \( M_v \) spatially fixed virtual sensors.

The input signals to the remote microphone technique are the control signals \( u(n) \in \mathbb{R}^L \) and the physical error signals \( e_p(n) \in \mathbb{R}^{M_p} \). As illustrated in Fig. 3.6, the physical primary disturbances \( d_p(n) \in \mathbb{R}^{M_p} \) are first retrieved from the physical error signals by subtracting the physical secondary disturbances \( y_p(n) \in \mathbb{R}^{M_p} \), such that

\[
d_p(n) = e_p(n) - y_p(n) = e_p(n) - G_{pu} u(n), \tag{3.80}
\]

where the physical secondary disturbances are thus generated by filtering the control signals with the physical secondary transfer path matrix \( G_{pu} \in \mathcal{RH}_{\infty}^{M_v \times L} \). Next, an estimate \( \hat{d}_v(n) \in \mathbb{R}^{M_v} \) of the virtual primary disturbances is computed by filtering the retrieved physical primary disturbances \( d_p(n) \) with a filter \( H \in \mathcal{RH}_{\infty}^{M_v \times M_p} \) as illustrated in Fig. 3.6. An optimal solution for this filter is computed in Section 3.4.2 by solving an optimal filtering problem. Finally, an estimate \( \hat{e}_v(n) \) of the virtual error signals is computed in Fig. 3.6 by superposing the estimated virtual primary disturbances and the virtual secondary disturbances \( y_v(n) \in \mathbb{R}^{M_v} \), such that

\[
\hat{e}_v(n) = \hat{d}_v(n) + y_v(n), \tag{3.81}
\]

where the virtual secondary disturbances are generated by filtering the control signals with the virtual secondary transfer path matrix \( G_{vu} \in \mathcal{RH}_{\infty}^{M_v \times (L + M_p)} \). A transfer function matrix \( G_{RMT} \in \mathcal{RH}_{\infty}^{M_v \times (L + M_p)} \) describing the input-output behaviour of the remote microphone technique can now be defined from Fig. 3.6 as

\[
\begin{bmatrix}
\hat{e}_v(n)
\end{bmatrix} = G_{RMT} \begin{bmatrix}
e_p(n) \\
u(n)
\end{bmatrix} = \begin{bmatrix}
H & G_{vu} - HG_{pu}
\end{bmatrix} \begin{bmatrix}
e_p(n) \\
u(n)
\end{bmatrix}. \tag{3.82}
\]
To implement the remote microphone technique, an optimal solution for the filter $H$ needs to be derived. As proposed by Popovich [104] and Roure and Albarrazin [112], this can be achieved using an adaptive system identification technique based on the LMS algorithm [51]. In this technique, physical sensors are temporarily located at the virtual locations in a preliminary identification stage, and the filter $H$ is modelled as a matrix of FIR filters. The coefficients of these filters are then adapted by the multi-channel LMS algorithm with the control sources switched off, such that the virtual output errors

$$
\epsilon_v(n) = d_v(n) - \hat{d}_v(n) = d_v - H d_p(n)
$$

(3.83)

are minimised. Here, the filter $H$ will be represented by a transfer function matrix rather than a matrix of FIR filters. An optimal solution that minimises the virtual output errors in Eq. (3.83) can then be derived using the factorisation approach presented in Chapter 2 to derive the causal Wiener filter defined in Theorem 2.1.

### 3.4.2 Optimal estimation without measurement noise

An optimal solution for the transfer function matrix $H$ is now derived assuming that there is no measurement noise on the physical sensors, including the ones located at the spatially fixed virtual locations in a preliminary identification stage. The virtual output error is again defined as

$$
\epsilon_v(n) = e_v(n) - \hat{e}_v(n),
$$

(3.84)

with the virtual error signals $e_v(n)$ given by

$$
e_v(n) = \begin{bmatrix} G_{vu} & G_{vs} \end{bmatrix} \begin{bmatrix} u(n) \\ s(n) \end{bmatrix},
$$

(3.85)

and with the estimate $\hat{e}_v(n)$ of the virtual error signals computed as defined in Eq. (3.82). The physical error signals $e_p(n)$ in Eq. (3.82) are again given by

$$
e_p(n) = \begin{bmatrix} G_{pu} & G_{ps} \end{bmatrix} \begin{bmatrix} u(n) \\ s(n) \end{bmatrix}.
$$

(3.86)

Using Eqs (3.82), (3.85) and (3.86), the virtual output error in Eq. (3.84) can simply be expressed as

$$
\epsilon_v(n) = (G_{vs} - H G_{ps}) s(n).
$$

(3.87)

The aim of the remote microphone technique is to minimise the virtual output error such that an accurate estimate of the virtual error signals is computed. This aim can be translated into minimising a cost function given by

$$
J_\epsilon = \text{tr} \left( \mathbb{E} \left[ \epsilon_v(n) \epsilon_v(n)^T \right] \right).
$$

(3.88)
Using Parseval’s Theorem [51], and the fact that the signals $s(n)$ have been defined in Eq. (3.2) as white and stationary random processes with unit covariance, an equivalent frequency domain expression of the time-domain cost function in Eq. (3.88) is given by

$$J_{\epsilon} = \|G_{vs} - HG_{ps}\|^2. \quad (3.89)$$

Given the primary physical and virtual transfer path matrices $G_{ps}$ and $G_{vs}$, respectively, the virtual output error can thus be minimised by minimising the cost function in Eq. (3.89) subject to $H \in \mathcal{RH}_{\infty}^{M_v \times M_p}$. A block diagram of this optimisation problem is depicted in Fig. 3.7.

![Block diagram for determining a causal Wiener solution for the filter $H$](image)

Figure 3.7: Block diagram for determining a causal Wiener solution for the filter $H$, with $S$ disturbance sources, $M_p$ physical sensors, and $M_v$ spatially fixed virtual sensors.

An optimal solution for the filter $H$ in Fig. 3.6 can now be computed by solving the optimal filter problem illustrated in Fig. 3.7, which can be formulated as follows.

**Problem 3.1** (Optimal filter problem for computing filter $H$).

*Given the primary transfer path matrices $G_{ps} \in \mathcal{RH}_{\infty}^{M_p \times S}$ and $G_{vs} \in \mathcal{RH}_{\infty}^{M_v \times S}$, determine an optimal filter $H_o \in \mathcal{RH}_{\infty}^{M_v \times M_p}$ such that*

$$H_o = \arg \min_{H \in \mathcal{RH}_{\infty}^{M_v \times M_p}} J_{\epsilon}(H), \quad (3.90)$$

*with the cost function $J_{\epsilon}(H)$ defined by*

$$J_{\epsilon}(H) = \|G_{vs} - HG_{ps}\|^2. \quad (3.91)$$

A solution to Problem 3.1 can be found using the factorisation approach adopted in Chapter 2 to derive the causal Wiener filter defined in Theorem 2.1. Thus, using the outer-inner factorisation defined in Lemma 2.2, the adjoint operator introduced in Definition 2.2, and the causality operator introduced in Definition 2.5, a solution to Problem 3.1 is given by the following theorem.
3.4 Remote microphone technique

**Theorem 3.2** (Causal Wiener solution for $H$).

Given the primary transfer function matrices $G_{ps} \in \mathcal{RH}_\infty^{M_p \times S}$ and $G_{vs} \in \mathcal{RH}_\infty^{M_v \times S}$, and assuming that $G_{ps}$ does not have any zeros on the unit circle, the following outer-inner factorisation can be defined

$$G_{ps} = G_{ps,co} G_{ps,ci},$$

where $G_{ps,co}$ has a stable left-inverse $G_{ps,co}^\dagger$. Furthermore, let $G_{ps,ci}^\perp$ be such that $[G_{ps,ci}^* G_{ps,ci}^\perp]^*$ is unitary. Then

$$H_0 = [G_{vs} G_{ps,ci}^*] + G_{ps,co}$$

minimises

$$J = \|G_{vs} - H G_{ps}\|_2^2, \text{ subject to } H \in \mathcal{RH}_\infty^{M_v \times M_p},$$

and its minimum value is given by

$$J_{\text{min}} = \|G_{vs} - H_0 G_{ps}\|_2^2 = \|G_{vs} G_{ps,ci}^*\|_2^2 + \|G_{vs} G_{ps,ci}^\perp\|_2^2.$$

**Proof.** A proof can be found in Vidyasagar [126], and an alternative proof based on the proof of Theorem 2.1 presented by Fraanje [34] has been included in Appendix A.

In Appendix B, a state-space model of the causal Wiener filter in Eq. (3.93) is derived given state-space models of the physical and virtual primary transfer paths $G_{ps}$ and $G_{vs}$, respectively. The resulting state-space description of the optimal filter $H_0$ is subsequently used in Eq. (3.82) to determine a state-space model of the transfer function matrix $G_{RMT}$. The order of the resulting state-space model of the remote microphone technique is equal to the order $N$ of the system defined in Eq. (3.1).

**Interpretation of minimum value of cost function**

The minimum value of the cost function defined in Eq. (3.95) indicates that the estimation performance of the remote microphone technique is in theory solely determined by the properties of the physical and virtual primary transfer path matrices $G_{ps}$ and $G_{vs}$, respectively. If there are less physical sensors than disturbance source signals, such that $M_p < S$, it may happen that $G_{ps,ci}^\perp$ is non-zero, such that the first term in Eq. (3.95) contributes to the minimum value of the cost function, provided that $G_{ps} G_{ps,ci}^\perp$ is also non-zero. In this situation, there are disturbances that contribute to the virtual primary disturbances $d_v(n)$, but these disturbances are not measured, or observed, at the physical sensors, and are thus not contained in the physical primary disturbances $d_p(n)$. This is related to the concept of unobservable modes of the active noise control system [130]. In this instance, the remote microphone technique is thus not able to provide estimates of these parts of the virtual primary disturbances that are not observed at the physical sensors. The first term in Eq. (3.95) is therefore related to the physical and virtual...
sensor configuration that is used in the active noise control system. The locations of the physical sensors should thus be chosen such that all the modes that contribute to the virtual primary disturbances are observable at the physical sensors.

The second term in Eq. (3.95) is related to the restriction that the filter matrix $H$ should be causal. Therefore, the second term is determined by delays and non-minimum-phase zeros in the physical primary transfer path $G_{ps}$, which result in non-causal terms in $G_{ps,ci}^*$. These non-causal terms contribute to the second term of the minimum value of the cost function defined in Eq. (3.95). To minimise the contribution of this second term, the physical and virtual sensor configuration should be chosen such that the physical primary disturbances contain time-advanced information about the virtual primary disturbances. This ensures that the virtual primary disturbances can be causally estimated from the physical primary disturbances.

**Pre-whitening of the physical primary disturbances**

Theorem 3.2 shows that an optimal estimate $\hat{d}_v(n)$ of the virtual primary disturbances can be calculated in Fig. 3.6 using the optimal filter defined in Eq. (3.93) as

$$\hat{d}_v(n) = \left[ G_{vs} G_{ps,ci}^* \right] + G_{ps,co}^* d_p(n) = \left[ G_{vs} G_{ps,ci}^* \right] + s'(n),$$

with $s'(n)$ the pre-whitened physical primary disturbances computed as

$$s'(n) = G_{ps,co}^* d_p(n) = G_{ps,co}^* G_{ps} s(n) = G_{ps,ci} G_{ps} s(n).$$  \hspace{1cm} (3.96)

The signal $s'(n)$ in Eq. (3.96) is a white noise process because its power spectral density matrix is given by

$$G_{ps,ci} s(n) s(n)^T G_{ps,ci}^* = G_{ps,ci} G_{ps,ci}^* = I,$$ \hspace{1cm} (3.97)

since the disturbance source signals $s(n)$ are defined as white and stationary processes with unit covariance in Eq. (3.2). The left-inverse $G_{ps,co}^*$ of the minimum-phase co-outer factor of the physical primary transfer path matrix is thus a pre-whitening filter for the physical primary disturbances. Note that these physical primary disturbances act as feedforward reference signals to the filter $H$ in Fig. 3.7, and that the pre-whitening of feedforward reference signals generally appears in optimal filter solutions [25, 60].

**Estimation for tonal disturbances**

It has now been shown that the broadband estimation performance of the remote microphone technique depends in theory on the properties of the physical and virtual primary transfer paths. The broadband estimation performance is limited due to the causality constraint placed on the design of the filter $H$. However, for tonal disturbances, causality is not an issue and a perfect estimate of the virtual error signals can in theory
be computed using the remote microphone technique. Writing the relevant variables in complex notation for a normalised frequency $\omega T_s$, a perfect estimate of the virtual error signal can thus be computed using Eq. (3.82) as

$$
e_v(e^{j\omega T_s}) = \left[ Z_{vu} - H(e^{j\omega T_s})Z_{pu} H(e^{j\omega T_s}) \right] \left[ u(e^{j\omega T_s}) e^{j\omega T_s} \right],$$

with $H(e^{j\omega T_s}) \in \mathbb{C}^{M_v \times M_p}$ the complex transfer impedance matrix between the complex physical and virtual primary disturbances, and $Z_{pu} \in \mathbb{C}^{M_p \times L}$ and $Z_{vu} \in \mathbb{C}^{M_v \times L}$ the complex physical and virtual secondary transfer impedance matrices, respectively, defined as

$$Z_{pu} \triangleq G_{pu}(e^{j\omega T_s}), \quad Z_{vu} \triangleq G_{vu}(e^{j\omega T_s}).$$

Perfect estimates of the virtual error signals can thus in theory be obtained for tonal disturbances when using a complex transfer impedance matrix $H(e^{j\omega T_s}) \in \mathbb{C}^{M_v \times M_p}$ instead of a broadband transfer function matrix $H \in \mathcal{RH}_{\infty}^{M_v \times M_p}$ in the remote microphone technique.

### 3.4.3 Optimal estimation with measurement noise

An optimal solution for the filter $H$ in Fig. 3.6 is now derived assuming that there is measurement noise on the physical sensors, including the ones located at the virtual locations in a preliminary identification stage. Fig. 3.8 shows a block diagram of the remote microphone technique including measurement noise $v(n) \in \mathbb{R}^V$.

The disturbance source signals $s(n)$ and the measurement noise signals $v(n)$ are assumed to be uncorrelated white and stationary random processes with zero-mean and unit covariance, such that

$$E \left[ \begin{bmatrix} s(n) \\ v(n) \end{bmatrix} \begin{bmatrix} s(k)^T \\ v(k)^T \end{bmatrix} \right] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \delta_{nk}.$$ 

In Fig. 3.8, the transfer function matrices $N_{pv}(z^{-1}) \in \mathcal{RH}_\infty^{M_p \times V}$ and $N_{vv}(z^{-1}) \in \mathcal{RH}_\infty^{M_v \times V}$ are noise shaping filters. It is assumed in this section that the physical measurement noise signals $v_p(n)$ and the virtual measurement noise signals $v_v(n)$ included in the state-space model defined in Eq. (3.1) result from filtering the measurement noise signals $v(n)$ with these noise shaping filters, such that

$$\begin{bmatrix} v_p(n) \\ v_v(n) \end{bmatrix} = \begin{bmatrix} N_{pv} \\ N_{vv} \end{bmatrix} v(n).$$

These noise shaping functions can thus be chosen to model various measurement noise situations [34]. As an example, to model the physical and virtual measurement noise
signals as zero-mean white noise sequences with covariances as defined in Eq. (3.2), the shaping functions can be chosen as direct feedthrough terms given by

$$
\begin{bmatrix}
N_{pv} \\
N_{vv}
\end{bmatrix} = 
\begin{bmatrix}
R_p^{1/2} & 0 \\
\tilde{R}_p^{1/2} & \tilde{R}_v^{1/2}
\end{bmatrix},
$$

(3.102)

such that \( V = M_p + M_v \) in Fig. 3.8, and where the matrix on the right-hand side is calculated from a lower-triangular Cholesky factorisation of the measurement noise covariance matrix in Eq. (3.2), such that

$$
\begin{bmatrix}
N_{pv} \\
N_{vv}
\end{bmatrix}^T 
\begin{bmatrix}
N_{pv} \\
N_{vv}
\end{bmatrix} = 
\begin{bmatrix}
R_p^{1/2} & \tilde{R}_p^{1/2} & R_v^{1/2} & \tilde{R}_v^{1/2} \\
R_p^{1/2} & R_v^{1/2} & \tilde{R}_p^{1/2} & R_v^{1/2} \\
\end{bmatrix} 
\begin{bmatrix}
R_p & R_{pv} \\
\tilde{R}_p & R_v
\end{bmatrix}. 
$$

(3.103)

The noise shaping functions also allow the modelling of more complicated measurement noise characteristics, such as coloured measurement noise [34].

It can now be shown that the virtual output error \( \varepsilon_v(n) \) defined in Eq. (3.84) can be written, with the estimate of the virtual error signals computed as shown in Fig. 3.8, as

$$
\varepsilon_v(n) = \begin{bmatrix}
G_{ps} - HG_{ps} & N_{vv} - HN_{vp}
\end{bmatrix} \begin{bmatrix}
s(n) \\
v(n)
\end{bmatrix}. 
$$

(3.104)
Eq. (3.104) indicates that the measurement noise signals \( \nu(n) \) enter the plant in exactly the same manner as the disturbance source signals \( \delta(n) \). The cost function in Eq. (3.88) can therefore be written in the frequency domain as [34]

\[
J_e = \| G_{\text{aug}}^{\text{ps}} - HG_{\text{aug}}^{\text{ps}} \|^2
\]

(3.105)

where the augmented physical and virtual primary transfer paths are defined as

\[
G_{\text{aug}}^{\text{ps}} = \begin{bmatrix} G_{\text{ps}} & N_{\text{pv}} \end{bmatrix}, \quad G_{\text{aug}}^{\text{vs}} = \begin{bmatrix} G_{\text{vs}} & N_{\text{vv}} \end{bmatrix}.
\]

(3.106)

Using Theorem 3.2, the optimal solution for the filter \( H \) for the case with measurement noise is now given by

\[
H_0 = \begin{bmatrix} G_{\text{aug}}^{\text{ps}} & G_{\text{aug}}^{\text{ps,ci}} \end{bmatrix} + G_{\text{ps,co}}^+, \quad (3.107)
\]

where \( G_{\text{aug}}^{\text{ps,co}} \) and \( G_{\text{aug}}^{\text{ps,ci}} \) are computed from an outer-inner factorisation of the augmented physical primary transfer path matrix \( G_{\text{aug}}^{\text{ps}} \).

Uncorrelated physical and virtual measurement noise

Eq. (3.107) can be simplified when the physical and virtual measurement noise signals are uncorrelated [34]. For this case, the noise shaping functions can be chosen as

\[
\begin{bmatrix} N_{\text{pv}} \\ N_{\text{vv}} \end{bmatrix} = \begin{bmatrix} N_{\text{pv1}} & 0 \\ 0 & N_{\text{vv2}} \end{bmatrix},
\]

(3.108)

where \( N_{\text{pv1}} \in \mathcal{RH}_{\infty}^{M_p \times V_1} \) and \( N_{\text{vv2}} \in \mathcal{RH}_{\infty}^{M_v \times V_2} \), with \( V_1 + V_2 = V \). The outer-inner factorisation of the augmented physical primary transfer path matrix \( G_{\text{aug}}^{\text{ps}} \) defined in Eq. (3.106) is then given by [34]

\[
G_{\text{ps}} N_{\text{pv1}} 0 = G_{\text{ps,co}}^{\text{aug}} \begin{bmatrix} G_{\text{ps,ci}}^{1\text{aug}} & G_{\text{ps,ci}}^{2\text{aug}} & 0 \end{bmatrix}.
\]

(3.109)

Using Eqs (3.106), (3.108) and (3.109), the term \( G_{\text{ps}}^{\text{aug}} G_{\text{ps,ci}}^{\text{aug}*} \) in Eq. (3.107) then simplifies to

\[
G_{\text{ps}}^{\text{aug}} G_{\text{ps,ci}}^{\text{aug}*} = \begin{bmatrix} G_{\text{ps}} & 0 & N_{\text{vv2}} \end{bmatrix} \begin{bmatrix} G_{\text{ps,ci}}^{1\text{aug}*} & G_{\text{ps,ci}}^{2\text{aug}*} & 0 \end{bmatrix} = G_{\text{ps}} G_{\text{ps,ci}}^{1\text{aug}*},
\]

(3.110)

such that the optimal filter solution defined in Eq. (3.107) reduces to

\[
H_0 = \begin{bmatrix} G_{\text{ps}} G_{\text{ps,ci}}^{1\text{aug}*} \end{bmatrix} + G_{\text{ps,co}}^+.
\]

(3.111)

For this case, the optimal solution is thus independent of \( N_{\text{vv2}} \), indicating that virtual measurement noise that is uncorrelated to the physical error signals cannot be cancelled, and thus remains in the virtual output errors \( \varepsilon_v(n) \).
Chapter 3  Spatially fixed virtual sensing algorithms

The structure of the solution given in Eq. (3.111) is similar to the structure of the measurement noise free solution defined in Eq. (3.93). This can be seen by replacing $G_{ps,ci}$ and $G_{ps,co}$ in Eq. (3.93) with $G_{ps,ci}^{aug}$ and $G_{ps,co}^{aug}$, respectively. Eq. (3.109) indicates that the physical primary transfer path matrix $G_{ps}$ is now factorised as

$$G_{ps} = G_{ps,co}^{aug} G_{ps,ci}^{aug}.$$  (3.112)

The factorisation in Eq. (3.112) is not determined from an outer-inner factorisation of $G_{ps}$, which is the case in the measurement noise free case, but rather from an outer-inner factorisation of the augmented physical primary transfer path matrix $G_{ps}^{aug}$ defined in Eq. (3.106). From the outer-inner factorisation Lemma 2.2 on page 55, the outer-inner factorisation of $G_{ps}^{aug}$ results in the following relationship [34]

$$G_{ps,co}^{aug} G_{ps,co}^{aug*} = G_{ps} G_{ps}^{*} + N_{p01} N_{p01}^{*}.$$  (3.113)

The above equation shows that the magnitude of $G_{ps,co}^{aug}$ is increased in comparison to the magnitude of $G_{ps}$ at frequencies where the magnitude of $N_{p01}$ is significantly large. Examining Eq. (3.111), the magnitude of the optimal filter $H_o$ is thus reduced at those frequencies where the measurement noise on the physical error signals $e_p(n)$ is significantly large in comparison to the measurement noise free physical error signals. This is to be expected because reducing the magnitude of the filter $H_o$ at these frequencies reduces the contribution of the physical measurement noise to the virtual output errors $\varepsilon_v(n)$, and therefore to the cost function in Eq. (3.105).

3.5  Virtual microphone arrangement

This section analyses the virtual microphone arrangement suggested by Elliott and David [27]. The concept of virtual sensing for active noise control was first suggested in their paper.

3.5.1  Algorithm structure

A block diagram of the virtual microphone arrangement is depicted in Fig. 3.9. Comparing this block diagram to the remote microphone technique [104, 112] illustrated in Fig. 3.6 indicates that the virtual microphone arrangement and the remote microphone technique are very similar algorithms. In fact, the virtual microphone arrangement can be seen as a simplified version of the remote microphone technique.

The virtual microphone arrangement has generally been implemented using the same number of physical and virtual sensors, such that $M_p = M_v$ in Fig. 3.9. The locations of these $2M_p$ error sensors are typically such that there are $M_p$ error sensor pairs, with each pair consisting of a physical and a virtual sensor. It is then assumed for
3.5 Virtual microphone arrangement

Figure 3.9: Block diagram of the virtual microphone arrangement without measurement noise, with \( L \) control sources, \( M_p \) physical sensors, and \( M_v \) spatially fixed virtual sensors.

each pair that the primary disturbances at the physical and virtual sensors are equal. This means that the estimate \( \hat{d}_v(n) \) of the virtual primary disturbances is assumed to be equal to the physical primary disturbances \( d_p(n) \) in Fig. 3.9. This assumption can be made provided the primary sound field changes relatively little between the physical and virtual sensors of each pair. The filter \( H \) in the block diagram of the remote microphone technique shown in Fig. 3.6 is thus assumed to be equal to the identity matrix \( I \) in the virtual microphone arrangement, as illustrated in Fig. 3.9. The virtual microphone arrangement therefore is a simplified version of the remote microphone technique. A transfer function matrix \( G_{VMA} \in \mathcal{RH}_{\infty}^{M_p \times (L+M_p)} \) that describes the input-output behaviour of the virtual microphone arrangement is now defined from Fig. 3.9 as

\[
\hat{e}_v(n) = G_{VMA} \begin{bmatrix} e_p(n) \\ u(n) \end{bmatrix} = \begin{bmatrix} I & G_{vu} - G_{pu} \end{bmatrix} \begin{bmatrix} e_p(n) \\ u(n) \end{bmatrix},
\]

which can also simply be found by substituting \( H = I \) into the input-output equation of the remote microphone technique defined in Eq. (3.82).

3.5.2 Estimation performance

The factors that limit the estimation performance of the virtual microphone arrangement are now derived. The virtual output error defined in Eq. (3.84) can be calculated for the virtual microphone arrangement by substituting \( H = I \) into Eq. (3.87), which results in

\[
e_v(n) = (G_{vs} - G_{ps}) s(n).
\]

The minimum value of the cost function defined in Eq. (3.88) is now found by substituting \( H = I \) into the cost function defined in Eq. (3.89), which results in

\[
J_{e,\text{min}} = \|G_{vs} - G_{ps}\|_2^2.
\]
The virtual output errors in Eq. (3.115) and the minimum value of the cost function $J_\varepsilon$ defined in Eq. (3.116) indicate that the estimation performance of the virtual microphone arrangement is in theory solely determined by the difference between the physical and virtual primary transfer path matrices $G_{ps}$ and $G_{vs}$, respectively. This is as expected because the estimation performance of the virtual microphone arrangement depends on the validity of the assumption that the primary disturbances at the physical and virtual sensors are equal.

3.6 Hybrid adaptive feedforward observer

In this section, the virtual sensing method proposed by Tran and Southward [119] is analysed, which is called the hybrid adaptive feedforward observer method. It is shown that if the number of physical sensors is less than the order of the state-space model used by the observer, which is usually the case in practice, the proposed method is only suitable for rejecting non-stationary primary disturbances at the physical sensors, and not for virtual sensing purposes. The discussion presented in this section has been accepted for publication in Petersen et al. [102].

3.6.1 Algorithm structure

A brief discussion of the hybrid adaptive feedforward observer method is presented in this section. For a detailed description of the proposed method, one is referred to the original paper [119], and subsequent papers [120, 121]. A block diagram of the hybrid adaptive feedforward observer is shown in Fig. 3.10 [119]. A state-space model that describes the plant in this figure is given in Ref. [119] by

$$
\begin{align*}
\mathbf{z}(n+1) &= A\mathbf{z}(n) + B_s\mathbf{u}(n) + \mathbf{w}(n) \\
\mathbf{e}_p(n) &= C_p\mathbf{z}(n) + D_{pu}\mathbf{u}(n) \\
\mathbf{e}_v(n) &= C_v\mathbf{z}(n) + D_{vu}\mathbf{u}(n),
\end{align*}
$$

(3.117)

with $\mathbf{z}(n) \in \mathbb{R}^N$ the states of the system, where $N$ is the system order, $\mathbf{e}_p(n) \in \mathbb{R}^{M_p}$ the physical error signals, $\mathbf{e}_v(n) \in \mathbb{R}^{M_v}$ the virtual error signals, $\mathbf{u}(n) \in \mathbb{R}^L$ the control signals, and $\mathbf{w}(n) \in \mathbb{R}^N$ a vector that contains the unknown external disturbances. Note that this state-space model is equivalent to the standard state-space model introduced in Eq. (3.1) by defining the external disturbances $\mathbf{w}(n) \triangleq \mathbf{B}_s\mathbf{s}(n)$, and by setting the measurement noise on the physical and virtual sensors and the direct feedthrough matrices $D_{ps}$ and $D_{vs}$ equal to zero. It is assumed that no a priori knowledge is available of how the external disturbances $\mathbf{w}(n)$ affect the states of the plant [119]. However, as indicated by the dashed line in Fig. 3.10, a feedforward reference signal that is correlated to the external disturbances $\mathbf{w}(n)$ is assumed to be available.
First, by initially assuming that there are no external disturbances \[ w(n) = 0 \] in Eq. (3.117), a conventional observer is designed \[ [119] \]. Note that to design such a conventional observer, the covariance of the external disturbances \[ w(n) \] generally needs to be defined. Because these external disturbances are assumed to be equal to zero in the first step, this will generally result in a very small Kalman gain matrix \[ [35] \]. Next, the conventional observer is augmented with an adaptive feedforward component that aims to provide estimates \[ \hat{w}(n) \] of the external disturbances in Eq. (3.117). A state-space description of the complete hybrid adaptive feedforward observer shown in Fig. 3.10 is given by \[ [119] \]

\[
\begin{align*}
\dot{z}(n+1) &= A\hat{z}(n) + Bu(n) + Ke_p(n) + \hat{w}(n) \\
\dot{e}_p(n) &= C_p\hat{z}(n) + D_pu(n) \\
\dot{e}_v(n) &= C_v\hat{z}(n) + D_vu(n),
\end{align*}
\] (3.118)

where \[ e_p(n) = e_p(n) - \hat{e}_p(n) \] is a vector of length \( M_p \) that contains the physical output errors, and \( K \) is the Kalman gain matrix. The aim of the observer defined in Eq. (3.118) is to obtain accurate estimates \( \hat{z}(n) \) of the true plant states. These estimates plus the deterministic control inputs \( u(n) \) are then used to obtain an estimate \( \hat{e}_v(n) \) of the virtual error signals as defined in Eq. (3.118). In order to obtain accurate estimates of the virtual error signals \( e_v(n) \), such that the virtual output errors \( e_v(n) = e_v(n) - \hat{e}_v(n) \) are small, the observer needs to provide accurate estimates of the true states \( z(n) \) of the plant. If the state estimation error is defined as \( \rho(n) = z(n) - \hat{z}(n) \), the estimator error equation can be written as \[ [119] \]

\[
\begin{align*}
\rho(n+1) &= (A - KC_p)\rho(n) + w(n) - \hat{w}(n) \\
e_p(n) &= C_p\rho(n).
\end{align*}
\] (3.119)
Eq. (3.119) shows that the state estimation error $\rho(n)$ can be reduced to zero if $\hat{w}(n)$ approaches $w(n)$ [119], provided that the poles of the matrix $A - KC_p$ are inside the unit circle. It is then suggested by Tran and Southward [119] that this objective can be achieved by minimising the physical output errors $\varepsilon_p(n)$ using an LMS-based adaptive algorithm that minimises the cost function

$$J_p = E[\varepsilon_p(n)^T \varepsilon_p(n)].$$  \hspace{1cm} (3.120)

Although the suggested hybrid observer indeed minimises this cost function to zero such that $\varepsilon_p(n) = 0$, it is proven in the next section that the hybrid observer will not always reduce the state estimation error $\rho(n)$ to zero as suggested by Tran and Southward. This is already illustrated by the numerical results presented by Tran and Southward in Figs. 6 and 7 of their paper [119]. The results depicted in Fig. 6 of their paper indicate that the physical output errors have indeed been minimised, such that $\varepsilon_p(n) = 0$ after convergence of the adaptive algorithm. If the observer works as suggested, the virtual output errors would be minimised to $\varepsilon_v(n) = 0$ as well, since the state estimation errors $\rho(n)$ are then equal to zero (according to Tran and Southward).

Comparing Figs. 6 and 7 in their paper, it can be seen that this is clearly not the case because the virtual output errors $\varepsilon_v(n)$ have not converged to zero in Fig. 7. This indicates that, although the physical output errors $\varepsilon_p(n)$ are indeed minimised to zero, accurate estimates $\hat{z}(n)$ of the plant states are not obtained, and the state estimation errors $\rho(n)$ are not equal to zero after convergence of the adaptive algorithm.

In the next section, it is shown that the adaptive algorithm can converge to infinitely many solutions $\hat{w}(n)$ that drive the cost function in Eq. (3.120) to zero, but that do not necessarily set the state estimation error equal to the desired solution $\rho(n) = 0$.

### 3.6.2 Analysis of proposed observer method

By substituting $\varepsilon_p(n) = C_p \rho(n)$ into the cost function $J_p$ in Eq. (3.120), this cost function can also be written as

$$J_p = E\left[\rho(n)^T C_p^T C_p \rho(n)\right].$$  \hspace{1cm} (3.121)

The LMS-based algorithm adjusts the estimate $\hat{w}(n)$ such that this cost function is minimised. In the case that the number of physical sensors $M_p$ is less than the number of states $N$, which is almost always the case, the matrix $C_p \in \mathbb{R}^{M_p \times N}$ in Eq. (3.118) will have rank $\leq M_p$. Due to the rank-nullity theorem [4], the matrix $C_p$ will therefore have a nullspace $\mathcal{N}(C_p)$ of dimension $\geq N - M_p$. Thus, the non-negative cost function in Eq. (3.121) is minimised for any $\rho(n) \in \mathcal{N}(C_p)$. An equivalent observation is that the matrix $C_p^T C_p$ in Eq. (3.121) will be positive semi-definite if $M_p < N$, but not positive definite as required to ensure that $\rho(n)$ converges to zero. It follows that the LMS-based
algorithm will only ensure $\rho(n)$ converges to the subspace $N(C_p)$, and not to the desired value $\rho(n) = 0$ as suggested by Tran and Southward [119].

It is now shown that once $\rho(n)$ has converged to $N(C_p)$, the structure of the hybrid observer will allow it to remain there. Suppose that a basis $S$ for $N(C_p)$ is given by [4]

$$ S = \{v_1, v_2, \ldots, v_k\}, $$

(3.122)

with $k = \text{nullity}(C_p)$. The values of $\rho(n) \in N(C_p)$ that minimise the cost function in Eq. (3.121) can now be written as a linear combination of the basis vectors $v_i \in \mathbb{R}^N$ as

$$ \rho(n) = a_1(n)v_1 + a_2(n)v_2 + \ldots + a_k(n)v_k $$

$$ = \mathbf{V}a(n), $$

(3.123)

where the columns of the matrix $\mathbf{V} \in \mathbb{R}^{N \times k}$ consist of the basis vectors $v_i$ defined in Eq. (3.122), and where $a_i(n) \in \mathbb{R}^k$ is a column vector with arbitrary real-valued elements $a_i(n)$. Substituting Eq. (3.123) into the estimator error equation defined in Eq. (3.119) gives

$$ \rho(n + 1) = A\mathbf{V}a(n) + \mathbf{w}(n) - \hat{\mathbf{w}}(n), $$

$$ \epsilon_p(n) = 0, $$

(3.124)

since $C_p\mathbf{V}a(n) = 0$ as $\mathbf{V}a(n)$ defines any arbitrary vector in $N(C_p)$. For $\rho(n)$ to remain in $N(C_p)$, such that the cost function in Eq. (3.120) has been minimised with $\epsilon_p(n) = 0$, the state estimation error $\rho(n + 1)$ at the next time step must be in $N(C_p)$ as well. This means that $\rho(n + 1)$ can then be expressed as a linear combination of the basis vectors $v_i$ in Eq. (3.122) as

$$ \rho(n + 1) = \beta_1(n)v_1 + \beta_2(n)v_2 + \ldots + \beta_k(n)v_k $$

$$ = \mathbf{V}\beta(n), $$

(3.125)

with $\beta(n) \in \mathbb{R}^k$ a column vector with arbitrary real-valued elements $\beta_i(n)$. Substituting Eq. (3.125) into Eq. (3.124) gives

$$ \mathbf{V}\beta(n) = A\mathbf{V}a(n) + \mathbf{w}(n) - \hat{\mathbf{w}}(n), $$

$$ \epsilon_p(n) = 0, $$

(3.126)

Eq. (3.126) shows that once the adaptive algorithm reaches a solution $\hat{\mathbf{w}}(n)$ given by

$$ \hat{\mathbf{w}}(n) = \mathbf{w}(n) + A\mathbf{V}a(n) - \mathbf{V}\beta(n), $$

(3.127)

the cost function in Eq. (3.121) has been minimised as both $\rho(n) \in N(C_p)$ and $\rho(n + 1) \in N(C_p)$, which means that the physical output error stays equal to $\epsilon_p(n) = 0$.

As the elements of $a(n)$ and $\beta(n)$ can be set to any arbitrary real value in Eq. (3.127), the adaptive algorithm can converge to infinitely many solutions $\hat{\mathbf{w}}(n)$ that are not
equal to $w(n)$. Once the adaptive algorithm has converged to one of these solutions, the state estimation error $\rho(n)$ enters and remains in $N(C_p)$. In other words, the adaptive algorithm can converge to infinitely many solutions $\hat{w}(n)$ that minimise the cost function in Eq. (3.120), but that do not necessarily set the state estimation error equal to the desired solution $\rho(n) = 0$.

In conclusion, if the number of physical sensors $M_p$ is less than the system order $N$, which is usually the case in practice, the method suggested by Tran and Southward indeed minimises the physical output errors such that $\varepsilon_p(n) = 0$ after convergence, but it does not guarantee that the state estimation errors converge to the desired values $\rho(n) = 0$. Since accurate state estimates are required to obtain accurate estimates of the virtual error signals $e_v(n)$, the suggested hybrid adaptive feedforward observer is only suitable for rejecting non-stationary disturbances at the physical sensor outputs, and not for spatially fixed virtual sensing purposes.

### 3.7 Kalman filter based spatially fixed virtual sensing algorithm

The common aim of the spatially fixed virtual sensing algorithms that have been analysed in Sections 3.3–3.6 is to compute an accurate estimate of the virtual error signals given the directly measured physical error signals. The difference between these algorithms is the way in which the estimate is computed, i.e. the assumed structure is what makes the virtual sensing algorithms proposed so far different from one another. Once the structure has been chosen, the optimal solutions for the unknown parameters of the algorithm can be computed by optimising the estimation performance as described in Sections 3.3–3.6. The question now arises as to whether there is an optimal structure that can be used to solve the spatially fixed virtual sensing for active noise control problem. It is well-known that the Kalman filter provides an optimal structure for solving linear estimation problems [60]. Since virtual sensing for active noise control is a linear estimation problem, an optimal solution to this problem is therefore derived in this section using Kalman filter theory [36, 60].

Kalman filters are usually computed given a state-space model of the dynamic system under consideration plus the covariance properties of its stochastic input signals [60]. Here, a state-space model of the active noise control system under consideration has been defined in Eq. (3.1). It is important to note that the control signals $u(n)$ into this state-space model are deterministic input signals, while the disturbance source signals $s(n)$, the physical measurement noise signals $v_p(n)$, and the virtual measurement noise signals $v_v(n)$ are stochastic input signals. In Section 3.7.1, the covariance properties of these stochastic input signals are introduced.
It is also important to note that the virtual error signals \( e_v(n) \) of the state-space model in Eq. (3.1) are not directly measured during real-time control. A Kalman filtering solution to the virtual sensing problem thus needs to be computed assuming that only the physical error signals are directly available as observations to the Kalman filter. In Section 3.7.2, the prediction and time/measurement update form of the Kalman filter [60] are therefore introduced assuming that only the physical error signals are directly measured during real-time control.

The two forms of the Kalman filter presented in Section 3.7.2 can be used to compute the innovations process [60] that is associated with the physical error signals, i.e. the observations. The innovations process is an important process that often occurs in Kalman filtering theory, and a brief discussion of this process is presented in Section 3.7.3. This discussion, the covariance properties introduced in Section 3.7.1, and the Kalman filter forms introduced in Section 3.7.2 are then used in Section 3.7.4 to solve the spatially fixed virtual sensing for active noise control problem using a Kalman filtering approach. A slightly different derivation of the Kalman filter based spatially fixed virtual sensing algorithm presented here has been accepted for publication in Petersen et al. [101].

In a practical situation, a standard state-space model [60] of the active noise control system as defined in Eq. (3.1) can generally not be estimated because the stochastic input signals are unknown. An innovations model [60] of the active noise control system can, however, be estimated in a practical situation using subspace identification techniques [50, 55, 124]. In Section 3.7.5, the practical implementation of the proposed virtual sensing algorithm given an innovations model of the active noise control system is therefore discussed.

### 3.7.1 Covariance and correlation properties

The active noise control system is again described by the following standard state-space model

\[
\begin{align*}
  z(n+1) &= Az(n) + B_u u(n) + B_s s(n) \\
  e_p(n) &= C_p z(n) + D_{pu} u(n) + D_{ps} s(n) + v_p(n) \quad n \geq 0 \\
  e_v(n) &= C_v z(n) + D_{vu} u(n) + D_{vs} s(n) + v_v(n),
\end{align*}
\]

with \( z(n) \in \mathbb{R}^N \) the states of the system, \( u(n) \in \mathbb{R}^L \) the control signals, \( e_p(n) \in \mathbb{R}^{M_p} \) the physical error signals, \( e_v(n) \in \mathbb{R}^{M_v} \) the unmeasured virtual error signals, \( v_p(n) \in \mathbb{R}^{M_p} \) the physical measurement noise signals, and \( v_v(n) \in \mathbb{R}^{M_v} \) the virtual measurement noise signals. Note that measurement noise on the virtual sensors is included to account for the fact that physical sensors are generally positioned at the spatially fixed virtual locations in a preliminary identification stage. Also note that the state-space matrices \( C_v \), \( D_{vu} \), and \( D_{vs} \) in Eq. (3.128) are not dependent on the time-index \( n \) because it is assumed
in this chapter that the virtual locations are \textit{spatially fixed} within the sound field. In this section, the covariance properties of the stochastic input signals \{s(n), v_p(n), v_v(n)\} of the state-space model in Eq. (3.128) are defined. Using these covariance properties, the cross-correlation properties between the stochastic input signals and the stochastic output signals \{e_p(n), e_v(n)\} of the state-space model in Eq. (3.128) can be derived. A complete derivation of these cross-correlation properties can, for instance, be found in Kailath et al. [60]. Only one example is therefore presented here that illustrates how to derive these cross-correlation properties. The presented covariance and correlation properties will be used in Section 3.7.4 to solve the virtual sensing for active noise control problem using a Kalman filtering approach.

\textbf{Covariance properties of the stochastic input signals}

The disturbance source signal \(s(n) \in \mathbb{R}^S\) in Eq. (3.128) is again assumed to be an unknown white and stationary random process with zero-mean and unit covariance. The physical measurement noise signals \(v_p(n)\) and the virtual measurement noise signals \(v_v(n)\) are also assumed to be zero mean white and stationary random processes, such that the following covariance matrices can be defined

\[
\mathbb{E} \left[ \begin{bmatrix} s(n) \\ v_p(n) \\ v_v(n) \\ z(0) \\ 1 \end{bmatrix} \begin{bmatrix} s(k) \\ v_p(k) \\ v_v(k) \\ z(0) \\ 1 \end{bmatrix}^T \right] = \begin{bmatrix} I & S_{sp}^T & S_{sv}^T & 0 & 0 \\ S_{sp} & R_p & R_{pv} & 0 & 0 \\ S_{sv} & R_{vp} & R_v & 0 & 0 \\ 0 & 0 & 0 & \Pi_0 & 0 \end{bmatrix} \delta_{nk}, \tag{3.129} \]

where \(z(0) \in \mathbb{R}^N\) is the initial state. In the state-space model in Eq. (3.128), the term \(B_s s(n)\) can be interpreted as a process noise signal \(w(n) \triangleq B_s s(n)\). The measurement noise and the direct feedthrough from the disturbance source signals \(s(n)\) to the physical and virtual error signals can be combined into an auxiliary measurement noise signal \(v(n)\), which is defined as

\[
v(n) \triangleq \begin{bmatrix} v_p(n) \\ v_v(n) \end{bmatrix} = \begin{bmatrix} D_{ps} s(n) + v_p(n) \\ D_{sv} s(n) + v_v(n) \end{bmatrix}. \tag{3.130} \]

Using these definitions of the process noise signals \(w(n)\), and the auxiliary measurement noise signals \(v(n)\), the following covariance matrices can be defined

\[
\mathbb{E} \left[ \begin{bmatrix} w(n) \\ v(n) \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}^T \right] = \begin{bmatrix} \bar{Q}_s & \bar{S}_s^T \\ \bar{S}_s & \bar{R} \end{bmatrix} \delta_{nk}, \tag{3.131} \]
Using the covariance properties defined in Eq. (3.129), the covariance matrix $\bar{Q}_s$ of the process noise signals $w(n)$ is therefore given by

$$
\bar{Q}_s = B_s B_s^T. 
$$

(3.132)

The covariance matrix $\bar{R}$ of the auxiliary measurement noise signals $v(n)$ is defined as

$$
\bar{R} = \begin{bmatrix}
\bar{R}_p & \bar{R}_{pv} \\
\bar{R}_{pv}^T & \bar{R}_v
\end{bmatrix},
$$

(3.133)

which can be written in expanded form as

$$
\bar{R} = \begin{bmatrix}
R_p + S_{ps} D_{ps}^T + D_{ps} S_{ps}^T + D_{ps} D_{ps}^T \\
R_{pv} + S_{ps} D_{ps}^T + D_{ps} S_{ps}^T + D_{ps} D_{ps}^T \\
R_{pv} + S_{vs} D_{vs}^T + D_{vs} S_{vs}^T + D_{vs} D_{vs}^T \\
R_v + S_{vs} D_{vs}^T + D_{vs} S_{vs}^T + D_{vs} D_{vs}^T
\end{bmatrix}. 
$$

(3.134)

The covariance matrix $\bar{S}_s$ between the auxiliary measurement noise signals $v(n)$ and the process noise signals $w(n)$ is given by

$$
\bar{S}_s = \begin{bmatrix}
\bar{S}_{ps} \\
\bar{S}_{vs}
\end{bmatrix} = \begin{bmatrix}
D_{ps} B_s^T + S_{ps} B_s^T \\
D_{vs} B_s^T + S_{vs} B_s^T
\end{bmatrix}. 
$$

(3.135)

Cross-correlation properties

To derive the proposed Kalman filter based spatially fixed virtual sensing algorithm, the cross-correlation properties between the stochastic input signals $\{s(n), v_p(n), v_v(n)\}$ and output signals $\{e_p(n), e_v(n)\}$ of the standard state-space model in Eq. (3.128) need to be derived. These cross-correlation properties are therefore presented here such that a more concise derivation of the virtual sensing algorithm based on Kalman filtering can be presented in Section 3.7.4. Because a derivation of these cross-correlation properties can, for instance, be found in Kailath et al. [60], only one example is presented here that illustrates how to derive these cross-correlation properties. This example is to derive that the current disturbance source signal $s(n)$ is uncorrelated to the current and past states $\{z(k), k = 1, \ldots, n\}$, such that

$$
E[s(n)z(k)^T] = 0, \quad k \leq n.
$$

(3.136)

This can be derived from Eq. (3.128) by writing the current state as

$$
z(k) = A^k z(0) + \sum_{m=1}^{k} A^{k-m} B_s s(m-1),
$$

(3.137)

where it has been assumed that the deterministic control signal $u(n) = 0$ for convenience. The state $z(k)$ is thus a linear combination of the initial state $z(0)$ and the
past disturbance source signals \{s(m), m = 1,\ldots,k-1\}. From the covariance properties defined in Eq. (3.129), the disturbance source signal \(s(n)\) is uncorrelated to all of these variables for \(k \leq n\), thereby arriving at the uncorrelatedness property defined in Eq. (3.136). The following cross-correlation properties can be derived in a similar way.

**Lemma 3.1** (Cross-correlation properties [60]).

The current stochastic input signals \(\{s(n), v_p(n), v_v(n)\}\) in Eq. (3.128) are uncorrelated to all past and current states \(\{z(k), k = 1,\ldots,n\}\), such that

\[
E[s(n)z(k)^T] = 0, \quad E[v_p(n)z(k)^T] = 0, \quad E[v_v(n)z(k)^T] = 0, \quad k \leq n.
\]  

(3.138)

This can be derived from Eq. (3.128) by noting that \(z(k)\) depends linearly only upon the random variables \(\{z(0), s(m), m \leq k-1\}\). From Eq. (3.129), the current stochastic input signals \(\{s(n), v_p(n), v_v(n)\}\) are uncorrelated to these variables for \(k \leq n\). Following a similar reasoning, it can be derived that the current stochastic input signals are uncorrelated to past outputs, such that

\[
\begin{align*}
E[s(n)e_p(k)^T] &= 0, \quad E[s(n)e_v(k)^T] = 0, \\
E[v_p(n)e_p(k)^T] &= 0, \quad E[v_p(n)e_v(k)^T] = 0, \quad k \leq n-1 \\
E[v_v(n)e_p(k)^T] &= 0, \quad E[v_v(n)e_v(k)^T] = 0.
\end{align*}
\]  

(3.139)

For the current output, however, it can be derived that

\[
\begin{align*}
E[s(n)e_p(n)^T] &= \mathbf{D}_ps^T + \mathbf{S}_{ps}^T, \quad E[s(n)e_v(n)^T] = \mathbf{D}_vs^T + \mathbf{S}_{vs}^T, \\
E[v_p(n)e_p(n)^T] &= \mathbf{S}_{ps}\mathbf{D}_ps^T + \mathbf{R}_p, \quad E[v_p(n)e_v(n)^T] = \mathbf{S}_{ps}\mathbf{D}_vs^T + \mathbf{R}_{pv}, \\
E[v_v(n)e_p(n)^T] &= \mathbf{S}_{vs}\mathbf{D}_ps^T + \mathbf{R}_{pv}, \quad E[v_v(n)e_v(n)^T] = \mathbf{S}_{vs}\mathbf{D}_vs^T + \mathbf{R}_v.
\end{align*}
\]  

(3.140)

**Proof.** See, for instance, Kailath et al. [60]. \(\square\)

### 3.7.2 Prediction and time/measurement update form

The Kalman filter can be described in at least two forms, which are usually called the prediction form, and the time/measurement update form [60]. These forms are presented here assuming that only the physical error signals \(e_p(n)\) in Eq. (3.128) are directly measured during real-time control. The Kalman filter thus needs to compute an estimate of the state given only the physical error signals. These signals will also be referred to as the observations in the following. The prediction form of the Kalman filter computes an estimate of the state \(z(n+1)\) given the observations \(e_p(i)\) up to time \(i = n\), with the state estimate denoted by \(\hat{z}(n+1|n)\). The observations up to time \(n\) are thus used to compute a prediction of the state one sample ahead at time \(n+1\), hence the name prediction form. The time/measurement update form of the Kalman filter computes an estimate of the state \(z(n)\) given the observations \(e_p(i)\) up to time \(i = n\), with the state estimate denoted by \(\hat{z}(n|n)\). The two forms are now described in more detail.
### Prediction form of Kalman filter

Using the prediction form of the Kalman filter, the predicted state estimate \( \tilde{z}(n+1|n) \) is computed as [60]

\[
\tilde{z}(n+1|n) = A\tilde{z}(n|n-1) + B_pu(n) + K_{ps}\varepsilon_p(n),
\]

with \( K_{ps} \in \mathbb{R}^{N \times M_p} \) the Kalman gain matrix, and with \( \varepsilon_p(n) \in \mathbb{R}^{M_p} \) the innovations given by

\[
\varepsilon_p(n) = e_p(n) - \hat{e}_p(n|n-1),
\]

where \( \hat{e}_p(n|n-1) \in \mathbb{R}^{M_p} \) is the predicted estimate of the physical error signals given the observations \( e_p(i) \) up to \( i = n-1 \), which is defined as [60]

\[
\hat{e}_p(n|n-1) = C_p\tilde{z}(n|n-1) + D_{pu}u(n).
\]

Note that in Eq. (3.141), only the directly measured physical error signals \( e_p(n) \) have been used in the state update equation, since the virtual error signals are not directly measured during real-time control. From Eqs (3.141)–(3.143), a state-space description of the Kalman filter in prediction form is given by

\[
\begin{align*}
\dot{z}(n+1) &= (A - K_{ps}C_p)\tilde{z}(n|n-1) + (B_p - K_{ps}D_{pu})u(n) + K_{ps}\varepsilon_p(n) \\
\dot{e}_p(n|n-1) &= C_p\tilde{z}(n|n-1) + D_{pu}u(n).
\end{align*}
\]

### Time/measurement update form of Kalman filter

An alternative form of the Kalman filter can be used that computes a current estimate \( \tilde{z}(n|n) \) of the state at sample \( n \) given the observations \( e_p(i) \) up to time \( i = n \). This results in a time/measurement update formulation, in which the current state estimate is computed as [60]

\[
\tilde{z}(n|n) = \tilde{z}(n|n-1) + M_{ps}\varepsilon_p(n),
\]

with \( M_{ps} \in \mathbb{R}^{N \times M_p} \) the innovations gain matrix. The current state estimate can now be used to obtain a current estimate \( \hat{e}_p(n|n) \in \mathbb{R}^{M_p} \) of the physical error signals given the observations \( e_p(i) \) up to \( i = n \) as

\[
\hat{e}_p(n|n) = C_p\tilde{z}(n|n) + D_{pu}u(n).
\]

A state-space model of the Kalman filter in time/measurement update form is now given from Eqs (3.141)–(3.146) by

\[
\begin{align*}
\dot{z}(n+1|n) &= (A - K_{ps}C_p)\tilde{z}(n|n-1) + (B_p - K_{ps}D_{pu})u(n) + K_{ps}\varepsilon_p(n) \\
\dot{e}_p(n|n) &= (C_p - C_pM_{ps}C_p)\tilde{z}(n|n-1) + (D_{pu} - C_pM_{ps}D_{pu})u(n) + C_pM_{ps}\varepsilon_p(n).
\end{align*}
\]
Kalman and innovations gain matrices

To implement the time/measurement update form of the Kalman filter in Eq. (3.147), optimal solutions for the Kalman gain matrix $K_{ps}$ and innovations gain matrix $M_{ps}$ need to be defined. The following theorem summarises the above discussions, and defines an optimal solution for the Kalman gain matrix $K_{ps}$ that minimises

$$\text{tr}\left( \mathbb{E}\left[ \rho(n|n-1)\rho(n|n-1)^T \right] \right),$$  
(3.148)

with $\rho(n|n-1) \in \mathbb{R}^N$ the predicted state estimation error defined as

$$\rho(n|n-1) = z(n) - \hat{z}(n|n-1).$$  
(3.149)

The theorem also defines an optimal solution for the innovation gain matrix $M_{ps}$ that minimises

$$\text{tr}\left( \mathbb{E}\left[ \rho(n)\rho(n)^T \right] \right),$$  
(3.150)

with $\rho(n) \in \mathbb{R}^N$ the current state estimation error defined as

$$\rho(n) = z(n) - \hat{z}(n|n).$$  
(3.151)

A detailed proof of the presented theorem can be found in [60].

**Theorem 3.3 (Kalman filter).**

Let a state-space realisation of the plant be given by Eq. (3.128), and let the covariance matrices $\bar{Q}_s$, $\bar{S}_{ps}$, and $\bar{R}_p$ be defined as in Eqs (3.129)–(3.135). Furthermore, let

- the pair $(C_p, A)$ be detectable;
- $\bar{R}_p > 0$, $\bar{Q}_s - \bar{S}_{ps}^T \bar{R}_p^{-1} \bar{S}_{ps} \geq 0$;
- $(A - \bar{S}_{ps}^T \bar{R}_p^{-1} C_p, \bar{Q}_s - \bar{S}_{ps}^T \bar{R}_p^{-1} \bar{S}_{ps})$ has no uncontrollable modes on the unit circle.

Then the prediction form of the Kalman filter, which gives optimal predicted estimates $\hat{e}_p(n|n-1)$ of the physical error signals given observations $e_p(i)$ of the physical error signals up to $i = n - 1$, is defined by the state-space realisation

$$\begin{bmatrix} \hat{z}(n+1|n) \\ \hat{e}_p(n|n-1) \end{bmatrix} = \begin{bmatrix} A - K_{ps} C_p & B_u - K_{ps} D_{pu} \\ C_p & D_{pu} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{z}(n|n-1) \\ u(n) \\ e_p(n) \end{bmatrix}. $$  
(3.152)

with $\hat{z}(n|n-1)$ the predicted estimate of the state $z(n)$ given observations $e_p(i)$ up to $i = n - 1$. The time/measurement update form of the Kalman filter, which gives optimal current estimates
\( \hat{e}_p(n|n) \) of the physical error signals given observations \( e_p(i) \) of the physical error signals up to \( i = n \), is defined by the state-space realisation

\[
\begin{bmatrix}
\dot{z}(n+1|n) \\
\hat{e}_p(n|n)
\end{bmatrix} = \begin{bmatrix}
A - K_{ps}C_p & B_u - K_{ps}D_{pu} & K_{ps} \\
C_p - C_pM_{ps}C_p & D_{pu} - C_pM_{ps}D_{pu} & C_pM_{ps}
\end{bmatrix} \begin{bmatrix}
\dot{z}(n|n-1) \\
u(n) \\
e_p(n)
\end{bmatrix}.
\]

(3.153)

The Kalman gain \( K_{ps} \in \mathbb{R}^{N \times M_p} \) and the innovation gain \( M_{ps} \in \mathbb{R}^{N \times M_p} \) in Eqs (3.152) and (3.153) are given by

\[
K_{ps} = (A P_{ps} C_p^T + \hat{S}_{ps}^T) R_{pe}^{-1},
M_{ps} = P_{pe} C_p^T R_{pe}^{-1},
\]

(3.154)

with \( P_{ps} = P_{ps}^T > 0 \) the unique stabilising solution to the discrete-time algebraic Ricatti equation (DARE) given by

\[
P_{ps} = A P_{ps} A^T - (A P_{ps} C_p^T + \hat{S}_{ps}^T)(C_p P_{ps} C_p^T + R_{p}^{-1})(A P_{ps} C_p^T + \hat{S}_{ps}^T)^T + \hat{Q}_p,
\]

(3.155)

and where \( R_{pe} \in \mathbb{R}^{M_p \times M_p} \) is the covariance matrix of the white innovations \( e_p(n) \) given by

\[
R_{pe} = E[e_p(n)e_p(n)^T] = C_p P_{ps} C_p^T + \hat{R}_p,
\]

(3.156)

with \( e_p(n) \) defined in Eq. (3.142).

**Proof.** See, for instance, Kailath et al. [60].

### 3.7.3 Linear estimation given the innovations process

The innovations process \( e_p(n) \) defined in Eq. (3.142) is an important process in linear least-mean-squares estimation theory [60]. In general, a linear least-mean-squares estimate of the physical error signals \( e_p(n) \) given the observations \( e_p(i) \) up to \( i = n - 1 \) can be computed as [60]

\[
\hat{e}_p(n|n-1) = \text{proj}\{e_p(n)\mid \mathcal{L}\{e_p(0),\ldots,e_p(n-1)\}\},
\]

(3.157)

where the operator \( \text{proj} \) denotes projection [60], and where

\[
\mathcal{L}\{e_p(0),\ldots,e_p(n-1)\}
\]

(3.158)

denotes the linear subspace spanned by the vectors \( \{e_p(0),\ldots,e_p(n-1)\} \). In Eq. (3.157), the predicted estimate \( \hat{e}_p(n|n-1) \) of the physical error signals is thus computed by projecting the current observation \( e_p(n) \) onto the linear subspace spanned by the set of vectors formed by the past observations \( \{e_p(0),\ldots,e_p(n-1)\} \), which generally form
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a set of nonorthogonal vectors. The innovations \( \{ \varepsilon_p(0), \ldots, \varepsilon_p(n-1) \} \) now form an equivalent set of orthogonal vectors that span the same subspace, such that [60]

\[
\mathcal{L}\{ \varepsilon_p(0), \ldots, \varepsilon_p(n-1) \} = \mathcal{L}\{ \varepsilon_p(0), \ldots, \varepsilon_p(n-1) \}.
\] (3.159)

From Eq. (3.157), an equivalent linear least-mean-squares estimate of the physical error signals \( \varepsilon_p(n) \) given the innovations \( \varepsilon_p(i) \) up to \( i = n - 1 \) can thus be computed as [60]

\[
\hat{\varepsilon}_p(n|n-1) = \text{proj}\{ \varepsilon_p(n)|\mathcal{L}\{ \varepsilon_p(0), \ldots, \varepsilon_p(n-1) \} \}.
\] (3.160)

Because the innovations are a set of orthogonal vectors, the projection in Eq. (3.160) is relatively easy to compute by separately projecting onto each of the orthogonal vectors \( \varepsilon_p(m) \), such that [60]

\[
\hat{\varepsilon}_p(n|n-1) = \sum_{m=0}^{n-1} \mathbb{E}\left[ \varepsilon_p(n)\varepsilon_p(m)^T \right] R_{pe}^{-1} \varepsilon_p(m),
\] (3.161)

with \( R_{pe} \) the covariance matrix of the innovations defined in Eq. (3.142). The predicted estimate \( \hat{\varepsilon}_p(n|n-1) \) of the physical error signals is thus that part of the new observation \( \varepsilon_p(n) \) that is determined from the past observations \( \{ \varepsilon_p(0), \ldots, \varepsilon_p(n-1) \} \). The remaining part \( \varepsilon_p(n) \) can be regarded as the new information in the observation \( \varepsilon_p(n) \), and is therefore called the innovations process associated with the physical error signals \( \varepsilon_p(n) \) [60]. The innovation \( \varepsilon_p(n) \) is orthogonal to, or uncorrelated with, all the previous innovations \( \{ \varepsilon_p(0), \ldots, \varepsilon_p(n-1) \} \), and is therefore a white-noise process. The advantage of the Kalman filter is that it can recursively and efficiently compute the innovations given the state-space model of the physical error signals in Eq. (3.128) [60]. Once the innovations are computed, a current estimate \( \hat{a}(n|n) \) of any other random variable \( a(n) \) can be computed by projecting it onto the linear subspace spanned by the innovations \( \{ \varepsilon_p(0), \ldots, \varepsilon_p(n) \} \), which, due to the orthogonality of the innovations, leads to the following lemma.

**Lemma 3.2** (Linear least-mean-squares estimation given the innovations [60]).

Given the innovations \( \varepsilon_p(m) \) up to \( m = n \) and the covariance matrix of the innovations \( R_{pe} \in \mathbb{R}^{M_p \times M_p} \), a current estimate \( \hat{a}(n|n) \) of any random variable \( a(n) \) can be obtained as

\[
\hat{a}(n|n) = \sum_{m=0}^{n} \mathbb{E}\left[ a(n)\varepsilon_p(m)^T \right] R_{pe}^{-1} \varepsilon_p(m)
\] (3.162)

**Proof.** See, for instance, Kailath et al. [60].

The linear least-mean-squares estimator given the innovations defined in Lemma 3.2 is used in the next section to solve the virtual sensing for active noise control problem using a Kalman filtering approach.
3.7.4 Virtual sensing algorithm

The linear least-mean-squares estimator defined in Lemma 3.2 is now used to compute a current estimate \( \hat{e}_v(n|n) \in \mathbb{R}^{M_v} \) of the virtual error signals given observations \( e_p(i) \) of the physical error signals up to \( i = n \) as

\[
\hat{e}_v(n|n) = \sum_{m=0}^{n} E \left[ e_v(n) e_p(m)^T \right] R_{pe}^{-1} e_p(m). \tag{3.163}
\]

From the state-space model in Eq. (3.128), a current estimate of the virtual error signals can thus be obtained as

\[
\hat{e}_v(n|n) = C_v \hat{z}(n|n) + D_{vu} u(n) + D_{vs} \hat{s}(n|n) + \hat{v}_v(n|n), \tag{3.164}
\]

where \( \hat{z}(n|n) \) is a current estimate of the state, \( \hat{s}(n|n) \) a current estimate of the disturbance source signal, and \( \hat{v}_v(n|n) \) a current estimate of the virtual measurement noise signal. The fact that the control signal \( u(n) \) is a deterministic input signal has also been used in Eq. (3.164), such that \( \hat{u}(n|n) = u(n) \). Note that the current state estimate \( \hat{z}(n|n) \) of the state has been defined in Eq. (3.145). Only the current estimate \( \hat{s}(n|n) \) of the disturbance source signal and the current estimate \( v_v(n|n) \) of the virtual measurement noise signal therefore need to be computed.

Current estimate \( \hat{s}(n|n) \) of the disturbance source signal

From Lemma 3.2, the current estimate \( \hat{s}(n|n) \) of the disturbance source signal can be computed as

\[
\hat{s}(n|n) = \sum_{m=0}^{n} E \left[ s(n) e_p(m)^T \right] R_{pe}^{-1} e_p(m). \tag{3.165}
\]

From the discussions in Section 3.7.3, it can be derived that the innovation \( e_p(m) \) in Eq. (3.165) can be represented as a vector in the linear subspace spanned by the observations \( \{ e_p(0), \ldots, e_p(m) \} \) [60], such that

\[
e_p(m) \in \mathcal{L} \{ e_p(0), \ldots, e_p(m) \}. \tag{3.166}
\]

From the correlation properties defined in Lemma 3.1, the disturbance source signal \( s(n) \) is orthogonal, or uncorrelated, to all such subspaces provided \( m \leq n - 1 \), such that

\[
E \left[ s(n) e_p(m)^T \right] = 0, \quad m \leq n - 1. \tag{3.167}
\]

The current estimate \( \hat{s}(n|n) \) of the disturbance source signal defined in Eq. (3.165) can therefore be reduced to

\[
\hat{s}(n|n) = E \left[ s(n) e_p(n)^T \right] R_{pe}^{-1} e_p(n). \tag{3.168}
\]
The innovation $\epsilon_p(n)$ in Eq. (3.168) can also be written as

$$\epsilon_p(n) = C_v \rho(n) + \bar{v}_p(n),$$  \hspace{1cm} (3.169)

with $\rho(n)$ the predicted state estimation error defined as

$$\rho(n) = z(n) - \hat{z}(n|n-1),$$  \hspace{1cm} (3.170)

and $\bar{v}_p(n)$ the auxiliary physical measurement noise signal defined in Eq. (3.130). The term $[s(n)\epsilon_p(n)^T]$ on the right-hand side in Eq. (3.168) can therefore be expanded as

$$E[s(n)\epsilon_p(n)^T] = E[s(n)\rho(n)^T]C_v^T + E[s(n)\bar{v}_p(n)^T],$$  \hspace{1cm} (3.171)

It can be shown [60] that the predicted state estimation error

$$\rho(n) \in \mathcal{L}\{z(0);s(0),\ldots,s(n-1);v_p(0),\ldots,v_p(n-1)\},$$  \hspace{1cm} (3.172)

and from the covariance properties defined in Eq. (3.129), the disturbance source signal $s(n)$ is orthogonal, or uncorrelated, to all such subspaces, and it follows that

$$E[s(n)\rho(n)^T] = 0.$$  \hspace{1cm} (3.173)

Using this uncorrelatedness property, Eq. (3.171) now reduces to

$$E[s(n)\epsilon_p(n)^T] = E[s(n)\bar{v}_p(n)^T].$$  \hspace{1cm} (3.174)

Using the covariance properties defined in Eq. (3.129), and the fact that the physical auxiliary measurement noise signal $\bar{v}_p(n)$ has been defined in Eq. (3.130) as

$$\bar{v}_p(n) = D_{ps}s(n) + v_p(n),$$  \hspace{1cm} (3.175)

Eq. (3.174) can also be written as

$$E[s(n)\epsilon_p(n)^T] = E[s(n)s(n)^T]D_{ps}^T + E[s(n)v_p(n)^T]$$

$$= D_{ps}^T + S_{ps}^T.$$  \hspace{1cm} (3.176)

From Eq. (3.168), a current estimate $\hat{s}(n|n)$ of the disturbance source signal is therefore computed as

$$\hat{s}(n|n) = \left(D_{ps}^T + S_{ps}^T\right)R_{pe}^{-1}\epsilon_p(n).$$  \hspace{1cm} (3.177)
Current estimate $\hat{v}_v(n|n)$ of the virtual measurement noise signal

From Lemma 3.2, a current estimate $\hat{v}_v(n|n)$ of the virtual measurement noise signal can be computed as

$$\hat{v}_v(n|n) = \sum_{m=0}^{n} E\left[v_v(n)e_p(m)^T\right] R_p^{-1} e_p(m). \quad (3.178)$$

From the discussions in Section 3.7.3, it can again be derived that the innovation $e_p(m)$ can be represented as a vector in the linear subspace spanned by the observations $\{e_p(0), \ldots, e_p(m)\}$ \[60\], such that

$$e_p(m) \in L\{e_p(0), \ldots, e_p(m)\}. \quad (3.179)$$

From the correlation properties defined in Lemma 3.1, the virtual measurement noise signal $v_v(n)$ is orthogonal, or uncorrelated, to all such subspaces provided $m \leq n - 1$, such that

$$E\left[v_v(n)e_p(m)^T\right] = 0, \quad m \leq n - 1. \quad (3.180)$$

The current estimate $\hat{v}_v(n|n)$ of the virtual measurement noise signal defined in Eq. (3.178) can therefore be reduced to

$$\hat{v}_v(n|n) = E\left[v_v(n)e_p(n)^T\right] R_p^{-1} e_p(n). \quad (3.181)$$

Using the expression for the innovation $e_p(n)$ in Eq. (3.169), the term $E\left[v_v(n)e_p(n)^T\right]$ on the right-hand side of Eq. (3.181) can be expanded as

$$E\left[v_v(n)e_p(n)^T\right] = E\left[v_v(n)p(n)^T\right] C_v^T + E\left[v_v(n)\bar{v}_p(n)^T\right]. \quad (3.182)$$

It can again be shown \[60\] that the predicted state estimation error

$$\rho(n) \in L\{z(0); s(0), \ldots, s(n-1); v_p(0), \ldots, v_p(n-1)\}, \quad (3.183)$$

and from the covariance properties defined in Eq. (3.129), the virtual measurement noise signal $v_v(n)$ is orthogonal, or uncorrelated, to all such subspaces, and it follows that

$$E\left[v_v(n)\rho(n)^T\right] = 0. \quad (3.184)$$

Using this uncorrelatedness property, Eq. (3.182) now reduces to

$$E\left[v_v(n)e_p(n)^T\right] = E\left[v_v(n)\bar{v}_p(n)^T\right]. \quad (3.185)$$
Using the covariance properties defined in Eq. (3.129), and the definition of the physical auxiliary measurement noise signal \( \bar{v}_p(n) \) given in Eq. (3.175), Eq. (3.185) can also be written as

\[
E[\bar{v}_v(n)\epsilon_p(n)^T] = E[\bar{v}_v(n)s(n)^T]D_{ps}^T + E[\bar{v}_v(n)v_p(n)^T] = S_{vs}D_{ps}^T + R_{pv}^v. \tag{3.186}
\]

From Eq. (3.181), a current estimate \( \hat{v}_v(n|n) \) of the virtual measurement noise signal is therefore computed as

\[
\hat{v}_v(n|n) = \left(S_{vs}D_{ps}^T + R_{pv}^v\right)R_{pv}^{-1}\epsilon_p(n). \tag{3.187}
\]

State-space model of spatially fixed virtual sensing algorithm

Using the current estimate \( \hat{s}(n|n) \) of the disturbance source signal defined in Eq. (3.177), and the current estimate \( \hat{v}_v(n|n) \) of the virtual measurement noise signal defined in Eq. (3.177), a current estimate of the virtual error signals is now computed from Eq. (3.164) as

\[
\hat{e}_v(n) = C_v\hat{z}(n|n) + D_{vu}u(n) + \hat{R}_{pv}^{-1}\epsilon_p(n), \tag{3.188}
\]

with \( \hat{R}_{pv}^T \) defined as

\[
\hat{R}_{pv}^T = R_{pv}^T + S_{vs}D_{ps}^T + D_{vs}S_{ps}^T + D_{vs}D_{ps}^T. \tag{3.189}
\]

Note that the matrix \( \hat{R}_{pv} \) has been defined in Eqs (3.133)–(3.134) as the covariance matrix between the auxiliary measurement noise signals \( \bar{v}_p(n) \) and \( \bar{v}_v(n) \). Using the expression for the current state estimate \( \hat{z}(n|n) \) given in Eq. (3.145), the current estimate of the virtual error signals Eq. (3.188) can also be written as

\[
\hat{e}_v(n|n) = C_v\hat{z}(n|n-1) + D_{vu}u(n) + M_{vs}\epsilon_p(n), \tag{3.190}
\]

with the virtual gain matrix \( M_{vs} \in \mathbb{R}^{M_v \times M_p} \) defined as

\[
M_{vs} = C_vM_{ps} + \hat{R}_{pv}^T R_{pv}^{-1} = (C_vP_{ps}C_p + \hat{R}_{pv}^T)R_{pv}^{-1}. \tag{3.191}
\]

Finally, using the definition of the innovations in Eq. (3.142), the current estimate of the virtual error signals defined in Eq. (3.190) can also be written as

\[
\hat{e}_v(n|n) = (C_v - M_{vs}C_p)\hat{z}(n|n-1) + (D_{vu} - M_{vs}D_{pu})u(n) + M_{vs}\epsilon_p(n). \tag{3.192}
\]

The above derivations are now summarised in the following theorem, which defines a state-space model of the proposed Kalman filter based spatially fixed virtual sensing algorithm.
Theorem 3.4 (Kalman filter based spatially fixed virtual sensing algorithm).

Let a state-space realisation of the plant be given by Eq. (3.128), and let the covariance matrices \( \bar{Q}_s, \bar{S}_ps, \bar{R}_p, \) and \( \bar{R}_{ps} \) be defined as in Eqs (3.2)–(3.135). Furthermore, let

- the pair \((C_p, A)\) be detectable;
- \( \bar{R}_p > 0, \bar{\bar{Q}}_s \bar{S}_ps^T \bar{R}_p^{-1} \bar{S}_ps \geq 0; \)
- \((A - \bar{S}_ps^T \bar{R}_p^{-1}C_p, \bar{Q}_s - \bar{S}_ps^T \bar{R}_p^{-1} \bar{S}_ps)\) has no uncontrollable modes on the unit circle.

Then a state-space realisation of the virtual sensing algorithm that gives optimal current estimates \( \hat{e}_v(n|n) \) of the virtual error signals, given observations \( e_p(i) \) of the physical error signals up to \( i = n \), is defined as

\[
\begin{bmatrix}
\hat{z}(n+1|n) \\
\hat{e}_v(n|n)
\end{bmatrix} =
\begin{bmatrix}
A - K_{ps}C_p & B_u - K_{ps}D_{pu} & K_{ps} \\
C_v - M_{vs}C_p & D_{vu} - M_{vs}D_{pu} & M_{vs}
\end{bmatrix}
\begin{bmatrix}
\hat{z}(n|n-1) \\
\hat{e}_v(n|n-1) \\
\end{bmatrix} +
\begin{bmatrix}
u(n) \\
e_p(n)
\end{bmatrix},
\]

where the Kalman gain matrix \( K_{ps}, \) and the virtual gain matrix \( M_{vs} \) are given by

\[
K_{ps} = (A_{ps}C_p^T + S_{ps}^T)R_{pe}^{-1}
\]
\[
M_{vs} = (C_vP_{ps}C_p^T + \bar{R}_{ps}^T)R_{pe}^{-1},
\]

with \( P_{ps} = P_{ps}^T > 0 \) the unique stabilising solution to the DARE given by

\[
P_{ps} = AP_{ps}A^T - (AP_{ps}C_p^T + S_{ps}^T)(C_pP_{ps}C_p^T + \bar{R}_p)^{-1}(AP_{ps}C_p^T + S_{ps}^T)^T + \bar{Q}_s,
\]

and where \( R_{pe} \in \mathbb{R}^{M_p \times M_p} \) is the covariance matrix of the white innovation signals \( e_p(n) \) given by

\[
R_{pe} = E[e_p(n)e_p(n)^T] = C_pP_{ps}C_p^T + \bar{R}_p,
\]

with \( e_p(n) \) defined in Eq. (3.142).

Uncorrelated physical and virtual auxiliary measurement noise

If the physical and virtual auxiliary measurement noise signals \( \bar{v}_p(n) \) and \( \bar{v}_v(n) \) defined in Eq. (3.130) are uncorrelated, such that \( \bar{R}_{pv} = 0 \) in Eq. (3.133), the virtual gain matrix \( M_{vs} \) in Eq. (3.194) is equal to

\[
M_{vs} = C_vM_{ps},
\]

with \( M_{ps} \) the innovation gain matrix defined in Eq. (3.145). For this case, a current estimate \( \hat{e}_v(n|n) \) of the virtual error signals is calculated from Eq. (3.193) as

\[
\hat{e}_v(n|n) = (C_v - C_vM_{ps}C_p)\hat{z}(n|n-1) + (D_{vu} - C_vM_{ps}D_{pu})u(n) + C_vM_{ps}e_p(n).
\]
Comparing this equation to the definition of the current state estimate $\hat{z}(n|n)$ given in Eq. (3.145), it can thus be seen that the current estimate of the virtual error signals is computed as

$$\hat{e}_v(n|n) = C_v \hat{z}(n|n) + D_{vu} u(n).$$

(3.199)

This result is intuitive because when the physical and virtual auxiliary measurement noise signals are uncorrelated, the virtual auxiliary measurement noise signal $\bar{v}_v(n)$ cannot be estimated from the observations of the physical error signals $e_p(n)$. In other words, in this case the physical error signals $e_p(n)$ only contain information that can be used to compute a current estimate $\hat{z}(n|n)$ of the state, and a current estimate $\hat{e}_v(n|n)$ of the virtual error signals is then computed as in Eq. (3.199).

### Block diagram of virtual sensing algorithm

In Fig. 3.11, a block diagram of the Kalman filter based spatially fixed virtual sensing algorithm presented in Theorem 3.4 is shown.

![Block diagram of the Kalman filter based spatially fixed virtual sensing algorithm](image)

Figure 3.11: Block diagram of the Kalman filter based spatially fixed virtual sensing algorithm, with $L$ control sources, $M_p$ physical sensors, and $M_v$ spatially fixed virtual sensors.

Fig. 3.11 illustrates that the input signals into the Kalman filter based spatially fixed virtual sensing algorithm are the physical error signals, or observations, $e_p(n)$ and the deterministic control signals $u(n)$. The prediction form of the Kalman filter is then used to compute the innovations $\varepsilon_p(n)$ and a predicted state estimate $\hat{z}(n|n-1)$. As shown in Fig. 3.11, these output signals of the Kalman filter plus the deterministic control signals $u(n)$ are then used to compute a current estimate $\hat{e}_v(n|n)$ of the virtual error signals given the state-space matrices $C_v$ and $D_{vu}$ of the state-space model defined in Eq. (3.128), and the virtual gain matrix $M_{vs}$ defined in Theorem 3.4.

### Comparison to remote microphone technique

When comparing the block diagram in Fig. 3.11 to the block diagram of the remote microphone technique [104, 112] shown in Fig. 3.6, one may wonder whether there is some sort of equivalence between the transfer function and state-space based virtual sensing.
solutions. In Appendix B, a state-space solution for the remote microphone technique is derived given state-space models of the transfer paths that describe the active noise control system, assuming there is no measurement noise on the sensors. It is then shown that the resulting state-space solution is equivalent to the Kalman filter based spatially fixed virtual sensing algorithm presented in Theorem 3.4. The derivations presented in Appendix B also show that a state-space solution for the optimal filter $H_o$ defined in Theorem 3 is given by

$$
H_o \sim \begin{bmatrix}
A - K_{ps}C_p \\
C_v - M_{vs}C_p
\end{bmatrix}
$$

(3.200)

where the Kalman gain matrix $K_{ps}$ and the virtual gain matrix $M_{vs}$ are defined in Theorem 3.4. These observations will be used in Chapter 5 to modify the remote microphone technique [104, 112] to allow for virtual locations that are moving through the sound field rather than being spatially fixed.

### 3.7.5 Practical implementation

To implement the virtual sensing algorithm presented in Theorem 3.4, the state-space matrices $A, B_u, C_p, C_v, D_{pu}, D_{vu}$ of the standard state-space model in Eq. (3.128), and the covariance matrices $\bar{Q}_s, \bar{S}_{ps}, \bar{R}_p, \bar{R}_{pv}$ of its stochastic input signals need to be known. These state-space and covariance matrices together describe the input-output behaviour of the active noise control system and the covariance properties of its stochastic input signals. In a practical situation, the input-output behaviour of the active noise control system can be estimated in a preliminary system identification stage in which physical sensors are temporarily located at the desired virtual locations. An innovations model [60] of the active noise control system can then be estimated using subspace identification techniques [50, 55, 124]. In this section, a method is therefore proposed that can be used to implement the virtual sensing algorithm presented in Theorem 3.4 in a practical situation, where an innovations model of the active noise control system can be estimated rather than a standard state-space model defined in Eq. (3.128). The proposed practical method will be used in the experiments presented in Chapter 8, where an innovations model of the experimental active noise control arrangement is estimated in a preliminary identification stage in which a physical sensor is temporarily located at the desired virtual location.

**Innovations model**

In a practical situation, an innovations model of the active noise control system under consideration can be estimated rather than a standard state-space model defined in
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Eq. (3.128). An innovations model of the active noise control system can be written as [60]

\[
\begin{align*}
\dot{z}(n+1|n) &= A\dot{z}(n|n-1) + B_u u(n) + K_s \begin{bmatrix} e_p(n)^T & e_v(n)^T \end{bmatrix}^T \\
e_p(n) &= C_p \dot{z}(n|n-1) + D_{pu} u(n) + e_p(n) \\
e_v(n) &= C_v \dot{z}(n|n-1) + D_{vu} u(n) + e_v(n),
\end{align*}
\] (3.201)

where the Kalman gain matrix \( K_s \in \mathbb{R}^{N \times (M_p + M_v)} \) is defined as

\[
K_s = (A P_s C^T + \tilde{S}_s) R_e^{-1},
\] (3.202)

with \( P_s = P_s^T > 0 \) the unique stabilising solution to the DARE given by

\[
P_s = A P_s A^T - (A P_s C^T + \tilde{S}_s) (C P_s C^T + \tilde{R})^{-1} (A P_s C^T + \tilde{S}_s)^T + \tilde{Q}_s,
\] (3.203)

and where the matrix \( C \in \mathbb{R}^{(M_p + M_v) \times N} \) is defined as

\[
C = \begin{bmatrix} C_p \\ C_v \end{bmatrix}.
\] (3.204)

The covariance matrices \( \tilde{Q}_s, \tilde{S}_s, \) and \( \tilde{R} \) in Eqs (3.202) and (3.203) have been defined in Eqs (3.131)–(3.135). The matrix \( R_e \in \mathbb{R}^{(M_p + M_v) \times (M_p + M_v)} \) in Eq. (3.202) is the covariance matrix of the white innovation signals

\[
\varepsilon(n) = \begin{bmatrix} e_p(n) \\ e_v(n) \end{bmatrix} = \begin{bmatrix} e_p(n) - \hat{e}_p(n|n-1) \\ e_v(n) - \hat{e}_v(n|n-1) \end{bmatrix},
\] (3.205)

and is given by

\[
R_e = E[\varepsilon(n)\varepsilon(k)^T] = \begin{bmatrix} \tilde{R}_{pe} & \tilde{R}_{pve} \\ \tilde{R}_{pve}^T & \tilde{R}_{ve} \end{bmatrix} \delta_{nk} = CP_s C^T + \tilde{R}.
\] (3.206)

Using the definition of the covariance matrix \( \tilde{R} \) given in Eq. (3.133), the covariance matrix of the innovation signals can be written in expanded form as

\[
\begin{bmatrix} \tilde{R}_{pe} & \tilde{R}_{pve} \\ \tilde{R}_{pve}^T & \tilde{R}_{ve} \end{bmatrix} = \begin{bmatrix} C_p P_s C_p^T + \tilde{R}_p & C_p P_s C_v^T + \tilde{R}_{pv} \\ C_v P_s C_p^T + \tilde{R}_{vp}^T & C_v P_s C_v^T + \tilde{R}_v \end{bmatrix}.
\] (3.207)

Using the definition of the covariance matrix \( \tilde{S}_s \) given in Eq. (3.135), the Kalman gain matrix \( K_s \) in Eq. (3.202) can be written in expanded form as

\[
K_s = \begin{bmatrix} A P_s C_p^T + \tilde{S}_{ps} & A P_s C_v^T + \tilde{S}_{vs} \end{bmatrix} R_e^{-1}.
\] (3.208)

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Computing $K_{ps}$ and $M_{vs}$ from innovations model

In a practical situation, subspace identification techniques [50, 55, 124] can be used to estimate an innovations model of the active noise control system defined in Eq. (3.201). These techniques can thus be used to compute estimates of the state-space matrices $A$, $B_{ss}$, $C_p$, $C_v$, $D_{ps}$, $D_{vs}$, the Kalman gain matrix $K_s$, and the covariance matrix $R_s$ of the innovation signals. To implement the virtual sensing algorithm presented in Eq. (3.193) of Theorem 3.4, the matrices that still need to be computed after the innovations model has been estimated are thus the Kalman gain matrix $K_{ps}$ and the virtual gain matrix $M_{vs}$. It is now explained how these matrices can be computed given the innovations model of the active noise control system defined in Eq. (3.201). The Kalman gain matrix $K_{ps}$ can be computed as

$$K_{ps} = \left( AXsC_p^T + K_s \begin{bmatrix} \hat{R}_{pe} \\ \hat{R}_{pve}^T \end{bmatrix} \right) \left( C_pXsC_p^T + \hat{R}_{pe} \right)^{-1},$$

(3.209)

with $X_s = X_s^T > 0$ the unique stabilising solution to the DARE given by

$$X_s = AXsA^T - K_{ps}(C_pXsC_p^T + \hat{R}_{pe})^{-1}K_{ps}^T + K_sR_sK_s^T.$$

(3.210)

This can be derived by first noting from the definition of the Kalman gain matrix $K_s$ in Eq. (3.208) that in Eq. (3.209), the term

$$K_s \begin{bmatrix} \hat{R}_{pe} \\ \hat{R}_{pve}^T \end{bmatrix} = \begin{bmatrix} AP_sC_p^T + \tilde{S}_{ps}^T \\ AP_vC_v^T + \tilde{S}_{vs}^T \end{bmatrix} R_s^{-1} \begin{bmatrix} \hat{R}_{pe} \\ \hat{R}_{pve}^T \end{bmatrix} = AP_sC_p^T + \tilde{S}_{ps},$$

(3.211)

because

$$R_s^{-1} \begin{bmatrix} \hat{R}_{pe} \\ \hat{R}_{pve}^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

(3.212)

Furthermore, from the definition of $\hat{R}_{pe}$ in Eq. (3.207), it can be seen that in Eq. (3.209), the term

$$C_pXsC_p^T + \hat{R}_{pe} = C_p(X_s + P_s)C_p^T + \tilde{R}_{pe}.$$

(3.213)

Using Eqs (3.211) and (3.213), the Kalman gain matrix $K_{ps}$ in Eq. (3.209) can therefore also be written as

$$K_{ps} = (AP_{ps}C_p^T + \tilde{S}_{ps}^T)(C_pP_{ps}C_p^T + R_p)^{-1}$$

(3.214)

where the matrix $P_{ps} = \hat{P}_{ps}^T > 0$ is defined as

$$P_{ps} \triangleq X_s + P_s,$$

(3.215)

with $X_s$ the solution to the DARE in Eq. (3.210), and $P_s$ the solution to the DARE in Eq. (3.203). Thus, if it can be shown that the matrix $P_{ps}$ is equivalent to the unique
stabilising solution $P_{ps}$ to the DARE given in Eq. (3.195) of Theorem 3.4, the Kalman gain matrix $K_{ps}$ in Eq. (3.209) is equivalent to the one defined in Eq. (3.194) of Theorem 3.4. To show this, first note that in Eq. (3.210), the term

$$K, R, K_s^T = A P_s A^T - P_s + Q_s$$

(3.216)

which can be easily derived from the DARE in Eq. (3.203). Using Eqs (3.213), (3.214) and (3.216), the DARE in Eq. (3.210) can thus be written as

$$\dot{P}_{ps} = A \dot{P}_{ps} A^T - (A \dot{P}_{ps} C_p + S_{ps}^T)(C_p \dot{P}_{ps} C_p + \dot{R}_p)^{-1}(A \dot{P}_{ps} C_p + S_{ps}^T)^T + Q_s.$$ (3.217)

This DARE is equivalent to the DARE that needs to be solved in Theorem 3.4. The matrix $\dot{P}_{ps}$ is therefore equivalent to the unique stabilising solution $P_{ps}$ to the DARE given in Eq. (3.195) of Theorem 3.4, and the Kalman gain matrix $K_{ps}$ in Eq. (3.209) is therefore equivalent to the optimal solution defined in Eq. (3.194) of Theorem 3.4.

It can now also be shown that the virtual gain matrix $M_{vs}$ in Eq. (3.194) of Theorem 3.4 can be computed from the innovations model in Eq. (3.201) as

$$M_{vs} = (C_v X_s C_p^T + R_{vce}^T)(C_p X_s C_p^T + R_{ps})^{-1},$$

(3.218)

with $X_s = X_s^T > 0$ the unique stabilising solution to the DARE given in Eq. (3.210). Using the definition of the covariance matrices $R_{ps}$ and $R_{vce}$ given in Eq. (3.207), the virtual gain matrix in Eq. (3.219) can be written as

$$M_{vs} = (C_v \dot{P}_{ps} C_p^T + \dot{R}_{ps}^T)(C_p \dot{P}_{ps} C_p^T + \dot{R}_p)^{-1},$$ (3.219)

with $\dot{P}_{ps}$ defined in Eq. (3.215). Because it has been shown that this matrix is equivalent to the unique stabilising solution $P_{ps}$ to the DARE given in Eq. (3.195) of Theorem 3.4, the virtual gain matrix $M_{vs}$ in Eq. (3.219) is equivalent to the optimal solution defined in Eq. (3.194) of Theorem 3.4. The practical implementation of the Kalman filter based spatially fixed virtual sensing algorithm presented in Theorem 3.4 is summarised in Table 3.1.

### 3.8 Conclusion

In this chapter, the virtual sensing for active noise control problem has been analysed assuming that the desired locations of the zones of quiet are spatially fixed within the sound field. The case of virtual locations that are moving through the sound field will be addressed in Chapter 5, where moving virtual sensing algorithms will be developed.

The adaptive LMS virtual microphone technique [14] has been analysed, and the optimal Wiener solution for the physical sensor weights has been presented. This technique has also been extended to the case of multiple spatially fixed virtual sensors. An optimal
1. Temporarily locate physical sensors at the desired spatially fixed virtual locations and measure an input-output data-set
\[
\{u(n), \begin{bmatrix} e_p(n) \\ e_v(n) \end{bmatrix} \}^N_{n=1}, \tag{3.220}
\]
with \( u(n) \) the deterministic control input signal, and \( e_p(n) \) and \( e_v(n) \) the stochastic physical and virtual error signals, respectively.

2. Use subspace identification methods \([50, 55, 124]\) to estimate an innovations model of the physical and virtual error signals
\[
\hat{z}(n+1|n) = \hat{A}\hat{z}(n|n-1) + \hat{B}_u u(n) + \hat{K}_s \begin{bmatrix} e_p(n)^T \\ e_v(n)^T \end{bmatrix}^T \tag{3.221}
\]
\[
e_p(n) = \hat{C}_p\hat{z}(n|n-1) + D_{pu} u(n) + e_p(n) \tag{3.222}
\]
\[
e_v(n) = \hat{C}_v\hat{z}(n|n-1) + D_{vu} u(n) + e_v(n), \tag{3.223}
\]
and to estimate the covariance matrix of the white innovation signals
\[
\hat{R}_e = \begin{bmatrix} \hat{R}_{pe} & \hat{R}_{pve} \\ \hat{R}_{ve} & \hat{R}_{ve} \end{bmatrix}. \tag{3.224}
\]

3. Implement the Kalman filter based spatially fixed virtual sensing algorithm as
\[
\begin{bmatrix} \hat{z}(n+1|n) \\ \hat{e}_v(n|n) \end{bmatrix} = \begin{bmatrix} \hat{A} - \hat{K}_{ps} \hat{C}_p & \hat{B}_u - \hat{K}_{ps} D_{pu} \\ \hat{C}_v - \hat{M}_{vs} \hat{C}_p & \hat{D}_{vu} - \hat{M}_{vs} D_{pu} \end{bmatrix} \begin{bmatrix} \hat{K}_{ps} \\ \hat{M}_{vs} \end{bmatrix} \begin{bmatrix} \hat{z}(n|n-1) \\ u(n) \end{bmatrix} + \begin{bmatrix} e_p(n) \\ e_v(n) \end{bmatrix}, \tag{3.225}
\]
where the Kalman gain matrix \( \hat{K}_{ps} \) and the virtual gain matrix \( \hat{M}_{vs} \) are computed as
\[
\hat{K}_{ps} = \left( \hat{A}X_s\hat{C}_p^T + \hat{K}_s \begin{bmatrix} \hat{R}_{pe} \\ \hat{R}_{pve} \end{bmatrix} \right) \left( \hat{C}_p X_s \hat{C}_p^T + \hat{R}_{pe} \right)^{-1}, \tag{3.226}
\]
\[
\hat{M}_{vs} = \left( \hat{C}_v X_s \hat{C}_p^T + \hat{R}_{pve} \right) \left( \hat{C}_p X_s \hat{C}_p^T + \hat{R}_{pe} \right)^{-1}, \tag{3.227}
\]
with \( X_s = X_s^T > 0 \) the unique stabilising solution to the DARE given by
\[
X_s = \hat{A}X_s\hat{A}^T - \hat{K}_{ps}(\hat{C}_p X_s \hat{C}_p^T + \hat{R}_{pe})^{-1}\hat{K}_{ps}^T + \hat{K}_s \hat{R}_s \hat{K}_s^T. \tag{3.228}
\]

Table 3.1: Practical implementation of the Kalman filter based spatially fixed virtual sensing algorithm presented in Theorem 3.4.
solution for the physical sensor weights has been derived given the standard state-space model of the active noise control system introduced in Chapter 2. This solution can be used to directly calculate the optimal weights when a state-space model of the considered active noise control system is available. This is especially convenient in a numerical analysis because the computationally more intensive adaptive method for computing the optimal weights, which was employed in previous research \cite{14, 80}, no longer needs to be used.

The remote microphone technique \cite{104, 112} has been analysed. The optimal estimation performance that can in theory be obtained with this technique has been derived using the factorisation approach introduced in Chapter 2. It has been shown that the optimal estimation performance is determined by the properties of the physical and virtual primary transfer path matrices. To obtain good estimation performance, the locations of the physical sensors should be chosen such that all the modes that contribute to the virtual primary disturbances are observable at the physical sensors. Furthermore, the physical and virtual sensor configuration should be chosen such that the physical primary disturbances contain time-advanced information about the virtual primary disturbances. This ensures that the virtual primary disturbances can be causally estimated from the physical primary disturbances.

It has been identified that the virtual microphone arrangement \cite{27} is a simplified version of the remote microphone technique \cite{104, 112}. The difference between these techniques is that the primary disturbances at the physical and virtual sensors are assumed to be equal in the virtual microphone arrangement. As a result, the estimation performance of the virtual microphone arrangement is determined by the validity of this assumption.

The hybrid adaptive feedforward observer method \cite{119} has been analysed. It has been shown that if the number of physical sensors is less than the order of the state-space model of the active noise control system considered, which is usually the case in practice, the proposed hybrid adaptive feedforward observer method is only suitable for rejecting non-stationary primary disturbances at the physical sensors, and not for virtual sensing purposes.

The common aim of the previously proposed spatially fixed virtual sensing algorithms that have been analysed in this chapter is to compute an accurate estimate of the virtual error signals given the directly measured physical error signals. The difference between these algorithms is the way in which the estimate is computed, i.e. the assumed structure is what makes the virtual sensing algorithms proposed so far different from one another. Once the structure has been chosen, the optimal solutions for the unknown parameters of the algorithm can be computed by optimising the estimation performance as described in this chapter. It has been shown here that an optimal solution to the spatially fixed virtual sensing problem can be derived using
Kalman filter theory [36, 60]. It is well-known that the Kalman filter provides an *optimal structure* for solving linear estimation problems [60]. Since virtual sensing for active noise control *is* a linear estimation problem, the presented Kalman filter based solution is the optimal solution to the spatially fixed virtual sensing for active noise control problem. The practical implementation of the proposed virtual sensing algorithm has also been discussed.
Chapter 4

Active noise control at spatially fixed virtual sensors

4.1 Introduction

In Chapter 2, active noise control algorithms have been introduced that can be used to create local zones of quiet at a number of virtual locations that are spatially fixed within the sound field. Provided that a suitable feedforward reference signal is available that is not contaminated by intrinsic feedback from the control sources, an optimal feedforward control approach can be adopted to minimise the virtual error signals without directly measuring them during real-time control. The general feedforward/feedback control approach can be adopted when the feedforward reference signals are contaminated by intrinsic feedback. However, the optimal feedback and adaptive feedforward control approaches presented in Chapter 2 cannot be implemented without directly measuring the virtual error signals during real-time control. This is because these control approaches require the feedback information contained in the virtual error signals to compute an appropriate control signal. To implement these control approaches without directly measuring the virtual error signals during real-time control, an estimate of the virtual error signals can instead be used as a feedback signal. This estimate can be computed using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3. In this chapter, these spatially fixed virtual sensing algorithms are therefore combined with the adaptive feedforward control algorithms introduced in Chapter 2. It is also shown that the optimal feedback control at virtual sensors problem can be solved by recognising that this problem is equivalent to the general feedforward/feedback control problem that has been solved in Chapter 2.

In Section 4.2, the optimal narrowband control performance that can be obtained at the spatially fixed virtual locations when minimising an estimate of the virtual error signals is analysed. The quadratic optimisation techniques that have been employed
in Section 2.3 to minimise the true virtual error signals for the narrowband case are now employed to minimise the estimate of the virtual error signals. The presented discussions are used in the numerical analyses presented in Chapters 7 and 8, where the narrowband control performance that can be obtained at a virtual location inside an acoustic duct arrangement is investigated.

As discussed previously, the adaptive feedforward control algorithms introduced in Chapter 2 cannot be implemented when the virtual error signals are not directly measured during real-time control. In Section 4.3, the standard implementation of the multi-channel filtered-x LMS algorithm introduced in Chapter 2 is therefore modified to account for the fact that the virtual error signals are not directly measured during real-time control but estimated using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3. A block diagram is included to illustrate the modified implementation, which has been used by a number of researchers that have investigated virtual sensing algorithms for active noise control [24, 42, 53, 80, 110, 112]. The only difference between this implementation and the standard implementation of the multi-channel filtered-x LMS algorithm is that an estimate of the virtual error signals is now adaptively minimised instead of the true virtual error signals. The question now arises as to whether this results in a decrease in the optimal control performance that can be obtained at the virtual locations after convergence of the adaptive algorithm. If this is the case, the next question is what degree of control performance is actually lost. The aim in Section 4.3 is to answer these questions for the case of broadband disturbances using the factorisation approach introduced in Chapter 2 and the analyses of the various spatially fixed virtual sensing algorithms [14, 27, 104, 112] presented in Chapter 3. The case of narrowband disturbances is effectively analysed in Section 4.2 using a quadratic optimisation technique. The factorisation approach is adopted because it allows an elegant interpretation of the factors that limit the optimal broadband control performance that can be obtained at the virtual locations when adaptively minimising an estimate of the virtual error signals.

The optimal feedback control approach introduced in Chapter 2, in which a feedback control signal is computed using the true virtual error signals, cannot be implemented when the virtual error signals are not directly measured during real-time control. Instead, a feedback control signal can only be computed from the physical error signals that are directly measured during real-time control by the physical sensors. The aim of the resulting optimal feedback control at virtual sensors problem is to design an optimal solution for the feedback controller that minimises the true virtual error signals. A block diagram of this problem is introduced in Section 4.4, which illustrates that an optimal solution can be derived by noting that this problem is equivalent to the general feedforward/feedback control problem introduced in Chapter 2.
4.2 Optimal narrowband control at virtual sensors

In Section 2.3, the optimal narrowband control performance that can in theory be obtained at the spatially fixed virtual locations when minimising the true virtual error signals has been derived using a quadratic optimisation technique. This technique is now employed to derive the optimal narrowband control performance that can be obtained when an estimate of the virtual error signals is minimised. This estimate can be computed using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3. The presented derivations can for instance be used to investigate the degree of narrowband control performance that is lost in comparison to minimising the true virtual error signals. The discussions presented in this section are used in Chapters 7 and 8 to analyse the narrowband control performance that can be obtained at a virtual location inside the experimental acoustic duct arrangement.

4.2.1 Quadratic optimisation

To use the quadratic optimisation method introduced in Section 2.3, the relevant single tone signals of normalised frequency $\omega T_s$ are again defined in a complex representation that contains the magnitude and phase information with respect to some arbitrary reference. The estimate $\hat{e}_v \in \mathbb{C}^{M_v}$ of the complex virtual error signals is then given by

$$\hat{e}_v = \hat{d}_v + \hat{y}_v = \hat{d}_v + \hat{Z}_{vu}u,$$  \hfill (4.1)

where $\hat{Z}_{vu} \in \mathbb{C}^{M_v \times L}$ is an estimate of the complex virtual secondary transfer impedance matrix between the control signals $u \in \mathbb{C}^L$ and the virtual secondary disturbances $y_v \in \mathbb{C}^{M_v}$. The estimated optimal control signal $\hat{u}_o \in \mathbb{C}^L$ that minimises the cost function

$$\hat{J} = \hat{e}_v^H \hat{e}_v$$ \hfill (4.2)

can now be calculated using the quadratic optimisation techniques described in Section 2.3. Similarly to Eq. (2.27), this results in

$$\hat{u}_o = -\hat{Z}_{vu}^+ \hat{d}_v,$$  \hfill (4.3)

with $\hat{Z}_{vu}^+$ the pseudo-inverse of the estimated complex virtual secondary transfer impedance matrix. As explained in Section 2.3, the computation of this pseudo-inverse depends on the relative number of control sources $L$ and virtual sensors $M_v$. If there are more virtual sensors than control sources, such that $M_v > L$, the pseudo-inverse in Eq. (4.3) is defined as

$$\hat{Z}_{vu}^+ = \left[\hat{Z}_{vu}^H \hat{Z}_{vu}\right]^{-1} \hat{Z}_{vu}^H.$$  \hfill (4.4)
If there are more control sources than virtual sensors, such that \( L > M_v \), the pseudo-inverse in Eq. (4.3) is defined as

\[
\hat{Z}_{vu}^\dagger = \hat{Z}_{vu}^H \left[ \hat{Z}_{vu} \hat{Z}_{vu}^H \right]^{-1}.
\] (4.5)

For the fully determined case, such that \( L = M_v \), the pseudo-inverse in Eq. (4.3) is simply equal to the normal inverse, such that

\[
\hat{Z}_{vu}^\dagger = \hat{Z}_{vu}^{-1}.
\] (4.6)

The theoretical limit on the narrowband control performance that can be obtained at the virtual locations while minimising an estimate \( \hat{e}_v \) of the virtual error signals is found by substituting the estimated optimal control signal \( \hat{u}_o \) defined in Eq. (4.3) into the true cost function that needs to be minimised, which has been defined in Eqs (2.22) and (2.23) as

\[
J = e_v^H e_v = u^H A u + u^H b + b^H u + c, \tag{4.7}
\]

with the matrix \( A \), the vector \( b \), and the scalar \( c \) defined in Eqs (2.24)–(2.26). Substituting the estimated optimal control signal \( \hat{u}_o \) defined in Eq. (4.3) into the cost function in Eq. (4.7) results in

\[
J(\hat{u}_o) = \hat{u}_o^H A \hat{u}_o + \hat{u}_o^H b + b^H \hat{u}_o + c. \tag{4.8}
\]

In the next section, it is shown that for the fully determined case, such that \( L = M_v \), and the under-determined case, such that \( L > M_v \), substituting the solution defined in Eq. (4.3) into the cost function defined in Eq. (4.2) results in

\[
J(\hat{u}_o) = 0. \tag{4.9}
\]

This means that the estimate \( \hat{e}_v \) of the virtual error signals is minimised to zero. The virtual output errors \( e_v \) are then given by

\[
e_v = e_v - \hat{e}_v = e_v - 0 = e_v, \tag{4.10}
\]

and the residual virtual error signals \( e_v \) are therefore equal to the virtual output errors \( e_v \) in this instance. Thus, for the fully determined and under-determined cases, the value of the cost function defined in Eq. (4.7) that is obtained when minimising the estimated virtual error signals \( \hat{e}_v \) is given by

\[
J(\hat{u}_o) = e_v^H e_v. \tag{4.11}
\]
The narrowband control performance that can in theory be obtained at the virtual locations while minimising an estimate of the virtual error signals is thus solely determined by the estimation accuracy of the virtual sensing algorithm for these cases. The residual virtual error signals $e_v$ at the virtual locations can now be expressed as

$$e_v = d_v + Z_{vu} \hat{u}_v = d_v - Z_{vu} Z^\dagger_{vu} d_v = d_v - \Psi d_v,$$  \hfill (4.12)

with the matrix $\Psi \in \mathbb{C}^{M_v \times M_v}$ defined as

$$\Psi = Z_{vu} Z^\dagger_{vu}.$$  \hfill (4.13)

The attenuation $\eta_i$ in decibels that is achieved at the $i^{th}$ virtual sensor is defined as

$$\eta_i \triangleq -20 \log_{10} \left| \frac{\hat{e}_{vi}}{d_{vi}} \right| \text{ dB.} \hfill (4.14)$$

Using Eq. (4.12), the attenuation $\hat{\eta}_{oi}$ that can in theory be obtained at the $i^{th}$ virtual location when minimising an estimate of the virtual error signals is now given by

$$\hat{\eta}_{oi} = -20 \log_{10} \left| 1 - \sum_{j=1}^{M_v} \Psi_{ij} \hat{d}_{vj} \right| \text{ dB.} \hfill (4.15)$$

Eq. (4.15) can be used to analyse the narrowband control performance that can in theory be obtained at the virtual locations when using one of the virtual sensing algorithms introduced in Chapter 3 to compute an estimate of the virtual error signals. This is now done for the remote microphone technique [104, 112] and the virtual microphone arrangement [27] that have been analysed in Section 3.4.

**Remote microphone technique**

The block diagram of the remote microphone technique shown in Fig. 3.6 is again illustrated in Fig. 4.1 but now for the case of single tone disturbances of normalised frequency $\omega T_s$. In this figure, the complex physical secondary transfer impedance matrix $Z_{pu}$ and the complex virtual secondary transfer impedance matrix $Z_{vu}$ are determined by evaluating the magnitude and phase of the physical and virtual secondary transfer path matrices $G_{pu}$ and $G_{vu}$, respectively, at the normalised frequency $\omega T_s$ of interest. When deriving the theoretical limit on the narrowband control performance that can be obtained when using the remote microphone technique, the estimate $\hat{Z}_{vu}$ of the complex virtual secondary transfer impedance matrix is thus assumed to be perfect, such that the matrix in Eq. (4.13) is assumed to be $\Psi = I$. An estimate of the virtual error signals is now effectively computed in Fig. 4.1 as

$$\hat{e}_v = H(e^{j\omega T_s}) d_p + Z_{vu} u = \hat{d}_v + y_{v'},$$  \hfill (4.16)
where $\mathbf{H}(e^{j\omega T_s}) \in \mathbb{C}^{M_v \times M_p}$ is a complex transfer impedance matrix. For single tone disturbances, this complex transfer impedance matrix can be designed such that it describes the magnitude and phase relationship between the physical and virtual primary disturbances at the normalised frequency $\omega T_s$ of interest. In this case, a perfect estimate $\hat{d}_v = d_v$ of the complex virtual primary disturbances is computed as

$$d_v = \mathbf{H}(e^{j\omega T_s}) d_p. \quad (4.17)$$

For this case, Eq. (4.15) indicates that infinite narrowband attenuations can in theory be achieved at the virtual locations for the fully determined case, since the matrix $\mathbf{Y} = \mathbf{I}$ and a perfect estimate of the virtual primary disturbances is computed. However, it is important to note that infinite narrowband attenuations can only be achieved in theory when the filter $\mathbf{H}$ is modelled as a complex transfer impedance matrix that describes the magnitude and phase relationship between the physical and virtual primary disturbances at the normalised frequency of interest.

In practice, the filter $\mathbf{H}$ in the block diagram of the remote microphone technique shown in Fig. 3.6 is often modelled as a broadband transfer function matrix, even for the case of single tone disturbances. The filter $\mathbf{H}$ is then designed to compute an estimate of the broadband virtual primary disturbances given the broadband physical primary disturbances, and causality then becomes an issue. As discussed in Section 3.4, the broadband estimation performance of the remote microphone technique is limited by the causality constraint placed on the design of the filter $\mathbf{H}$. Thus, if the optimal broadband solution for the filter $\mathbf{H}$ defined in Theorem 3.2 is used, an estimate of the virtual error signals is effectively computed in Fig. 4.1 for the single tone case as defined in Eq. (4.16), with $\mathbf{H}(e^{j\omega T_s})$ determined by evaluating the broadband filter $\mathbf{H}$ at the normalised frequency $\omega T_s$ of interest. Using Eq. (4.16), it can be shown that Eq. (4.15)
then reduces to
\[ \hat{\eta}_oi = -20 \log_{10} \left| 1 - \frac{\hat{d}_{vi}}{d_{vi}} \right| \text{ dB}, \] (4.18)

with the estimate \( \hat{d}_{vi} \) of the virtual primary disturbance computed from Eq. (4.16) as
\[ \hat{d}_{vi} = \sum_{j=1}^{M_p} H_{ij}(e^{j\omega T_s})d_{pj}, \] (4.19)

where \( H_{ij}(e^{j\omega T_s}) \) is the entry \((i,j)\) of the complex transfer impedance matrix \( H(e^{j\omega T_s}) \).

As stated previously, this matrix is determined by evaluating the magnitude and phase of the broadband filter \( H \) at the normalised frequency \( \omega T_s \) of interest. The estimated virtual primary disturbance \( \hat{d}_{vi} \) in Eq. (4.18) may not always be equal to the true virtual primary disturbance \( d_{vi} \) at the normalised frequency of interest due to the causality constraint put on the broadband filter \( H \). Eq. (4.18) indicates that this causality constraint thus limits the narrowband attenuation that can in theory be obtained at the virtual locations when using the remote microphone technique.

As an example, the remote microphone technique was applied by Roure and Albarrazin [112] to the active noise control of narrowband disturbances at virtual sensors located in a room with metal walls simulating an airplane interior. The filter \( H \) was modelled as a matrix of broadband FIR filters whose coefficients were estimated in a preliminary identification stage using an adaptive system identification technique [51]. For this case, the FIR filter matrix should thus approximate the impulse response matrix of the causal Wiener solution for the filter \( H \) defined in Theorem 3.2. Since a broadband solution for the filter \( H \) was used by Roure and Albarrazin [112], the discussions presented in this section partly explain (partly, because modelling error most likely occurred as well) as to why the narrowband control performance achieved at the virtual locations was observed by Roure and Albarrazin [112] to be less than or equal to the narrowband control performance achieved with physical sensors at these locations.

Virtual microphone arrangement

As discussed in Section 3.5, the virtual microphone arrangement [27] can be seen as a simplified version of the remote microphone technique [104, 112]. In the virtual microphone arrangement, the primary disturbances at the physical and virtual sensors are assumed to be equal. It is thus assumed that the filter \( H(e^{j\omega T_s}) = I \) in Fig. 4.1 such that Eq. (4.18) reduces to
\[ \hat{\eta}_oi = -20 \log_{10} \left| 1 - \frac{d_{pi}}{d_{vi}} \right| \text{ dB}. \] (4.20)
This indicates that, theoretically, the narrowband control performance that can be obtained at the virtual locations when using the virtual microphone arrangement is solely determined by the magnitude and phase differences between the primary disturbances at the physical and virtual sensors. This is to be expected because the estimation performance of the virtual microphone arrangement depends on the validity of the assumption of equal primary disturbances at the physical and virtual sensors.

**Single input single output case**

In the acoustic duct simulations and experiments presented in Chapters 7 and 8, one spatially fixed virtual sensor and one control source are used. For this case, Eq. (4.15) reduces to

$$\hat{\eta}_o = -20 \log_{10}\left|1 - \frac{Z_{vu}d_v}{\hat{Z}_{vu}d_v}\right| \text{ dB}. \quad (4.21)$$

Eq. (4.21) can also be written as

$$\hat{\eta}_o = -20 \log_{10}\left|1 - \frac{T_y}{T_d}\right| \text{ dB}, \quad (4.22)$$

with $T_d \in \mathbb{C}$ the complex transfer impedance between the virtual primary disturbance $d_v$ and its estimate, and $T_y \in \mathbb{C}$ as the complex transfer impedance between the virtual secondary disturbance $y_v$ and its estimate, such that

$$T_d \triangleq \frac{d_v}{d_v}, \quad T_y \triangleq \frac{y_v}{\hat{y}_v}, \quad (4.23)$$

where the fact that $y_v = Z_{vu}u$ and $\hat{y}_v = \hat{Z}_{vu}u$ has been used. Eq. (4.22) indicates that to obtain a large narrowband attenuation, the complex transfer impedance

$$\Delta = \frac{T_y}{T_d} \quad (4.24)$$

should have a magnitude $|\Delta| \simeq 1$ and a phase $\angle \Delta \simeq 0^\circ$. In Fig. 4.2, a contour plot of the attenuation $\hat{\eta}_o$ is shown for different values of the magnitude and phase of the complex transfer impedance $\Delta$ defined in Eq. (4.24). This figure illustrates that in order to obtain a narrowband attenuation at the virtual location of 20 dB or more, the magnitude of $\Delta$ should be between approximately $-0.9$ dB and $0.8$ dB, and the phase should be between approximately $-6^\circ$ and $6^\circ$. If larger narrowband attenuations are required, these bounds become even smaller. These results indicate that large narrowband attenuations at the virtual locations can only be obtained when the virtual sensing algorithm computes a very accurate estimate of the virtual error signal. The discussions presented in this section are used in Chapters 7 and 8 to analyse the narrowband control performance that can be obtained at a virtual location inside an experimental acoustic duct arrangement.
4.3 Adaptive feedforward control at virtual sensors

The *adaptive feedforward control algorithms* introduced in Chapter 2 cannot be implemented when the virtual error signals are not directly measured during real-time control. In this section, the standard implementation of the multi-channel filtered-x LMS algorithm illustrated in Fig. 2.7 is therefore modified to account for the fact that the virtual error signals are not directly measured during real-time control but estimated using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3. A block diagram is introduced in Section 4.3.1 to illustrate the modified implementation of the adaptive feedforward control algorithms. This implementation has been used by a number of researchers that have investigated virtual sensing algorithms for active noise control [24, 42, 53, 80, 110, 112]. The only difference between this implementation and the standard implementation of the multi-channel filtered-x LMS algorithm shown in Fig. 2.7 is that an *estimate* of the virtual error signals is adaptively minimised instead of the *true* virtual error signals. The question now arises as to whether this results in a decrease in the optimal control performance that can be obtained at the spatially fixed virtual locations. If this is the case, the next question is what degree of control performance is actually lost. These questions are answered in Sections 4.3.2–4.3.5 for the case of using the spatially fixed virtual sensing algorithms [14, 27, 104, 112] introduced in Chapter 3. Only the case of broadband disturbances is considered in these sections because the case of narrowband disturbances has effectively been analysed in Section 4.2.

To derive the optimal broadband control performance that can be obtained at the spatially fixed virtual locations when adaptively minimising an estimate of the virtual error signals, an equivalent optimal feedforward control problem is formulated in Section 4.3.1. This problem is then solved in Sections 4.3.2–4.3.5 for the case of computing an estimate of the virtual error signals with one of the spatially fixed virtual

Figure 4.2: Contour plot of expected attenuation $\hat{\eta}$ in dB plotted against $|\Delta|$ and $\angle \Delta$. 

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sensing algorithms introduced in Chapter 3. The solution is calculated using the causal Wiener filter theory introduced in Chapter 2. The factorisation approach adopted in this theory allows a straightforward interpretation of the factors that limit the optimal control performance that can be obtained at the virtual locations when adaptively minimising an estimate of the virtual error signals.

### 4.3.1 Block diagram of modified implementation

In Fig. 4.3, the block diagram of the standard implementation of the multi-channel filtered-x LMS algorithm shown in Fig. 2.7 is modified\(^1\) to account for the fact that the virtual error signals are no longer directly measured during real-time control but estimated using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3.

The block diagram shown in Fig. 4.3 can be compared to the block diagram of the standard implementation of the multi-channel filtered-x LMS algorithm shown in Fig. 2.7. This comparison indicates that the main difference between these two block diagrams is that the estimated virtual error signals \(\hat{e}_v(n)\) are now adaptively minimised instead of the true virtual error signals \(e_v(n)\). The multi-channel filtered-x LMS algorithm can thus be implemented as described in Section 2.5, with the update equation for the filter coefficients of the feedforward controller \(W\) in Fig. 4.3 now given by

\[
    w(n+1) = w(n) - \mu R_v(n) \hat{e}_v(n),
\]

\(^1\) Note that when the normalised filtered-x LMS or the filtered-x RLS algorithm is used instead of the filtered-x LMS algorithm, the block diagram in Fig. 4.3 can simply be modified by replacing the block denoting the LMS algorithm by a block denoting the normalised LMS or RLS algorithm.
with \( \mathbf{R}_v(n) \) the matrix of virtual filtered-reference signals generated as defined in Eqs (2.92)–(2.96). The update equation defined in Eq. (4.25) is simply found by replacing the true virtual error signals \( e_v(n) \) with their estimates \( \hat{e}_v(n) \) in Eq. (2.91), which defines the update equation for the standard implementation of the multi-channel filtered-x LMS.\(^2\) The virtual filtered-reference signals \( \mathbf{r}_v(n) \) in Fig. 4.3 are again generated by filtering the reference signals \( x(n) \) with the virtual secondary transfer path matrix \( \mathbf{G}_{vu} \) as described in Section 2.5. This transfer path matrix can be estimated in a preliminary identification stage in which physical sensors are temporarily located at the desired virtual locations.

### Adaptively minimising the true virtual error signals

When the multi-channel filtered-x LMS algorithm uses the true virtual error signals \( e_v(n) \) to update the feedforward controller as illustrated in Fig. 2.7, the cost function that is adaptively minimised is given by

\[
J = \text{tr} \left( \mathbf{E} \left[ e_v(n) e_v(n)^T \right] \right),
\]

which can be written in the frequency domain as defined in Eq. (2.36) as

\[
J = \| \mathbf{G}_{vs} + \mathbf{G}_{vu} \mathbf{W}_s \|_2^2.
\]

For this case, the multi-channel filtered-x LMS algorithm shown in Fig. 2.7 adapts the feedforward controller such that its response converges to the response of the causal Wiener solution defined in Theorem 2.1 as

\[
\mathbf{W}_o = - \mathbf{G}_{vu}^\dagger \left[ \mathbf{G}_{vu}^*_v \mathbf{G}_{vs}^*_{x,s} \right] + \mathbf{G}_{vs,co}^\dagger.
\]

From Theorem 2.1, the minimum value of the cost function in Eq. (4.27) that can be achieved after convergence of the multi-channel filtered-x LMS algorithm is thus given by

\[
J_{\text{min}} = \| \mathbf{G}_{vs} \mathbf{G}_{x,s,ci}^\dagger \|_2^2 + \| \mathbf{G}_{vu,j}^\dagger \mathbf{G}_{vu,j} \mathbf{G}_{vs}^*_{x,s,ci} \|_2^2 + \| \mathbf{G}_{vu,j}^\dagger \mathbf{G}_{vu,j} \mathbf{G}_{vs,co}^* \|_2^2,
\]

which can be found by substituting the causal Wiener solution \( \mathbf{W}_o \) in Eq. (4.28) into the cost function in Eq. (4.27). An interpretation of the terms on the right-hand side of Eq. (4.29), which determine the optimal control performance that can be achieved at the spatially fixed virtual locations, has been presented after Theorem 2.1.

---

\(^2\) Note that the update equations for the normalised filtered-x LMS or the filtered-x RLS algorithms are also simply found by replacing the true virtual error signals \( e_v(n) \) with their estimates \( \hat{e}_v(n) \) in the update equations for these algorithms defined in Eqs (2.99) and (2.104), respectively.
Adaptively minimising an estimate of the virtual error signals

When the multi-channel filtered-x LMS algorithm uses an estimate $\hat{e}_v(n)$ of virtual error signals to update the feedforward controller as illustrated in Fig. 4.3, the cost function that is adaptively minimised is given by

$$\hat{J} = \text{tr}\left( \mathbb{E} [\hat{e}_v(n)\hat{e}_v(n)^\top] \right). \quad (4.30)$$

To derive the optimal control performance that can in theory be obtained at the virtual locations for this case, an equivalent optimal feedforward control problem can be formulated as shown in Fig. 4.4.

Figure 4.4: Block diagram of the equivalent optimal feedforward control problem that is solved to determine the optimal control performance that can be obtained at the virtual locations when adaptively minimising an estimate of the virtual error signals as illustrated in Fig. 4.3, with $S$ disturbance sources, $L$ control sources, $K$ reference sensors, $M_p$ physical sensors, and $M_v$ spatially fixed virtual sensors.

In Fig. 4.4, the physical primary disturbances $d_p(n)$ and the feedforward reference signals $x(n)$ shown in Fig. 4.3 are assumed to be generated by filtering the disturbance source signals $s(n)$ with the physical primary transfer path matrix $G_{ps}$ and the detector transfer path matrix $G_{xs}$, respectively. The estimate of the virtual error signals is computed using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3. Given the transfer function matrices that describe the input-output behaviour of these virtual sensing algorithms, the equivalent optimal feedforward control problem shown in Fig. 4.4 can be solved using the causal Wiener filter theory introduced in Chapter 2. In the next sections, this is done for the case of using the spatially fixed virtual sensing algorithm introduced in Chapter 2. The derived optimal control performance is compared with the optimal control performance that can be obtained when adaptively minimising the true virtual error signals as illustrated in Fig. 2.7, which has been defined in Eq. (4.29).
4.3.2 Adaptive LMS virtual microphone technique

In this section, the optimal control performance that can in theory be obtained at the virtual locations when adaptively minimising an estimate \( \hat{e}_v(n) \) of the virtual error signals as illustrated in Fig. 4.3 is analysed, assuming this estimate is computed using the adaptive LMS virtual microphone technique [14] introduced in Section 3.3. The transfer function matrix \( G_{LMS} \) that describes the input-output behaviour of the adaptive LMS virtual microphone technique illustrated in Fig. 3.4 has been defined in Eq. (3.14) as

\[
G_{LMS} = \begin{bmatrix} H_{so}^T & (H_{u0} - H_{so})^T G_{pu} \end{bmatrix}, \quad (4.31)
\]

with \( H_{so} \) and \( H_{u0} \) the optimal solutions for the physical sensor weights for the primary and secondary sound fields, respectively. These optimal solutions have been defined in Theorem 3.1. Note that when the optimal solutions for the secondary and primary sound fields are equal, Eq. (4.31) reduces to

\[
G_{LMS} = \begin{bmatrix} H_{so}^T & 0 \end{bmatrix}, \quad (4.32)
\]

which is the implementation used in previous research on the adaptive LMS virtual microphone technique [80]. Using Eq. (4.31), it can be shown that the estimate \( \hat{e}_v(n) \) of the virtual error signals in the block diagram of the equivalent optimal feedforward control problem shown in Fig. 4.4 can be written as

\[
\hat{e}_v(n) = (H_{so}^T G_{ps} + H_{u0}^T G_{pu} W G_{xs}) s(n) = (\hat{G}_{vs} + \hat{G}_{vu} W G_{xs}) s(n), \quad (4.33)
\]

where the estimates \( \hat{G}_{vs} \) and \( \hat{G}_{vu} \) of the virtual primary and secondary transfer path matrices, respectively, are thus computed as

\[
\hat{G}_{vs} = H_{so}^T G_{ps}, \quad \hat{G}_{vu} = H_{u0}^T G_{pu}. \quad (4.34)
\]

Using the expression for the estimate \( \hat{e}_v(n) \) of the virtual error signals derived in Eq. (4.33), an equivalent frequency domain expression of the cost function in Eq. (4.30) that is adaptively minimised in Fig. 4.3 is given by

\[
\hat{f} = \| \hat{G}_{vs} + \hat{G}_{vu} W G_{xs} \|^2, \quad (4.35)
\]

where Parseval’s Theorem and the fact that the signals \( s(n) \) are assumed to be white noise sequences with unit covariance have been used. The block diagram of the equivalent optimal feedforward control problem illustrated in Fig. 4.4 can therefore be redrawn as shown in Fig. 4.5. The block diagram in this figure can be compared to the block diagram of the optimal feedforward control problem shown in Fig. 2.3, where the true virtual error signals \( e_v(n) \) are minimised instead of their estimates. The only difference between these two block diagrams is that the virtual primary and secondary transfer
path matrices in Fig. 2.3 have been replaced with their estimates \( \hat{G}_{\text{vs}} \) and \( \hat{G}_{\text{vu}} \) in Fig. 4.5, which are computed as defined in Eq. (4.34). The equivalent optimal feedforward control problem illustrated in Fig. 4.5 can now be solved using the causal Wiener solution defined in Theorem 2.1. The causal Wiener solution \( \hat{W}_o \in \mathcal{RH}_{\infty}^{L \times K} \) that minimises the cost function in Eq. (4.35) is given by

\[
\hat{W}_o = -\hat{G}_{\text{vu},o}^\dagger \left[ \hat{G}_{\text{vu},d}^* \hat{G}_{\text{vs}}^* G_{\text{xs},ci}^* \right] + G_{\text{xs},ci}^*.
\]  

(4.36)

When using the adaptive LMS virtual microphone technique as a virtual sensing algorithm in Fig. 4.3, the response of the feedforward controller \( W \) is thus adapted such that it approximates the impulse response of the causal Wiener solution defined in Eq. (4.36).

The optimal control performance that can in theory be obtained at the virtual locations when adaptively minimising the estimated virtual error signals \( \hat{e}_v(n) \) is now found by substituting Eq. (4.36) into the true cost function that needs to be minimised defined in Eq. (4.27). It is shown in Appendix A that this results in

\[
J(\hat{W}_o) = \| G_{\text{vs}} + G_{\text{vu}} \hat{W}_o G_{\text{xs}} \|_2^2
\]

\[
= J_{\text{min}} + \| \left[ G_{\text{vu},d}^* G_{\text{vs}}^* G_{\text{xs},ci}^* \right] + - G_{\text{vu},o} \hat{G}_{\text{vu},d}^* \hat{G}_{\text{vs}}^* G_{\text{xs},ci}^* \|_2^2,
\]

(4.37)

with \( J_{\text{min}} \) defined in Eq. (4.29) as the minimum value of the true cost function obtained when minimising the true virtual error signals. In the following, the value of the cost function defined in Eq. (4.37) is denoted by \( \bar{J}_{\text{min}} \triangleq J(\hat{W}_o) \) for notational convenience. Eq. (4.37) shows that \( \bar{J}_{\text{min}} = J_{\text{min}} \) if the estimates \( \hat{G}_{\text{vs}} \) and \( \hat{G}_{\text{vu}} \) of the virtual primary and secondary transfer path matrices defined in Eq. (4.34) are perfect. This equation also shows that \( \bar{J}_{\text{min}} > J_{\text{min}} \) if these estimates are not perfect. In other words, the optimal control performance that can in theory be obtained at the virtual locations when adaptively minimising an estimate of the virtual error signals as illustrated in
Fig. 4.3 is always less than or equal to the optimal control performance that can be obtained when adaptively minimising the true virtual error signals as illustrated in Fig. 2.7. Eq. (4.37) indicates that the degree of optimal control performance that is lost depends on the estimation accuracy of the adaptive LMS virtual microphone technique, which is to be expected.

### 4.3.3 Remote microphone technique

In this section, the optimal control performance that can in theory be obtained at the virtual locations when adaptively minimising an estimate \( \hat{e}_v(n) \) of the virtual error signals as illustrated in Fig. 4.3 is analysed, assuming this estimate is computed using the remote microphone technique \([104, 112]\) introduced in Section 3.4.\(^3\) The transfer function matrix \( G_{RMT} \) that describes the input-output behaviour of the remote microphone technique illustrated in Fig. 3.6 has been defined in Eq. (3.82) as

\[
G_{RMT} = \begin{bmatrix} H & G_{vu} - HG_{pu} \end{bmatrix}.
\]  

(4.38)

Using Eq. (4.38), it can be shown that the estimate \( \hat{e}_v(n) \) of the virtual error signals in the block diagram of the equivalent optimal feedforward control problem shown in Fig. 4.4 can be written as

\[
\hat{e}_v(n) = (HG_{ps} + G_{vs}WG_{xs}) s(n).
\]  

(4.39)

Using the expression for the estimate \( \hat{e}_v(n) \) of the virtual error signals derived in Eq. (4.39), an equivalent frequency domain expression of the cost function in Eq. (4.30) that is adaptively minimised in Fig. 4.3 is given by

\[
\hat{J} = \|HG_{ps} + G_{vs}WG_{xs}\|_2^2,
\]  

(4.40)

where Parseval’s Theorem and the fact that the signals \( s(n) \) are assumed to be white noise sequences with unit covariance have been used. The block diagram of the equivalent optimal feedforward control problem illustrated in Fig. 4.4 can therefore be redrawn as shown in Fig. 4.6. The block diagram in this figure can be compared to the block diagram of the optimal feedforward control problem illustrated in Fig. 2.3, where the true virtual error signals \( e_v(n) \) are minimised instead of their estimates. The only difference between these two block diagrams is that the virtual primary transfer path matrix \( G_{vs} \) in Fig. 2.3 has been replaced by an estimate in Fig. 4.6 which is computed as

\[
\hat{G}_{vs} = HG_{ps}.
\]  

(4.41)

---

\(^3\) Note that it is shown in Appendix B that the remote microphone technique is an equivalent transfer function description of the Kalman filter based spatially fixed virtual sensing algorithm presented in Theorem 3.4, and that the discussions presented in this section thus also apply to this algorithm.
The optimal feedforward control problem illustrated in Fig. 4.6 can thus be solved using the causal Wiener solution defined in Theorem 2.1. The causal Wiener solution \( \hat{W}_o \in \mathcal{RH}_{\infty}^{L \times K} \) that minimises the cost function in Eq. (4.40) is now given by

\[
\hat{W}_o = -G_{vu,o}^\dagger \left[ G_{vu,i}^* HG_{ps} G_{xs,i}^* \right] + G_{xs,co}^*.
\]  

When using the remote microphone technique as a virtual sensing algorithm in the implementation illustrated in Fig. 4.3, the response of the feedforward controller \( W \) is thus adapted such that it approximates the impulse response of the causal Wiener solution defined in Eq. (4.42).

The optimal control performance that can in theory be obtained at the virtual locations when adaptively minimising the estimated virtual error signals \( \hat{e}_v(n) \) as illustrated in Fig. 4.3 is now found by substituting Eq. (4.42) into the true cost function that needs to be minimised defined in Eq. (4.27). In Appendix A, it is shown that this results in

\[
J(\hat{W}_o) = \| G_{vs} + G_{vu,\hat{W}_o} G_{xs} \|^2_2 = J_{\text{min}} + \| [G_{vu,i}^* (G_{vs} - HG_{ps}) G_{xs,i}]^* \|^2_2.
\]  

In the following, the value of the cost function defined in Eq. (4.43) is again denoted by \( J_{\text{min}} \triangleq J(\hat{W}_o) \) for notational convenience. Eq. (4.43) shows that \( J_{\text{min}} = J_{\text{min}} \) if the estimate \( \hat{G}_{vs} = HG_{ps} \) of the virtual primary transfer path matrix is perfect. This equation also shows that \( J_{\text{min}} > J_{\text{min}} \) if this estimate is not perfect. In other words, the optimal control performance that can in theory be obtained at the virtual locations when adaptively minimising an estimate of the virtual error signals as illustrated in Fig. 4.3 is always less than or equal to the optimal control performance that can be obtained when adaptively minimising the true virtual error signals as illustrated in Fig. 2.7. Eq. (4.43) indicates that for the remote microphone technique, the degree of optimal control performance.
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that is lost depends on the minimum value of the term

\[ \| G_{vs} - H G_{ps} \|_2^2. \]  

(4.44)

As discussed in Section 3.4, the minimum value of this term defines the optimal broadband estimation performance of the remote microphone technique, and has been defined in Theorem 3.2 by deriving an optimal solution for the filter \( H \) for the filter problem illustrated in Fig. 3.7. Following the discussions presented after Theorem 3.2, the difference in the optimal control performance thus depends on the properties of the physical and virtual primary transfer path matrices \( G_{ps} \) and \( G_{vs} \). An example is now given to illustrate the presented discussions.

Example 4.1.

In this example, the transfer path matrices that describe the input-output behaviour of the active noise control system are single-input single-output transfer functions arbitrarily chosen as

\[
\begin{bmatrix}
G_{pu} & G_{ps} \\
G_{vu} & G_{vs}
\end{bmatrix} = \begin{bmatrix}
0.0512 z - 0.04871 & 0.04746 z - 0.05245 \\
\frac{z^2 - 1.997 z + 0.999}{z^2 - 1.997 z + 0.999} & \frac{z^2 - 1.997 z + 0.999}{z^2 - 1.997 z + 0.999} \\
0.09866 z - 0.1012 & 0.1037 z - 0.09616 \\
\frac{z^2 - 1.997 z + 0.999}{z^2 - 1.997 z + 0.999} & \frac{z^2 - 1.997 z + 0.999}{z^2 - 1.997 z + 0.999}
\end{bmatrix}.
\]  

(4.45)

To simplify the presented example, the detector transfer path is thus assumed to be \( G_{xs} = 1 \) in Figs 2.3 and 4.6. The transfer paths in Eq. (4.45) share the same denominator corresponding to a complex pole pair with an eigenfrequency of 1 Hz and a damping ratio of \( \zeta = 0.01 \), where a sample frequency of \( f_s = 20 \) Hz is employed. To illustrate the presented theory, the virtual secondary transfer path \( G_{vu} \) has a non-minimum-phase zero at \( z = 1.0253 \), and the physical primary transfer path \( G_{ps} \) has a non-minimum-phase zero at \( z = 1.1053 \). An inner-outer factorisation of \( G_{vu} \), and an outer-inner factorisation of \( G_{ps} \), are given by [130]

\[
\begin{bmatrix}
G_{vu,o} & G_{vu,i} \\
G_{ps,o} & G_{ps,i}
\end{bmatrix} = \begin{bmatrix}
0.1012 z^2 - 0.09866 z & 0.9753 z - 1 \\
\frac{z^2 - 1.997 z + 0.999}{z^2 - 1.997 z + 0.999} & \frac{z^2 - 0.9753 z}{z^2 - 0.9753 z} \\
0.05245 z^2 - 0.04746 z & 0.9047 z - 1 \\
\frac{z^2 - 1.997 z + 0.999}{z^2 - 1.997 z + 0.999} & \frac{z^2 - 0.9047 z}{z^2 - 0.9047 z}
\end{bmatrix}.
\]  

(4.46)

Adaptively minimising the true virtual error signal

First, an optimal controller \( W_o \) is calculated that minimises the true virtual error signal \( e_v(n) \) in Fig. 2.3. Using Theorem 2.1, this results in

\[ W_o = \frac{0.5387 z - 0.6214}{z - 0.9753}. \]  

(4.47)
The minimum value of the true cost function $J$ in Eq. (4.27) can be calculated using Eq. (4.29), which results in

$$J_{\text{min}} = \|G_{vs} + G_{vu}W_o\|_2^2 = \| [G_{vu,i}^* G_{vs}] - \|^2 = 0.5040,$$  \hspace{1cm} (4.48)

where use has been made of the fact that the complementary inner factor $G_{vu,i}^\perp = 0$, because the virtual secondary transfer path $G_{vu}$ is a single-input single-output system [130]. Since $\|G_{vs}\|_2^2 = 16.2467$, the virtual primary disturbance has been attenuated, but perfect attenuation, such that $J_{\text{min}} = 0$, is not possible. As explained after Theorem 2.1, this is caused by the non-minimum-phase zero in the virtual secondary transfer path $G_{vu}$, located outside the unit circle at $z = 1.0253$.

Adaptively minimising an estimate of the virtual error signal

Next, an optimal controller $\hat{W}_o$ is calculated that minimises the estimated virtual error signal $\hat{e}_v(n)$ in Fig. 4.6. To calculate this controller using Eq. (4.42), a causal Wiener solution for the filter $H$ first needs to be computed using Theorem 3.2. This results in

$$H_o = -\frac{3.392z^2 + 3.381}{z - 0.9047},$$ \hspace{1cm} (4.49)

with the minimum value of the cost function in Eq. (3.89) given by

$$J_{l,\text{min}} = \|G_{vs} - H_o G_{ps}\|_2^2 = \| [G_{vs} G_{ps,ci}^*] - \|^2 = 0.3859,$$ \hspace{1cm} (4.50)

where use has been made of the fact that the complementary co-inner factor $G_{ps,ci}^\perp = 0$, because the physical primary transfer path $G_{ps}$ is a single-input single-output system [130]. Since $J_{l,\text{min}}$ is not equal to zero in Eq. (4.50), the estimate of the virtual primary transfer path $\hat{G}_{vs} = H G_{ps}$ is not perfect. This can also be seen in Fig. 4.7(a), where Bode diagrams of the virtual primary transfer path and its estimate are shown. As explained after Theorem 3.2, the estimate is not perfect due to the non-minimum-phase zero in the physical primary transfer path $G_{ps}$, located outside the unit circle at $z = 1.1053$.

Using the causal Wiener solution for the filter $H$ defined in Eq. (4.49), the controller $\hat{W}_o$ can now be calculated using Eq. (4.42), which results in

$$\hat{W}_o = \frac{2.108z^2 - 4.242z + 2.13}{z^2 - 1.88z + 0.8824}.$$ \hspace{1cm} (4.51)

Bode diagrams of the controllers $\hat{W}_o$ and $W_o$ are shown in Fig. 4.7(b). This figure shows that these controllers are not equal due to the fact that the estimate $\hat{G}_{vs} = H G_{ps}$ of the virtual primary transfer path is not perfect. As can be seen from Eq. (4.43), this introduces an extra term into the minimum value of the true cost function that needs to be minimised defined in Eq. (4.27), such that this minimum value is now given by

$$\tilde{J}_{\text{min}} = J_{\text{min}} + \|[G_{vu,i} (G_{vs} - H G_{ps})]_+\|^2 = 0.5040 + 0.1389 = 0.6429.$$ \hspace{1cm} (4.52)
Thus, the control performance that theoretically can be achieved at the virtual location is reduced by 0.1389 when adaptively minimising the estimated virtual error signal $\hat{e}_v(n)$ as illustrated in Fig. 4.3 instead of the true virtual error signal $e_v(n)$ as illustrated in Fig. 2.7. The obtained control performance for both cases is illustrated in Fig. 4.7(c).

### 4.3.4 Virtual microphone arrangement

In this section, the optimal control performance that can in theory be obtained when the virtual microphone arrangement [25] introduced in Section 3.5 is employed in Fig. 4.3 to compute an estimate $\hat{e}_v(n)$ of the virtual error signals is analysed. The virtual microphone arrangement can be seen as a simplified version of the remote microphone arrangement.
technique [104, 112], with the filter $H$ assumed equal to the unity matrix $I$. The discussions presented in the previous section can thus be applied to the virtual microphone arrangement by substituting $H = I$ in the presented derivations. Thus, from Eq. (3.85), the estimate $\hat{e}_v(n)$ of the virtual error signals in the block diagram of the equivalent optimal feedforward control problem shown in Fig. 4.4 is now given by

$$\hat{e}_v(n) = (G_{ps} + G_{vu}WG_{xs})s(n),$$  \hspace{1cm} (4.53)

and the block diagram of the equivalent optimal feedforward control problem illustrated in Fig. 4.6 can now be redrawn as shown in Fig. 4.8.

Figure 4.8: Block diagram for determining the optimal broadband feedforward control performance that can be obtained at the virtual sensors when using the virtual microphone arrangement in Fig. 4.3, with $S$ disturbance sources, $L$ control sources, $K$ reference sensors, and $M_v$ spatially fixed virtual sensors.

Fig. 4.8 illustrates that when the virtual microphone arrangement is used as a virtual sensing algorithm, the cost function that is adaptively minimised can be written in the frequency domain as

$$\hat{J} = \|G_{ps} + G_{vu}WG_{xs}\|_2^2.$$  \hspace{1cm} (4.54)

Using Theorem 2.1, a causal Wiener solution $\hat{W}_o \in \mathcal{RH}_{M_v}^{M_v}$ that minimises this cost function is given by

$$\hat{W}_o = -G_{vu}^+[G_{vu,i}^*G_{ps}G_{xs,ci}^*] + G_{xs,co}^*.$$  \hspace{1cm} (4.55)

Substituting Eq. (4.55) into the true cost function that needs to be minimised defined in Eq. (4.27) results in

$$\hat{J}_{\text{min}} = \|G_{ps} + G_{vu}\hat{W}_oG_{xs}\|_2^2 = J_{\text{min}} + \|[G_{vu,i}^*(G_{vs} - G_{ps})G_{xs,ci}^*]\|_2^2,$$  \hspace{1cm} (4.56)

which can also simply be derived by replacing the filter $H$ in Eq. (4.43) with the unity matrix $I$. Eq. (4.56) shows that $\hat{J}_{\text{min}} = J_{\text{min}}$ if the physical primary transfer path matrix $G_{ps}$ is equal to the virtual primary transfer path matrix $G_{vs}$. Eq. (4.56) also indicates that $\hat{J}_{\text{min}} > J_{\text{min}}$ if these primary transfer path matrices are not equal, such that $G_{ps} \neq G_{vs}$. In other words, the optimal control performance that can in theory be obtained at the
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Virtual locations when adaptively minimising an estimate of the virtual error signals as illustrated in Fig. 4.3 is always less than or equal to the optimal control performance that can be obtained when adaptively minimising the true virtual error signals as illustrated in Fig. 2.7. Eq. (4.56) indicates that for the virtual microphone arrangement, the degree of optimal control performance that is lost depends on the difference between the physical and virtual primary transfer path matrices \( G_{ps} \) and \( G_{vs} \), respectively. This is to be expected because the estimation performance of the virtual microphone arrangement depends on the validity of the assumption of equal primary disturbances at the physical and virtual sensors.

### 4.3.5 Virtual secondary transfer path equalisation

Kuo et al. [67] suggested another adaptive feedforward control method that aims to minimise the virtual error signals without directly measuring them during real-time control. The proposed algorithm is based on a method referred to as virtual secondary transfer path equalisation. As in the virtual microphone arrangement [27], the primary disturbances at the physical and virtual sensors are assumed to be equal, and an equal number of physical and virtual sensors is therefore generally used. Fig. 4.9 shows a block diagram of the proposed algorithm.

Figure 4.9: Block diagram of the virtual sensing algorithm based on virtual secondary transfer path equalisation, with \( L \) control sources, \( K \) reference sensors, \( M_p \) physical sensors, and \( M_v \) spatially fixed virtual sensors, with \( M_p = M_v \) [66, 67].

Fig. 4.9 illustrates that the control signal \( u(n) \) is pre-filtered by a compensation filter \( H_u \in \mathbb{RH}^{L \times L} \) before being sent to the control sources, with the pre-filtered control signal \( \tilde{u}(n) \) given by

\[
\tilde{u}(n) = H_u u(n).
\] (4.57)
The compensation filter in Fig. 4.9 is designed to reshape the physical secondary transfer path matrix $G_{pu}$ such that an estimate $\hat{G}_{vu}$ of the virtual secondary transfer path matrix is obtained, with

$$\hat{G}_{vu} = H_u G_{pu}. \quad (4.58)$$

The estimated virtual secondary transfer path matrix $\hat{G}_{vu}$ is indicated by the grey box in Fig. 4.9. Furthermore, as in the virtual microphone arrangement, it is assumed that the virtual primary disturbance $d_v(n)$ is equal to the physical primary disturbance $d_p(n)$. Under this assumption, the block diagram shown in Fig. 4.9 is thus an approximation of the multi-channel filtered-x LMS algorithm shown in Fig. 2.7 on page 64, which can be implemented if the virtual error signals are directly measured during real-time control. The algorithm shown in Fig. 4.9 thus aims to minimise the virtual error signals by reshaping the physical secondary transfer path matrix using the compensation filter $H_u$, while assuming that the primary disturbances at the physical and virtual sensors are equal. The compensation filter is designed in a preliminary identification stage in which physical sensors are temporarily located at the virtual locations [67]. A block diagram of the resulting optimal filtering problem is depicted in Fig. 4.10.

From Fig. 4.10, the cost function $J_ε(H_u)$ that needs to be minimised to derive an optimal solution for the compensation filter $H_u$ in Fig. 4.9 can be written in the frequency domain as

$$J_ε(H_u) = \|G_{vu} - G_{pu}H_u\|_2^2. \quad (4.59)$$

From the causal Wiener solution defined in Theorem 2.1, a causal Wiener solution $H_{wo} \in \mathcal{RH}_L^{L \times L}$ that minimises the cost function in Eq. (4.59) is defined as

$$H_{wo} = G^+_{pu,\rho} \left[ G_{pu,\rho}^* G_{vu} \right]_+,$$  \quad (4.60)

and the minimum value of the cost function in Eq. (4.59) is given by

$$\|G_{vu} - G_{pu}H_{wo}\|_2^2 = \|G_{pu,\rho}^{\perp,\ast} G_{vu}\|_2^2 + \|\left[ G_{pu,\rho}^* G_{vu} \right]_+\|_2^2. \quad (4.61)$$
Once an optimal solution for the filter $H_u$ has been derived, the optimal control performance that can in theory be obtained at the virtual locations when using the adaptive algorithm shown in Fig. 4.9 can be computed. To derive the optimal control performance, the estimate $\hat{e}_v(n)$ of the virtual error signals in Fig. 4.9 can be written as

$$\hat{e}_v(n) = (G_{ps} + G_{pu}H_uWG_{xs})s(n) = (G_{ps} + \hat{G}_{vu}WG_{xs})s(n),$$

with $\hat{G}_{vu}$ the estimated virtual secondary transfer path matrix defined in Eq. (4.58). The cost function that is adaptively minimised in Fig. 4.9 can now be written in the frequency domain as

$$\hat{J} = \|G_{ps} + \hat{G}_{vu}WG_{xs}\|^2_2.$$  

(4.63)

Using Theorem 2.1, a causal Wiener solution that minimises the cost function in Eq. (4.63) is now given by

$$\hat{W}_o = -\hat{G}_{vu}^+\left[\hat{G}_{vu}^*G_{ps}G_{xs}^*\right] + G_{xs}^*co.$$  

(4.64)

If the optimal solution $H_{uo}$ for the compensation filter defined in Eq. (4.60) is such that the estimate $\hat{G}_{vu} = G_{pu}H_u$ of the virtual secondary transfer path matrix is perfect, the controller in Eq. (4.64) reduces to the controller defined in Eq. (4.55) for the virtual microphone arrangement. In this instance, the optimal control performance that can theoretically be achieved at the virtual locations with the algorithm shown in Fig. 4.9 is equivalent to the optimal control performance that can theoretically be achieved when using the virtual microphone arrangement [27] in the algorithm illustrated in Fig. 4.3. This indicates that these two virtual sensing for active noise control methods are very similar, which is due to the fact that both algorithms assume that the primary disturbances at the physical and virtual sensors are equal.

### 4.4 Optimal feedback control at virtual sensors

In Section 2.4.2, a solution to the optimal feedback control problem, in which the true virtual error signals are assumed to be directly measured during real-time control, has been presented. It is now assumed that this is no longer the case such that the feedback control signal $u(n)$ can no longer be computed as illustrated in Fig. 2.4 as

$$u(n) = Ce_v(n).$$

(4.65)

Instead, the feedback control signal can only be computed from the physical error signals $e_p(n)$ which are directly measured by the physical sensors during real-time control, such that

$$u(n) = Ce_p(n).$$

(4.66)
The aim of the resulting optimal feedback control problem is to design an optimal solution for the feedback controller $C$ in Eq. (4.66) that minimises the true virtual error signals $e_v(n)$. A block diagram of the resulting optimal feedback control at virtual sensors problem is shown in Fig. 4.11.

![Block diagram of the optimal feedback control at virtual sensors problem](image)

Figure 4.11: Block diagram of the optimal feedback control at virtual sensors problem, with $S$ disturbance sources, $L$ control sources, $M_p$ physical sensors, and $M_v$ spatially fixed virtual sensors.

The feedback control problem illustrated in Fig. 4.11 has been analysed previously by Rafaely et al. [108] in the context of active noise control in an active headrest. A robust feedback control solution was designed assuming that the primary disturbances at the physical and virtual sensors were equal. In the following, an optimal feedback control solution is presented assuming that the primary disturbances at the physical and virtual sensors are not necessarily equal. This solution is derived by showing that the optimal feedback control at virtual sensors problem illustrated in Fig. 4.11 is equivalent to the general feedforward/feedback control problem introduced in Section 2.4.3.

### 4.4.1 Causal Wiener solution

Comparing the block diagram of the optimal feedback control at virtual sensors problem shown in Fig. 4.11 to the block diagram of the general feedforward/feedback control problem shown in Fig. 2.6 on page 62 indicates that these two optimal control problems are equivalent. This can easily be shown by replacing the physical secondary transfer path matrix $G_{pu}$ in Fig. 4.11 with the intrinsic feedback path matrix $G_{ux}$, and the physical primary transfer path matrix $G_{ps}$ with the detector transfer path matrix $G_{xs}$. The physical sensors in the optimal feedback control at virtual sensors problem therefore simply act as reference sensors, and the physical error signals $e_p(n)$ in Fig. 4.11 can thus be viewed as feedforward reference signals that are distorted by intrinsic feedback from the control source via the physical secondary transfer path matrix $G_{ps}$. As discussed in Section 2.4.1, the idea of the optimal feedback control at virtual sensors approach is thus that the physical error signals in Fig. 4.11 contain time-advanced information that can...
be used to compute a causal estimate of the virtual primary disturbances $d_v(n)$. Note that this is also the aim of the virtual sensing algorithms introduced in Chapter 3. Using the theory presented in Section 2.4.3, the optimal feedback control at virtual sensors problem can now be reformulated as an equivalent feedforward control problem as shown in Fig. 4.12.

The equivalent feedforward control problem illustrated in Fig. 4.12 has been derived using the Youla parameterisation introduced in Section 2.4.2. Thus, the Youla parameterisation can be used in the optimal feedback control at virtual sensors problem shown in Fig. 4.11 to parameterise all internally stabilising controllers $C \in \mathcal{RH}_{L \times M_p}$ as

$$C = (I + WG_{pu})^{-1}W, \quad \forall W \in \mathcal{RH}_{L \times M_p}.$$  \hspace{1cm} (4.67)

With the controller $C$ parameterised as in Eq. (4.67), the virtual error signals in Fig. 4.11 can be written as

$$e_v(n) = (G_{vs} + G_{vu}WG_{ps})s(n),$$  \hspace{1cm} (4.68)

which results in the equivalent feedforward control problem depicted in Fig. 4.12. The virtual error signals are now independent of the physical secondary transfer path $G_{pu}$, indicating that the distortion of the physical error signals by intrinsic feedback from the control sources has been cancelled. The equivalent feedforward control problem shown in Fig. 4.12 can now be solved using Theorem 2.1 on page 57. The optimisation problem for the optimal feedback control at virtual sensors problem is thus to minimise

$$J = \|G_{vs} + G_{vu}WG_{ps}\|_2^2,$$  \hspace{1cm} (4.69)

subject to $W \in \mathcal{RH}_{L \times M_p}$. From Theorem 2.1, a solution to this problem is defined by

$$W_0 = -G_{vu,0}^\dagger \left[G_{vu}^* G_{vs} G_{ps}^* \right]_{+} G_{ps,0}^{-1},$$  \hspace{1cm} (4.70)

Figure 4.12: Block diagram of the optimal feedback control at virtual sensors problem reformulated as an equivalent feedforward control problem, with $S$ disturbance sources, $L$ control sources, $M_p$ physical sensors, and $M_v$ spatially fixed virtual sensors.
and the minimum value of the cost function in Eq. (4.69) is given by

\[
J_{\text{min}} = \| G_{vs,G_{ps,ci}} \|^2 + \| G_{vu,G_{ps,ci}}^* \|^2 + \| G_{vu,G_{ps,ci}}^* G_{ps,ci}^* \|^2 - 2.
\]  

(4.71)

Note that this value does not depend on the physical secondary transfer path matrix \( G_{pu} \). The terms on the right-hand side in Eq. (4.71) can now be interpreted using the discussions presented after Theorem 2.1.

**Interpretation of minimum value of cost function**

If the number of physical error sensors \( M_p \) is smaller than the number of disturbance source signals \( S \), the term \( G_{ps,ci}^* \) may not be equal to zero. The first term on the right-hand side in Eq. (4.71) then contributes to the minimum value of the cost function, provided that \( G_{vs} G_{ps,ci}^* \) is not equal to zero [34]. In this case, there are disturbances that contribute to the virtual primary disturbances at the virtual sensors, but these disturbances are not measured, or observed, at the physical sensors. These parts of the virtual primary disturbances can thus not be predicted from the physical error signals \( e_p(n) \), and can therefore not be controlled. This is related to the concept of unobservable modes of the active noise control system [34, 130]. Note that the first term on the right-hand side in Eq. (4.71) also occurs in Theorem 3.2 where the optimal estimation performance of the remote microphone technique [104, 112] is defined. This is not a coincidence since the aim of a virtual sensing algorithm is also to predict the virtual primary disturbances using the time-advanced information contained in the physical error signals.

If the number of control sources \( L \) is smaller than the number of virtual sensors \( M_v \), the term \( G_{vu,ci}^* \) may not be equal to zero. The second term on the right-hand side in Eq. (4.71) then contributes to the minimum value of the cost function, provided that \( G_{vu} G_{ps,ci}^* \) is not equal to zero [34]. In this case, there are disturbances that contribute to the primary disturbances at the virtual sensors, but these disturbances cannot be controlled with the chosen actuator configuration. This is related to the concept of uncontrollable modes of the active noise control system [34, 130].

The first and second term that contribute to the minimum value of the cost function in Eq. (4.71) are related to the actuator and sensor configurations. The third term is related to the fact that the controller is constrained to be causal [34]. This term is determined by delays and non-minimum-phase zeros in the virtual secondary transfer path matrix and the physical primary transfer path matrix, which result in non-causal terms in \( G_{vu,ci}^* \) and \( G_{ps,ci}^* \). These non-causal terms contribute to the third term of the minimum value of the cost function defined in Eq. (4.71). Note that the non-causal terms in \( G_{ps,ci}^* \) also limit the optimal estimation performance of the remote microphone technique [104, 112] defined in Theorem 3.2. Again, this is not a coincidence since the aim of a virtual sensing algorithm is also to causally estimate the virtual primary.
disturbances using the time-advanced information contained in the physical error signals.

### 4.4.2 State-space solution

In the previous section, the optimal feedback control at virtual sensors problem has been solved using the causal Wiener filter solution defined in Theorem 2.1, after reformulating this problem as an equivalent feedforward control problem using an internal model control approach. The optimal solution for the feedback controller $C$, which is found by substituting the optimal feedforward controller defined in Eq. (4.70) into Eq. (4.67), can be calculated given models of the transfer paths in either transfer function matrix or state-space form. When models of the transfer paths are given in state-space form, the theory presented in Appendix B can be used to derive a state-space solution for the optimal feedback controller.

To derive state-space solutions for optimal feedback control problems, another approach is often followed in the literature [3, 20, 60, 130], which is based on Kalman filtering and linear quadratic (LQ) control. An optimal state-space solution [18, 19] for the optimal feedback control at virtual sensors problem can be derived using a general control problem formulation [114] as shown in Fig. 4.13.

![Generalised plant formulation for the optimal feedback control at virtual sensors problem](image)

**Figure 4.13:** Generalised plant formulation for the optimal feedback control at virtual sensors problem, with $S$ disturbance sources, $L$ control sources, $M_p$ physical sensors, and $M_v$ spatially fixed virtual sensors.

The generalised plant $G$ illustrated in Fig. 4.13 is defined in transfer function matrix form as

$$ G = \begin{bmatrix} G_{pu} & G_{ps} \\ G_{vu} & G_{vs} \end{bmatrix}, \quad (4.72) $$

with a state-space realisation given by

$$ \begin{bmatrix} z(n+1) \\ e_p(n) \\ e_v(n) \end{bmatrix} = \begin{bmatrix} A & B_u & B_s \\ C_p & 0 & D_{ps} \\ C_v & D_{vu} & D_{vs} \end{bmatrix} \begin{bmatrix} z(n) \\ u(n) \\ s(n) \end{bmatrix}. \quad (4.73) $$
The disturbance source signal $s(n)$ in Fig. 4.13 is the exogenous input, $u(n)$ the control input, the physical error signal $e_p(n)$ the sensed output, and the virtual error signal $e_v(n)$ the exogenous output. As stated in Skogestad and Postlethwaite [114], the overall control objective in the general control problem formulation is to find a controller $C$, which, based on the information in the sensed output $e_p(n)$, generates a control signal $u(n)$ that counteracts the disturbance source signal $s(n)$ on the exogenous output $e_v(n)$, thereby minimising the closed-loop norm from $s(n)$ to $e_v(n)$. This is exactly what the objective is here, since the aim is to counteract the virtual primary disturbances at the virtual sensors caused by the disturbance source signal $s(n)$, by generating an optimal feedback control signal $u(n)$ based on the physical error signals $e_p(n)$. The following theorem defines a state-space solution for the controller $C$ that minimises the cost function defined in Eq. (4.69).

**Theorem 4.1** (LQG control at virtual sensors [18, 19].

Let a state-space realisation of the plant be given by Eq. (4.73), and define the following matrices

$$
\begin{align*}
\Tilde{Q}_s &= B_sB_s^T, & \Tilde{R}_p &= D_{ps}D_{ps}^T, & \Tilde{S}_{ps} &= D_{ps}B_s^T, \\
\Tilde{Q}_v &= C_v^TC_v, & \Tilde{R}_{vu} &= D_{vu}D_{vu}^T, & \Tilde{S}_{vu} &= D_{vu}C_v.
\end{align*}
$$

Furthermore, let

- the pair $(C_p, A)$ be detectable, and the pair $(A, B_u)$ be stabilisable;
- $\Tilde{R}_p > 0$, $\Tilde{Q}_s - \Tilde{S}_{ps}^T\Tilde{R}_p^{-1}\Tilde{S}_{ps} \geq 0$, and $\Tilde{R}_{vu} > 0$, $\Tilde{Q}_v - \Tilde{S}_{vu}^T\Tilde{R}_{vu}^{-1}\Tilde{S}_{vu} \geq 0$;
- the pair $(A - \Tilde{S}_{ps}^T\Tilde{R}_p^{-1}C_p, \Tilde{Q}_s - \Tilde{S}_{ps}^T\Tilde{R}_p^{-1}\Tilde{S}_{ps})$ has no uncontrollable modes on the unit circle, and the pair $(\Tilde{Q}_v - \Tilde{S}_{vu}^T\Tilde{R}_{vu}^{-1}\Tilde{S}_{vu}, A - B_u\Tilde{R}_{vu}^{-1}\Tilde{S}_{vu})$ has no unobservable modes on the unit circle.

Then the optimal state-space controller that minimises Eq. (4.69) is given by

$$
\begin{bmatrix}
\hat{z}(n|n-1) \\
u(n)
\end{bmatrix} =
\begin{bmatrix}
A + B_uK_p^0C_p - B_uF_{vu} - K_{ps}C_p \\
F_{vu} - K_{ps}^0C_p
\end{bmatrix}^{-1}
\begin{bmatrix}
A + B_uK_p^0C_p - B_uF_{vu} - K_{ps}C_p \\
F_{vu} - K_{ps}^0C_p
\end{bmatrix}
\begin{bmatrix}
\hat{z}(n|n-1) \\
e_p(n)
\end{bmatrix},
$$

with $\hat{z}(n|n-1)$ the estimate of $z(n)$ given the measurements $e_p(i)$ up to $i = n - 1$, and with

$$
\begin{align*}
F_{vu} &= (B_u^TP_{vu}B_u + \Tilde{R}_{vu})^{-1}(B_u^TP_{vu}A + \Tilde{S}_{vu}), \\
F_{vu}^0 &= (B_u^TP_{vu}B_u + \Tilde{R}_{vu})^{-1}(B_u^TP_{vu}B_s + D_{vu}D_{vu}^T),
\end{align*}
$$

and

$$
\begin{align*}
K_{ps} &= (AP_{ps}C_p^T + \Tilde{S}_{ps})(C_pP_{ps}C_p^T + \Tilde{R}_p)^{-1}, \\
K_{ps}^0 &= (F_{vu}P_{ps}C_p^T + F_{vu}^0D_{ps}^T)(C_pP_{ps}C_p^T + \Tilde{R}_p)^{-1},
\end{align*}
$$

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with $P_{vu} = P_{vu}^\top > 0$ and $P_{ps} = P_{ps}^\top > 0$ the unique stabilising solution to the following DAREs

$$
P_{vu} = A^T P_{vu} A - (A^T P_{vu} B_u + \tilde{S}_{vu}^T) (B_u^T P_{vu} B_u + R_{vu})^{-1} (A^T P_{vu} B_u + \tilde{S}_{vu}^T)^T + \tilde{Q}_v,
$$

$$
P_{ps} = A P_{ps} A^T - (A P_{ps} C_p^T + \tilde{S}_{ps}^T) (C_p P_{ps} C_p^T + \tilde{R}_p)^{-1} (A P_{ps} C_p^T + \tilde{S}_{ps}^T)^T + \tilde{Q}_s.
$$

**Proof.** See, for instance, Chen and Francis [18, 19].

It has been shown by Fraanje [34] that the state-space solution in Eq. (4.75) is equivalent to the state-space solution that can be derived from Eq. (4.70) when models of the transfer paths in Eq. (4.72) are given in state-space form.

Note that the Kalman gain matrix $K_{ps}$ defined in Theorem 4.1 also appears in the Kalman filter based spatially fixed virtual sensing algorithm presented in Theorem 3.4. Again, this is not a coincidence because the state-space solution in Theorem 4.1 is usually derived in two stages using Kalman filtering and linear quadratic control (LQG). In the first stage, a Kalman filter is designed that computes the predicted state estimate $\hat{z}(n|n-1)$ in Eq. (4.75) given the physical error signals $e_p(i)$ up to $i = n - 1$. As discussed in Section 3.7, this stage also needs to be performed to compute the Kalman filter based spatially fixed virtual sensing algorithm presented in Theorem 3.4.

### 4.5 Conclusion

In Chapter 2, active noise control algorithms have been introduced that can be used to create local zones of quiet at a number of virtual locations that are spatially fixed within the sound field. The presented optimal feedback and adaptive feedforward control approaches cannot be implemented without directly measuring the virtual error signals during real-time control. This is because these control approaches require the feedback information contained in the virtual error signals to compute an appropriate control signal. To implement these control approaches without directly measuring the virtual error signals during real-time control, an estimate of the virtual error signals can instead be used as a feedback signal. This estimate can be computed using one of the spatially fixed virtual sensing algorithms introduced in Chapter 3. In this chapter, these spatially fixed virtual sensing algorithms have therefore been combined with the adaptive feedforward control algorithms introduced in Chapter 2. The optimal narrowband and broadband control performance that can be obtained at the virtual locations when adaptively minimising an estimate of the virtual error signals instead of the true virtual error signals has been derived. It has been shown that this optimal control performance is always smaller than or equal to the optimal control performance that can be obtained when adaptively minimising the true virtual error signals during real-time control. The degree of optimal control performance that is lost is determined by the estimation performance of the spatially fixed virtual sensing algorithm that is used, and has been
derived for the various virtual sensing algorithms introduced in Chapter 3. It has also been shown that the *optimal feedback control at virtual sensors problem* can be solved by recognising that this problem is equivalent to the *general feedforward/feedback control problem* introduced in Chapter 2.
Chapter 5

Active noise control at moving physical and virtual sensors

5.1 Introduction

The local active noise control algorithms introduced in Chapter 2 and the spatially fixed virtual sensing algorithms presented in Chapter 3 can be combined as described in Chapter 4 to create zones of quiet at a number of virtual locations that are spatially fixed within the sound field. Because an observer is very likely to move their head, the desired locations of the zones of quiet are generally moving through the sound field rather than being spatially fixed. The performance of a local active noise control system can thus be improved by creating moving zones of quiet that track the observer’s ears. In this chapter, methods are therefore proposed that aim to create moving zones of quiet that track the desired locations of maximum attenuation.

In a practical application of the proposed methods, an important issue that would need to be addressed is how to determine the desired locations of maximum attenuation, i.e. the moving virtual locations. This could for instance be done using a 3D head tracking system based on camera vision or on ultrasonic position sensing such as the Logitech® head tracker. However, the issue of determining the desired locations of the zones of quiet is beyond the scope of this thesis. The aim here is therefore to develop methods that can be used to create zones of quiet at a number of virtual locations that move through the sound field, while assuming that an exact measurement of these locations is available.

In Section 5.2, the considered local active noise control problem is introduced and the assumptions made to analyse this problem are presented. The analysis is based on the standard state-space model of the considered active noise control system introduced in Chapter 2. The covariance properties of the stochastic input signals of this model are again defined as these properties are used frequently throughout this chapter. It is
noted that the virtual primary disturbances that need to be attenuated are non-stationary signals due to the movement of the virtual locations through the sound field. When the primary disturbances are non-stationary, the common approach in active noise control is to use an adaptive control algorithm [25, 68]. Adaptive control algorithms are able to track the changes in the statistical properties of the primary disturbances and adjust the controller accordingly.

The adaptive feedforward control algorithms introduced in Chapter 2 can be used to minimise the true virtual error signals at a number of virtual locations that are spatially fixed within the sound field. These adaptive algorithms can thus be implemented when the virtual error signals are directly measured during real-time control by a number of spatially fixed physical sensors. To implement these adaptive feedforward control algorithms for the case considered in this chapter, the following two issues need to be addressed.

The first issue that needs to be addressed is that the virtual locations are moving through the sound field rather than being spatially fixed. In Section 5.3, it is therefore assumed that the virtual error signals are directly measured during real-time control by a number of moving physical sensors that track the desired moving locations of maximum attenuation. The adaptive feedforward control algorithms introduced in Chapter 2 are then modified to minimise the true virtual error signals at the desired moving locations of maximum attenuation. Effectively, the aim of the adaptive feedforward control implementation developed in Section 5.3 is thus to attenuate the non-stationary primary disturbances at a number of moving physical sensors that track the desired locations of maximum attenuation. The proposed method will be implemented on an acoustic duct arrangement in Chapter 9, where a moving zone of quiet at a moving physical sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances.

The second issue that needs to be addressed is that the virtual error signals are not directly measured during real-time control and the true virtual error signals are therefore not available to adapt the feedforward controller as described in Chapter 2. In Section 5.4, the adaptive feedforward control implementation developed in Section 5.3 is therefore modified to allow for virtual error signals that are not directly measured during real-time control. The difference with the implementation developed in Section 5.3 is that an estimate of the virtual error signals is adaptively minimised instead of the true virtual error signals directly measured by moving physical sensors that track the desired moving locations of maximum attenuation. To obtain this estimate, a moving virtual sensing method is developed that is able to compute an estimate of the virtual error signals when the virtual locations are moving through the sound field. The previously proposed spatially fixed virtual sensing algorithms analysed in Chapter 3 are extended to the case of virtual locations that are moving through the sound field rather
than being spatially fixed. It is then shown that an optimal solution to the moving virtual sensing problem can be derived using Kalman filtering theory [60]. This approach has also been used in Chapter 3 to derive an optimal solution to the spatially fixed virtual sensing problem. The proposed method will be implemented on an acoustic duct arrangement in Chapter 10, where a moving zone of quiet at a virtual sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances.

5.2 Problem description and assumptions

In this section, the considered active noise control problem is introduced and the assumptions made to analyse this problem are presented. The analysis is based on the standard state-space model of the considered active noise control system introduced in Chapter 2. The covariance properties of the stochastic input signals of this model are again defined as these properties are used frequently throughout this chapter.

5.2.1 Desired moving locations of zones of quiet

It is assumed that a perfect measurement of the desired locations of the zones of quiet, i.e. the moving virtual locations, is available during real-time control. These locations are contained in a time-variant matrix of moving virtual locations \( X_v(n) \in \mathbb{R}^{3 \times M_v} \), which is defined as

\[
X_v(n) = \begin{bmatrix}
x_{v1}(n) & x_{v2}(n) & \ldots & x_{vM_v}(n)
\end{bmatrix},
\]

(5.1)

where each column \( x_{vm(n)} \in \mathbb{R}^3 \) contains the time-variant three-dimensional spatial coordinates of the \( m_{v}^{th} \) moving virtual location with respect to some reference frame, such that

\[
x_{vm(n)} = \begin{bmatrix}
x_{vm_x(n)} & y_{vm_y(n)} & z_{vm_z(n)}
\end{bmatrix}^T.
\]

(5.2)

It will be assumed that the \( M_v \) moving virtual locations each stay within confined regions of the sound field which will be called the target zones. To illustrate these definitions, consider the practical example of the active control of engine-induced noise inside a mining vehicle cabin illustrated in Fig. 5.1. The aim in this practical example is to reduce the primary disturbances at the driver’s ears, such that \( M_v = 2 \) desired moving locations of maximum attenuation can be identified. These locations are indicated by the grey dots in Fig. 5.1, and are contained in the matrix of moving virtual locations defined from Eq. (5.1) as

\[
X_v(n) = \begin{bmatrix}
x_{v1}(n) & x_{v2}(n) \\
y_{v1}(n) & y_{v2}(n) \\
z_{v1}(n) & z_{v2}(n)
\end{bmatrix}.
\]

(5.3)
Figure 5.1: Illustration of the active control of engine-induced noise inside a mining vehicle cabin using two moving virtual sensors \( \hat{x}_1(n) \) that track the driver’s ears. The driver’s ears are assumed to stay within the indicated target zones during real-time control.

The spatial coordinates defined by Eq. (5.3) are time-variant due to the possible movement of the driver’s head. During operation, the driver’s ears will most likely only move throughout relatively small regions of the mining vehicle cabin, which are defined by the two target zones illustrated in Fig. 5.1. The active noise control system is thus designed to only work over the regions in which the driver’s ears will be located during operation.

5.2.2 Plant model and covariance properties

In this section, the state-space model of the considered active noise control system and the covariance properties of its stochastic input signals are introduced.
### 5.2 Problem description and assumptions

#### Standard state-space model

The active noise control system that is considered here is described by the following standard state-space model

\[
\begin{align*}
\mathbf{z}(n+1) &= \mathbf{A}\mathbf{z}(n) + \mathbf{B}_u\mathbf{u}(n) + \mathbf{B}_s\mathbf{s}(n) \\
\mathbf{e}_p(n) &= \mathbf{C}_p\mathbf{z}(n) + \mathbf{D}_{pu}\mathbf{u}(n) + \mathbf{D}_{ps}\mathbf{s}(n) + \mathbf{v}_p(n) \\
\mathbf{e}_v(n) &= \mathbf{C}_v\mathbf{z}(n) + \mathbf{D}_{vu}\mathbf{u}(n) + \mathbf{D}_{vs}\mathbf{s}(n) + \mathbf{v}_v(n) \\
\mathbf{x}(n) &= \mathbf{C}_x\mathbf{z}(n) + \mathbf{D}_{xu}\mathbf{u}(n) + \mathbf{D}_{xs}\mathbf{s}(n) + \mathbf{v}_x(n),
\end{align*}
\]

(5.4)

with \( \mathbf{z}(n) \in \mathbb{R}^N \) the states of the system, \( \mathbf{s}(n) \in \mathbb{R}^S \) the disturbance source signals, \( \mathbf{u}(n) \in \mathbb{R}^L \) the control signals, \( \mathbf{e}_p(n) \in \mathbb{R}^{M_p} \) the physical error signals, \( \mathbf{e}_v(n) \in \mathbb{R}^{M_v} \) the virtual error signals at the moving virtual locations, \( \mathbf{x}(n) \in \mathbb{R}^K \) the reference signals, \( \mathbf{v}_p(n) \in \mathbb{R}^{M_p} \) the physical measurement noise signals, \( \mathbf{v}_v(n) \in \mathbb{R}^{M_v} \) the virtual measurement noise signals, and \( \mathbf{v}_x(n) \in \mathbb{R}^K \) the reference measurement noise signals. The state-space matrices in Eq. (5.4) are real-valued and of appropriate dimensions. Note that the state-space matrices \( \mathbf{C}_v(n) \), \( \mathbf{D}_{vu}(n) \), and \( \mathbf{D}_{vs}(n) \) are time-variant due to the movement of the virtual locations through the sound field. This is in contrast to the discussions presented in Chapters 2–4, where these matrices are time-invariant because the virtual locations were assumed spatially fixed within the sound field.

#### Covariance properties

The physical measurement noise signals \( \mathbf{v}_p(n) \), the virtual measurement noise signals \( \mathbf{v}_v(n) \), and the reference measurement noise signals \( \mathbf{v}_x(n) \) in Eq. (5.4) are assumed to be zero mean white and stationary random processes. The disturbance source signals \( \mathbf{s}(n) \) are again assumed to be either single tone signals of the same normalised frequency \( \omega T_s \), or unknown white and stationary random processes with zero-mean and unit covariance such that the following covariance matrices can be defined

\[
\mathbf{E} \begin{bmatrix} \mathbf{s}(n) \\ \mathbf{v}_p(n) \\ \mathbf{v}_v(n) \\ \mathbf{v}_x(n) \\ \mathbf{z}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{s}(k) \\ \mathbf{v}_p(k) \\ \mathbf{v}_v(k) \\ \mathbf{v}_x(k) \\ \mathbf{z}(0) \end{bmatrix}^T = \begin{bmatrix} \mathbf{I} & \mathbf{S}_{ps}^T & \mathbf{S}_{es}^T & \mathbf{S}_{zs}^T & 0 & 0 \\ \mathbf{S}_{ps} & \mathbf{R}_p & \mathbf{R}_{pc} & \mathbf{R}_{px} & 0 & 0 \\ \mathbf{S}_{es} & \mathbf{R}_{ep}^T & \mathbf{R}_e & \mathbf{R}_{ex} & 0 & 0 \\ \mathbf{S}_{zs} & \mathbf{R}_{zp}^T & \mathbf{R}_{zx} & \mathbf{R}_x & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_0 & 0 \end{bmatrix} \delta_{nk}. \quad (5.5)
\]

In Eq. (5.4), the term \( \mathbf{B}_s\mathbf{s}(n) \) can again be interpreted as process noise \( \mathbf{w}(n) \), with \( \mathbf{w}(n) \triangleq \mathbf{B}_s\mathbf{s}(n) \). The influence of the measurement noise signals and the direct feedthrough

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from the signals $s(n)$ on the physical and virtual error signals can again be combined into an auxiliary measurement noise signal $v(n)$, which is defined as

$$v(n) \triangleq \begin{bmatrix} v_p(n) \\ v_v(n) \\ \bar{v}_x(n) \end{bmatrix} = \begin{bmatrix} D_{ps}s(n) + v_p(n) \\ D_{vs}s(n) + v_v(n) \\ D_{xs}s(n) + v_x(n) \end{bmatrix}. \tag{5.6}$$

Using these definitions of the process noise signals $w(n)$, and the auxiliary measurement noise signals $v(n)$, the following covariance matrix can be defined

$$E \left[ \begin{bmatrix} w(n) \\ v(n) \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}^T \right] = \begin{bmatrix} \bar{Q}_s & \bar{S}_s(n)^T \\ \bar{S}_s(n) & \bar{R}(n) \end{bmatrix} \delta_{nk}. \tag{5.7}$$

Using Eq. (5.5), the covariance matrix $\bar{Q}_s$ of the process noise $w(n)$ is therefore given by

$$\bar{Q}_s = B_s B_s^T. \tag{5.8}$$

The covariance matrix $\bar{R}(n)$ of the auxiliary measurement noise $v(n)$ on the physical, virtual and reference sensors is defined as

$$\bar{R}(n) = \begin{bmatrix} \bar{R}_p & \bar{R}_{pv}(n) & \bar{R}_{px} \\ \bar{R}_{pv}(n)^T & \bar{R}_x(n) & \bar{R}_{vx}(n) \\ \bar{R}_{px}^T & \bar{R}_{vx}(n)^T & \bar{R}_x \end{bmatrix}, \tag{5.9}$$

with the elements of this matrix written in expanded form as

$$\begin{align*}
\bar{R}_p &= R_p + S_{ps} D_{ps}^T + D_{ps} S_{ps}^T + D_{ps} D_{ps}^T \\
\bar{R}_{pv}(n) &= R_{pv} + S_{ps} D_{vs}^T(n) + D_{ps} S_{ps}^T + D_{ps} D_{vs}^T(n)^T \\
\bar{R}_{px} &= R_{px} + S_{xs} D_{xs}^T + D_{xs} S_{xs}^T + D_{xs} D_{xs}^T \\
\bar{R}_x &= R_x + S_{xs} D_{xs}^T + D_{xs} S_{xs}^T + D_{xs} D_{xs}^T \\
\bar{R}_{vx}(n) &= R_{vx} + S_{xs} D_{xs}^T(n) + D_{xs} S_{xs}^T(n) + D_{xs} D_{xs}^T(n)^T. \tag{5.10}
\end{align*}$$

Note that some of the elements of the covariance matrix $\bar{R}(n)$ of the auxiliary measurement noise defined in Eq. (5.9) are time-variant due to the movement of the virtual locations through the sound field, which causes the state-space matrix $D_{vs}(n)$ to be time-variant in Eq. (5.4). The covariance matrix $\bar{S}_s(n)$ between the auxiliary measurement noise $v(n)$ and the process noise $w(n)$ is given by

$$\bar{S}_s(n) = \begin{bmatrix} S_{ps}^T & S_{vs}(n)^T & S_{xs}^T \end{bmatrix}^T, \tag{5.11}$$
with the elements of this matrix written in expanded form as

\[
\begin{align*}
\bar{S}_{ps} & = D_{ps}B_s^T + S_{ps}B_s^T \\
\bar{S}_{vs}(n) & = D_{vs}(n)B_s^T + S_{vs}B_s^T \\
\bar{S}_{xs} & = D_{xs}B_s^T + S_{xs}B_s^T.
\end{align*}
\] (5.12)

Note that in Eq. (5.11), the covariance matrix \(\bar{S}_{vs}(n)\) between the process noise and virtual auxiliary measurement noise is time-variant due to the movement of the virtual locations through the sound field, which causes the state-space matrix \(D_{vs}(n)\) to be time-variant in Eq. (5.4).

### 5.2.3 Non-stationary virtual primary disturbances

The difference between the linear estimation and control problems discussed in Chapters 2–4 and the problem currently considered is that the virtual locations are now assumed to be moving through the sound field rather than being spatially fixed. As a result, the virtual primary and secondary transfer paths are now linear time-variant systems instead of linear time-invariant systems, with state-space realisations given from Eq. (5.4) by

\[
G_{vs}(n) \sim \begin{bmatrix} A & B_s \\ C_v(n) & D_{vs}(n) \end{bmatrix}, \quad G_{vu}(n) \sim \begin{bmatrix} A & B_u \\ C_v(n) & D_{vu}(n) \end{bmatrix}.
\] (5.13)

The virtual primary disturbances \(d_v(n)\) that need to be attenuated are again assumed to be generated in Eq. (5.4) by filtering the disturbance source signals \(s(n)\) with the time-variant virtual primary transfer path defined in Eq. (5.13), such that

\[
d_v(n) = G_{vs}(n)s(n).
\] (5.14)

When the disturbance source signals in Eq. (5.14) are single tone stationary signals of the same normalised frequency \(\omega T_s\), the phases and magnitudes of the virtual primary disturbances \(d_v(n)\) at the moving virtual locations will thus be time-variant with respect to some arbitrary reference. When the disturbance source signals are unknown white and stationary random processes with zero-mean and unit covariance as defined in Eq. (5.5), the virtual primary disturbances \(d_v(n)\) are random process with time-variant stochastic properties. The virtual primary disturbances are thus non-stationary signals when the virtual locations move through the sound field, even when the disturbance source signals are assumed to be stationary single tone or random signals.

When the primary disturbances are non-stationary, the common approach in active noise control is to use an adaptive control algorithm \([25, 68]\). Adaptive control algorithms are able to track the changes in the statistical properties of the primary
disturbances and adjust the controller accordingly. In Chapter 2, the adaptive feedforward control algorithms known as the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms have been introduced. These algorithms can be implemented as described in that chapter when the virtual error signals are directly measured during real-time control, and when the virtual locations are spatially fixed within the sound field. To implement these algorithms for the case considered here, the following two issues need to be addressed.

**Generating the virtual filtered-reference signals**

The first issue that needs to be addressed is the generation of the virtual filtered-reference signals. These signals are required in the adaptive feedforward control algorithms introduced in Chapter 2, where the reference signals $x(n)$ are filtered with the time-invariant virtual secondary transfer paths $G_{vu}$ between the control sources and the spatially fixed virtual locations, as illustrated in Fig. 2.7. However, when the virtual locations are moving through the sound field, the virtual secondary transfer paths are time-variant as defined in Eq. (5.13). In Section 5.3, the generation of the virtual filtered-reference signals for the case of a time-variant virtual secondary transfer path is therefore discussed. Note that for this case, the virtual filtered-reference signals are non-stationary signals due to the movement of the virtual locations through the sound field, even when the reference signals are stationary signals. It is assumed that the true virtual error signals are directly measured during real-time control by moving physical sensors that track the moving virtual locations. Effectively, the adaptive feedforward control algorithms are thus modified in Section 5.3 to enable active noise control at a number of moving physical sensors that track the desired moving locations of maximum attenuation.

**Moving virtual sensing algorithms**

The second issue that needs to be addressed is the assumption that the virtual error signals are not directly measured during real-time control, such that the adaptive feedforward control algorithms cannot be implemented as described in Chapter 2. This is because to adapt the feedforward controller in response to the non-stationarities in the virtual primary disturbances and the virtual filtered-reference signals, the true virtual error signals at the moving virtual locations are generally needed. In Section 5.4, a moving virtual sensing method is therefore developed that can be used to compute an estimate of the virtual error signals at virtual locations that are moving through the sound field. These moving virtual sensing algorithms can be combined with the method presented in Section 5.3 for generating the virtual filtered-reference signals for the case of a time-variant virtual secondary transfer path. The proposed method
will be implemented on an acoustic duct arrangement in Chapter 10, where a moving zone of quiet at a virtual sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances.

**Previous research on tracking of non-stationary primary disturbances**

A few papers have been written previously that investigate the tracking performance of adaptive feedforward control algorithms for the active control of non-stationary primary disturbances [46, 70, 71, 74, 78, 92, 93, 111, 123]. Most of these papers investigate the active control of noise originating from a moving primary disturbance source, such as an airplane during take-off and landing. The aim of these papers is to reduce the non-stationary primary disturbances at a number of physical sensors that are spatially fixed. For this case, the physical and virtual primary transfer paths are time-variant systems, with state-space realisations given by

\[
G_{ps}(n) \sim \begin{bmatrix} A & B_s(n) \\ C_p & D_{ps}(n) \end{bmatrix}, \quad G_{vs}(n) \sim \begin{bmatrix} A & B_s(n) \\ C_v & D_{vs}(n) \end{bmatrix}.
\] (5.15)

Note that the state-space matrices \(B_s, D_{ps},\) and \(D_{vs}\) in Eq. (5.4) are then time-variant in Eq. (5.15) due to the movement of the primary disturbance source. This in contrast to the problem analysed here, where the state-space matrices \(C_v(n), D_{vs}(n),\) and \(D_{vs}(n)\) are assumed to be time-variant in Eq. (5.4) due to the movement of the virtual locations through the sound field. In previous research [46, 70, 71, 74, 78, 92, 93, 111, 123], the secondary transfer paths between the control sources and the locations of control are assumed time-invariant, because the non-stationary primary disturbances that originate from the moving primary disturbance sources are controlled at a number of spatially fixed physical sensors. Here, the secondary transfer paths between the control sources and the locations of control are time-variant due to the movement of the virtual locations through the sound field. The intent of the previous research that investigated the tracking performance of adaptive feedforward control algorithms for active noise control is therefore different from the research considered here.

### 5.3 Active noise control at moving physical sensors

In this section, it is assumed that the true virtual error signals \(e_v(n)\) in the state-space model of the active noise control system defined in Eq. (5.4) are directly measured during real-time control by moving physical sensors. The adaptive feedforward control algorithms introduced in Chapter 2, which can be implemented when the virtual locations are spatially fixed, are then modified to allow for virtual locations that are
moving through the sound field. Effectively, the aim is therefore to attenuate the non-stationary primary disturbances at a number of moving physical sensors that track the desired moving locations of maximum attenuation.

In Section 5.3.1, it is assumed that the time-variant virtual secondary transfer path matrix $G_{vu}(n)$ defined in Eq. (5.13) is known at every sample $n$. In other words, the state-space matrices $C_v(n)$ and $D_{vu}(n)$ of the state-space model defined in Eq. (5.4) are assumed known at every sample $n$. The generation of the virtual filtered-reference signals given this state-space model is then discussed. These signals are needed to implement the adaptive feedforward control algorithms introduced in Chapter 2.

In practice, the time-variant virtual secondary transfer path defined in Eq. (5.13) is generally not known at every sample $n$, since the moving virtual locations will not be known a priori for every sample $n$. In Section 5.3.2, a practical method for generating the virtual filtered-reference signals is therefore introduced based on spatial interpolation.

In Section 5.3.3, the tracking of non-stationary primary disturbances is briefly discussed. As stated by Haykin [51], the tracking details of a time-variant system are very problem specific. As a result, the tracking behaviour of the algorithms proposed here will be problem specific as well, and general statements on how fast the virtual locations can move through the sound field while still obtaining effective local control cannot easily be made. The proposed method will therefore be implemented on an acoustic duct arrangement in Chapter 9 and the real-time tracking performance of the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms will be analysed and compared for various speeds of movement and various spatial characteristics of the narrowband sound field.

Active noise control at a moving physical sensor, which is considered in this section, has been investigated previously by Michalczyk and Czyz [78]. However, the standard implementation of the filtered-x LMS algorithm introduced in Chapter 2 was used in their research rather than the implementation proposed in this section. A time-invariant secondary transfer path model was estimated for only one spatially fixed location in the previous research [78], and this model was used to generate the filtered-reference signals. Narrowband control experiments were then conducted at a frequency for which the variations in the secondary transfer path over the trajectory of movement were small, i.e. variations in magnitude of a few decibels and negligible variations in phase. This is because the implementation adopted in the previous research [78] cannot account for large variations in the secondary transfer path because the method used for generating the filtered-reference signals cannot accommodate this situation. This is therefore a substantial difference between the previous research and the methods developed in the following sections.
Active noise control at moving physical sensors

5.3.1 Adaptive feedforward control at moving physical sensors

In this section, the adaptive feedforward control algorithms introduced in Chapter 2 are modified to account for the fact that the virtual locations are moving through the sound field rather than being spatially fixed. It is assumed that the virtual error signals are directly measured by moving physical sensors that track the moving virtual locations. For this case, a block diagram of the multi-channel filtered-x LMS algorithm is shown in Fig. 5.2.1

Figure 5.2: Block diagram of the multi-channel filtered-x LMS algorithm without measurement noise, with L control sources, K reference sensors, and \( M_v \) moving physical sensors that directly measure the virtual error signals at the moving virtual locations \( X_v(n) \) during real-time control.

The difference between the implementation illustrated in Fig. 5.2 and the standard implementation of the multi-channel filtered-x LMS algorithm illustrated in Fig. 2.7 is that the virtual secondary transfer path \( G_{vu}(n) \) in Fig. 5.2 is now time-variant, as defined in Eq. (5.13). As discussed previously, this is due to the movement of the physical sensors that directly measure the true virtual error signals at the moving virtual locations. The update equation for the control filter coefficients \( w(n) \in \mathbb{R}^{KL} \) of the feedforward controller in Fig. 5.2 is now given by

\[
w(n + 1) = w(n) - \mu R_v(n)e_v(n), \tag{5.16}
\]

with \( \mu \in \mathbb{R}^+ \) the convergence coefficient, \( e_v(n) \in \mathbb{R}^{M_v} \) the true virtual error signals directly measured by the moving physical sensors, and \( R_v(n) \in \mathbb{R}^{KL \times M_v} \) the matrix of virtual filtered-reference signals defined in Eq. (2.94). The update equation defined

---

1. Note again that when the normalised filtered-x LMS or the filtered-x RLS algorithm is used instead of the filtered-x LMS algorithm, the block diagram in Fig. 5.2 can simply be modified by replacing the block denoting the LMS algorithm by a block denoting the normalised LMS or RLS algorithm.
in Eq. (5.16) is equivalent to the update equation defined in Eq. (2.91) for the standard implementation of the multi-channel filtered-x LMS algorithm illustrated in Fig. 2.7.

**Generating the virtual filtered-reference signals**

The difference between the implementation illustrated in Fig. 5.2 and the standard implementation illustrated in Fig. 2.7 is that the matrix $R_v(n)$ of virtual filtered-reference signals in Eq. (5.16) is now generated by filtering the reference signals $x(n)$ with a virtual secondary transfer path matrix $G_{vu}(n)$ that is time-variant rather than time-invariant. The $KLM_v$ virtual filtered-reference signals $r_{klm_v}(n)$, which are needed to construct the matrix $R_v(n)$ in Eq. (5.16), can be generated using the state-space model of the time-variant virtual secondary transfer path matrix $G_{vu}(n)$ defined in Eq. (5.18). Thus, similar to the discussions presented in Chapter 2, the virtual filtered-reference signals $r_{kl}(n) \in \mathbb{R}^{M_v}$, with

$$r_{kl}(n) = \begin{bmatrix} r_{kl1}(n) & r_{kl2}(n) & \ldots & r_{klM_v}(n) \end{bmatrix}^T,$$  \hspace{1cm} (5.17)

can be generated by filtering the reference signal $x_k(n)$ with the linear time-variant state-space model $G_{vu,l}(n)$

$$G_{vu,l}(n) \sim \begin{bmatrix} A & B_{vl, l} \\ C_v(n) & D_{vu,l}(n) \end{bmatrix},$$  \hspace{1cm} (5.18)

such that

$$z_{kl}(n+1) = A z_{kl}(n) + B_{vl} x_k(n)$$

$$r_{kl}(n) = C_v(n) z_{kl}(n) + D_{vu,l}(n) x_k(n),$$  \hspace{1cm} (5.19)

where $B_{vl} \in \mathbb{R}^N$ and $D_{vu,l}(n) \in \mathbb{R}^{M_v}$ are the $l$th column of the state-space matrices $B_v$ and $D_{vu}(n)$, respectively. The virtual secondary transfer path matrix $G_{vu,l}(n)$ defined in Eq. (5.18) thus contains the linear time-variant virtual secondary transfer paths between the $l$th control source and the $M_v$ moving virtual locations.

### 5.3.2 Practical implementation

The modified implementation of the multi-channel filtered-x LMS algorithm shown in Fig. 5.2 can only be implemented when the time-variant virtual secondary transfer path matrix $G_{vu}(n)$ is known at every sample $n$. This is necessary to generate the virtual filtered-reference signals as defined in Eqs (5.17)–(5.19), where the state-space matrices $C_v(n)$ and $D_{vu}(n)$ need to be known at every sample $n$. However, the time-variant virtual secondary transfer path matrix $G_{vu}(n)$ is generally not known at every sample $n$, and practical methods for generating the virtual filtered-reference signals are therefore considered in this section.
Online secondary path identification

A possible method to obtain an estimate of the time-variant virtual secondary transfer path matrix $G_{vu}(n)$ at every sample $n$ is to use online secondary path identification techniques [25, 68]. Online secondary path identification is the process of modelling the secondary path response while the active noise control system is kept in operation [25, 68]. This technique is generally used when the secondary transfer paths are time-variant due to changes in the acoustic environment. This is in contrast to the case considered here, where the secondary path is time-variant due to the movement of the error sensors through an acoustic environment that is assumed stationary.

The most common approach to online secondary path identification is to add a white noise identification signal to the control signal $u(n)$ that is uncorrelated to the reference signal $x(n)$ [25]. A negative side-effect of this method is that it introduces additional noise at the error sensors, which reduces the attenuation that can be achieved. Unfortunately, high levels of identification noise are generally needed to accurately track rapid changes in the secondary path [25]. For the case considered here, rapid changes in the secondary path might occur when the spatial rate of change of the secondary sound field is large, and/or the temporal rate of change $\dot{X}_v(n)$ of the moving virtual location is large, i.e. the observer is moving their head relatively quickly. If the spatial rate of change of the secondary sound field over the region through which the virtual location moves is small, and/or the virtual location is moving slowly (and provided that the virtual error signals are directly measured by moving physical sensors), online secondary path identification could possibly be used to obtain a reasonably accurate estimate of the time-variant virtual secondary transfer paths $G_{vu}(n)$ at every sample $n$. For this case, the virtual filtered-reference signals can be generated as described in the previous section.

However, if the spatial rate of change of the secondary sound field over the region through which the virtual location moves is large, and/or the virtual location is moving relatively quickly, the online identification algorithm may not converge fast enough to obtain a reasonably accurate estimate of the time-variant virtual secondary transfer paths. Furthermore, if the virtual error signals at the moving virtual locations are not directly measured by physical sensors, but estimated using a moving virtual sensing method, online secondary path identification cannot be used. This is the case in the algorithms presented in Section 5.4. In this instance, the time-variant virtual secondary transfer paths need to be estimated in a preliminary offline identification procedure.

Offline secondary path identification

As stated previously, the virtual secondary transfer paths are time-variant due to the movement of the virtual locations through a sound field that is assumed stationary,
not because the sound field itself is non-stationary. Therefore, if the moving virtual locations are known a priori for every sample \( n \), the time-variant virtual secondary transfer paths could be modelled by a number of time-invariant virtual secondary transfer paths, each corresponding to the positions of the moving virtual locations at the sample \( n \). Relating this to the state-space model defined in Eq. (5.4), this means that the state-space matrices \( C_v(n) \) and \( D_{vu}(n) \) would be known a priori at every sample \( n \). However, the moving virtual locations will generally not be known a priori for every sample \( n \) in a practical situation. Furthermore, even if this is the case, an impractically large number of time-invariant virtual secondary transfer path models would probably need to be estimated and stored. Therefore, a more practical method for generating the virtual filtered-reference signals based on *spatial interpolation* is presented in the next section.

**Spatial interpolation technique**

The assumption introduced in Section 5.2, i.e. that the moving virtual locations \( X_v(n) \) are confined to relatively small regions called *target zones*, is now used to develop a more practical method for generating the virtual filtered-reference signals. Fig. 5.3 shows a block diagram of the proposed practical implementation of the multi-channel filtered-x LMS algorithm.

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**Figure 5.3:** Block diagram of the practical implementation of the multi-channel filtered-x LMS algorithm, with \( K \) reference sensors, \( L \) control sources, \( M_v \) spatially fixed virtual sensors located at \( X_v \) within the target zones, and \( M_v \) moving physical sensors that directly measure the true virtual error signals at the moving virtual locations \( X_v(n) \).
As shown in Fig. 5.3, the approach taken here is based on using a time-invariant virtual secondary transfer path matrix $\mathbf{G}_{vu} \in \mathcal{RH}_{\infty}^{\tilde{M}_v \times L}$ that contains the virtual secondary transfer paths between the $L$ control sources and the virtual secondary disturbances at $\tilde{M}_v$ spatially fixed virtual locations, which are suitably positioned throughout the target zones. As indicated in Fig. 5.3, these locations are contained in a matrix $\mathbf{X}_v \in \mathbb{R}^{3 \times \tilde{M}_v}$, which is defined as

$$\mathbf{X}_v = \begin{bmatrix} \bar{x}_{v1} & \bar{x}_{v2} & \cdots & \bar{x}_{v\tilde{M}_v} \end{bmatrix},$$

(5.20)

where each column $\bar{x}_{v\tilde{m}_v} \in \mathbb{R}^3$ contains the spatial coordinates of the $m_{\tilde{m}_v}^{th}$ spatially fixed virtual location with respect to some reference frame, such that

$$\bar{x}_{\tilde{m}_v} = \begin{bmatrix} \bar{x}_{\tilde{m}_v} & \bar{y}_{\tilde{m}_v} & \bar{z}_{\tilde{m}_v} \end{bmatrix}^T.$$  

(5.21)

In practice, a model of the time-invariant virtual secondary transfer path matrix $\mathbf{G}_{vu}$ can be estimated in a preliminary identification stage in which physical sensors are temporarily located at the spatially fixed virtual locations $\mathbf{X}_v$. A state-space model of the virtual secondary transfer path matrix $\mathbf{G}_{vu}$ is given by

$$\mathbf{G}_{vu} \sim \begin{bmatrix} \mathbf{A} & \mathbf{B}_u \\ \bar{C}_u & \bar{D}_{vu} \end{bmatrix}.$$  

(5.22)

As illustrated in Fig. 5.3, this virtual secondary transfer path matrix $\mathbf{G}_{vu}$ is used to generate $KLM_v$ virtual filtered-reference signals $\bar{r}_v(n)$. Using the state-space model defined in Eq. (5.22), these virtual filtered-reference signals can again be generated as described in Chapter 2. Thus, the virtual filtered-reference signals $\bar{r}_{kl}(n) \in \mathbb{R}^{\tilde{M}_v}$, with

$$\bar{r}_{kl}(n) = \begin{bmatrix} \bar{r}_{kl1}(n) & \bar{r}_{kl2}(n) & \cdots & \bar{r}_{kl\tilde{M}_v}(n) \end{bmatrix}^T,$$

(5.23)

are generated by filtering the reference signal $x_k(n)$ with that part of the state-space model of the virtual secondary transfer path matrix $\mathbf{G}_{vu}$ that corresponds to the $l^{th}$ control source, such that

$$\begin{align*}
\bar{z}_{kl}(n+1) &= \mathbf{A}\bar{z}_{kl}(n) + \mathbf{B}_{ul}x_k(n) \\
\bar{r}_{kl}(n) &= \bar{C}_u\bar{z}_{kl}(n) + \bar{D}_{vu}\bar{x}_k(n).
\end{align*}$$

(5.24)

where $\mathbf{B}_{ul} \in \mathbb{R}^N$ and $\mathbf{D}_{vu} \in \mathbb{R}^{\tilde{M}_v}$ are the $l^{th}$ column of the state-space matrices $\mathbf{B}_u$ and $\bar{\mathbf{D}}_{vu}$, respectively. Implementing Eqs (5.23) and (5.24) for all the reference signals $x_k(n)$, with $k = 1, \ldots, K$, and all $L$ control sources gives the $KLM_v$ virtual filtered-reference signals $\bar{r}_v(n)$ that are needed in the implementation of the multi-channel filtered-x LMS algorithms illustrated in Fig. 5.3.

As shown in Fig. 5.3, an estimate $\hat{r}_{kl}(n) \in \mathbb{R}^{\tilde{M}_v}$ of the virtual filtered-reference signals $r_{kl}(n)$, which have been defined in Eq. (5.17) for the moving virtual locations $\mathbf{X}_v(n)$, is

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now computed using \textit{spatial interpolation} between the virtual filtered-reference signals \( \tilde{r}_v(n) \) for the \textit{spatially fixed} virtual locations \( \tilde{X}_v \), which are defined in Eq. (5.23). Note that to compute the spatial interpolation, it is assumed in Fig. 5.3 that the moving virtual locations \( X_v(n) \) are known at every sample \( n \). Implementing the described procedure for all the reference signals \( x_k(n) \), with \( k = 1, \ldots, K \), and all \( L \) control sources, gives an estimate of the \( KLM_v \) virtual filtered-reference signals needed in the implementation of the multi-channel filtered-x LMS algorithms illustrated in Fig. 5.3.

The update equation for the control filter coefficients \( w(n) \in \mathbb{R}^{KL} \) of the feedforward controller in Fig. 5.3 is now given by

\[
    w(n+1) = w(n) - \mu \tilde{\hat{R}}(n) e_v(n),
\]

where \( \tilde{\hat{R}}(n) \in \mathbb{R}^{KL \times M_v} \) is the matrix of \textit{estimated} virtual filtered-reference signals defined similarly to Eq. (2.94). The proposed method will be implemented on an acoustic duct arrangement in Chapter 9, where a moving zone of quiet at a \textit{physical} sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances.

\textbf{Spatial interpolation techniques}

The algorithms developed in this chapter will be implemented on an acoustic duct arrangement in Chapters 9 and 10 using \textit{linear} spatial interpolation techniques. Note that more sophisticated spatial interpolation methods, such as \textit{polynomial} or \textit{spline} interpolation, can be used to increase the accuracy of the interpolated estimate of the virtual filtered-reference signals in Fig. 5.3. However, \textit{linear} spatial interpolation techniques are generally easier to implement and computationally less intensive, and will prove to provide sufficient accuracy in the real-time acoustic duct experiments presented in Chapters 9 and 10. Of course, the accuracy of the spatial interpolation is also determined by the choice of the spatially fixed virtual locations \( \tilde{X}_v \) defined in Eq. (5.20). For instance, when the spatial variations in magnitude and phase of the secondary sound field are large over a certain section of the target zones, this section needs to be covered by spatially fixed virtual locations that are relatively closely spaced when using a linear spatial interpolation technique. Thus, a suitable interpolation technique and choice of spatially fixed virtual locations \( \tilde{X}_v \) can be made by investigating the spatial characteristics of the secondary sound field over the target zones.

\textbf{Practical example}

To clarify the proposed method, let us again consider the example of the active control of engine-induced noise inside a mining vehicle cabin illustrated in Fig. 5.4, which was introduced in Section 5.2. The aim is to reduce the primary disturbances at the driver’s
ears, such that $M_v = 2$ moving virtual sensors are used, as indicated by the grey dots in Fig. 5.4. The desired locations of these two moving virtual sensors are defined by the moving virtual locations $X_v(n)$ defined in Eq. (5.1). The engine-induced noise is generally narrowband, and a tonal reference signal $x(n)$ is assumed available with a frequency equal to the engine’s firing frequency, such that $K = 1$. Furthermore, it is assumed that $L = 2$ control sources are used as illustrated in Fig. 5.4.

Figure 5.4: Illustration of the active control of engine-induced noise inside a mining vehicle cabin, using $M_v = 2$ moving virtual sensors that track the driver’s ears. The driver’s ears are assumed to stay within the indicated target zones during operation. An estimate of the virtual filtered-reference signals is computed using spatial interpolation between the virtual filtered-reference signals that are computed for the spatially fixed virtual locations closest to each of the moving virtual locations.

As shown in Fig. 5.4, a rectangular grid consisting of $5 \times 8 = 40$ spatially fixed virtual locations $\mathbf{X}_v$ is used to cover the two target zones. A state-space model of the time-invariant virtual secondary transfer path matrix $\mathbf{G}_{vu} \in \mathcal{RH}_\infty^{40 \times 2}$ defined in Eq. (5.22) is assumed available, which models the transfer paths between the two control sources and the virtual secondary disturbances at the 40 spatially fixed virtual locations $\mathbf{X}_v$. The virtual filtered-reference signals $\mathbf{f}_{11}(n) \in \mathbb{R}^2$ and $\mathbf{f}_{12}(n) \in \mathbb{R}^2$ in Eq. (5.23) are
now generated for these locations as defined in Eq. (5.24), such that

\[
\begin{align*}
\mathbf{z}_{11}(n+1) &= \mathbf{A} \mathbf{z}_{11}(n) + \mathbf{B}_{u1} x(n) \\
\mathbf{r}_{11}(n) &= \mathbf{C}_v \mathbf{z}_{11}(n) + \mathbf{D}_{vu1} x(n),
\end{align*}
\]

(5.26)

and

\[
\begin{align*}
\mathbf{z}_{12}(n+1) &= \mathbf{A} \mathbf{z}_{12}(n) + \mathbf{B}_{u2} x(n) \\
\mathbf{r}_{12}(n) &= \mathbf{C}_v \mathbf{z}_{12}(n) + \mathbf{D}_{vu2} x(n).
\end{align*}
\]

(5.27)

Estimates of the \(KLM_v = 4\) virtual filtered-reference signals for the moving virtual locations, which are denoted by

\[
\begin{align*}
\hat{\mathbf{r}}_{11}(n) &= \begin{bmatrix} \hat{r}_{111}(n) \\ \hat{r}_{112}(n) \end{bmatrix}, \\
\hat{\mathbf{r}}_{12}(n) &= \begin{bmatrix} \hat{r}_{121}(n) \\ \hat{r}_{122}(n) \end{bmatrix},
\end{align*}
\]

(5.28)

are now computed using spatial interpolation between the discrete points closest to the moving virtual locations, which have been indicated by black dots in Fig. 5.4. Once estimates of the virtual filtered-reference signals are available, the next step is to implement the multi-channel filtered-x LMS algorithm defined in Eq. (5.25). In this example, it is assumed that \(I = 2\) control filter coefficients are used because the disturbances are narrowband in this example. From Eq. (2.86) on page 65, the two control signals are computed as

\[
\begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix} = \begin{bmatrix} x(n) & 0 \\ x(n-1) & 0 \\ 0 & x(n) \\ 0 & x(n-1) \end{bmatrix}^T \begin{bmatrix} w_{11,0}(n) \\ w_{11,1}(n) \\ w_{21,0}(n) \\ w_{21,1}(n) \end{bmatrix},
\]

(5.29)

where the \(KLI = 4\) control filter coefficients \(\mathbf{w}(n)\) are updated at every sample \(n\) using the multi-channel filtered-x LMS algorithm defined in Eq. (5.25), such that

\[
\begin{bmatrix} w_{11,0}(n+1) \\ w_{11,1}(n+1) \\ w_{21,0}(n+1) \\ w_{21,1}(n+1) \end{bmatrix} = \begin{bmatrix} w_{11,0}(n) \\ w_{11,1}(n) \\ w_{21,0}(n) \\ w_{21,1}(n) \end{bmatrix} - \mu \begin{bmatrix} \hat{r}_{111}(n) & \hat{r}_{112}(n) \\ \hat{r}_{111}(n-1) & \hat{r}_{112}(n-1) \\ \hat{r}_{121}(n) & \hat{r}_{122}(n) \\ \hat{r}_{121}(n-1) & \hat{r}_{122}(n-1) \end{bmatrix} \begin{bmatrix} e_{v1}(n) \\ e_{v2}(n) \end{bmatrix},
\]

(5.30)

where the estimated virtual filtered-reference signals defined in Eq. (5.28) are used, and where the two virtual error signals \(e_{v1}(n)\) and \(e_{v2}(n)\) are directly measured during real-time control using moving physical sensors that track the observer’s ears. In the next section, these virtual error signals will be estimated using a moving virtual sensing algorithm.
5.3.3 Tracking of non-stationarities

The aim of the practical implementation of the multi-channel filtered-x LMS algorithm illustrated in Fig. 5.3 is to track the changes in the statistical properties of the virtual primary disturbances at the moving virtual locations, and adjust the control filter coefficients \( w(n) \) accordingly. The amount and speed of tracking needed is dependent on both the temporal rate of change \( \dot{X_v}(n) \) of the moving virtual locations, and the spatial rate of change of the relative magnitude and phase between the primary and secondary sound fields over the target zone. This spatial rate of change determines to what degree the control filter coefficients need to be adjusted over the region through which the virtual locations are moving. This information, together with the temporal rate of change of the moving virtual locations, determines the amount and speed of tracking that is needed to successfully create zones of quiet that track the desired locations of maximum attenuation.

However, as stated by Haykin [51], the tracking details of a time-variant system are very problem specific. As a result, the tracking behaviour of the proposed algorithm will be problem specific as well, and general statements on how quickly the virtual locations can move through the sound field while still obtaining effective local control cannot easily be made. In Chapter 9, the proposed method will therefore be implemented on an acoustic duct arrangement and the tracking performance of the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms will be analysed and compared for various speeds of movement and various spatial characteristics of the sound field.

As discussed by Haykin [51], the convergence rate and the tracking capability of an adaptive algorithm are generally two different properties. Whereas convergence is a transient phenomenon, tracking is a steady-state phenomenon. For an adaptive algorithm to exercise its tracking capability, it must therefore first pass from the transient mode to the steady-state mode [51]. In the experiments presented in Chapter 9, the narrowband performance obtained with the proposed method will be measured after the adaptive algorithms have passed from the transient mode to the steady-state mode. Thus, it is the tracking capability of the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms that will be investigated in Chapter 9 for narrowband disturbances at a moving physical sensor.

5.4 Moving virtual sensing algorithms

In the previous section, it has been assumed that the virtual error signals are directly measured by moving physical sensors that track the moving virtual locations. In this section, it is assumed that this is not the case, and the adaptive feedforward control method developed in the previous section can therefore not be implemented as illustrated in the block diagram in Fig. 5.3. The true virtual error signals \( e_v(n) \) in this
Chapter 5  Active noise control at moving physical and virtual sensors

block diagram are now thus not directly available to the adaptive control algorithm that updates the control filter coefficients. This block diagram is therefore modified as shown in Fig. 5.5, where an estimate \( \hat{e}_v(n) \) of the virtual error signals is used instead to update the feedforward controller.²

![Figure 5.5: Block diagram of the multi-channel filtered-x LMS algorithm including a moving virtual sensing algorithm, with \( K \) reference signals, \( L \) control sources, \( M_v \) spatially fixed virtual sensors suitably positioned throughout the target zones at \( \bar{X}_v \), and \( M_v \) moving virtual sensors that track the moving virtual locations \( X_v(n) \).](image)

The estimate \( \hat{e}_v(n) \) of the virtual error signals is computed in Fig. 5.5 using a moving virtual sensing algorithm. This estimate is adaptively minimised as described in Section 5.3 and illustrated in Fig. 5.3. In the following sections, the moving virtual sensing for active noise control problem is defined and the assumptions made to analyse this problem are introduced. The previously proposed spatially fixed virtual sensing algorithms, which have been analysed in Chapter 3, are modified to allow for virtual locations that are moving through the sound field rather than being spatially fixed. It is then shown that an optimal solution to the moving virtual sensing problem can be derived using Kalman filtering theory [60].

² Note again that when the normalised filtered-x LMS or the filtered-x RLS algorithm is used instead of the filtered-x LMS algorithm, the block diagram in Fig. 5.5 can simply be modified by replacing the block denoting the LMS algorithm by a block denoting the normalised LMS or RLS algorithm.
5.4 Moving virtual sensing algorithms

5.4.1 Problem definition and assumptions

In this section, the moving virtual sensing for active noise control problem is defined and the assumptions made to analyse this problem are introduced. As shown in Fig. 5.5, the moving virtual sensing algorithm uses the physical error signals \( e_p(n) \), which are measured by a number of physical sensors, to compute an estimate of the virtual error signals. From Eq. (5.4), a state-space model of the linear estimation problem considered here is given by

\[
\begin{align*}
\mathbf{z}(n+1) &= A \mathbf{z}(n) + B_u \mathbf{u}(n) + B_s \mathbf{s}(n) \\
\mathbf{e}_p(n) &= C_p \mathbf{z}(n) + D_{pu} \mathbf{u}(n) + D_{ps} \mathbf{s}(n) + \mathbf{v}_p(n) \\
\mathbf{e}_v(n) &= C_v(n) \mathbf{z}(n) + D_{vu}(n) \mathbf{u}(n) + D_{vs}(n) \mathbf{s}(n) + \mathbf{v}_v(n).
\end{align*}
\]

(5.31)

The aim of the moving virtual sensing algorithm is to compute an accurate estimate \( \hat{e}_v(n) \in \mathbb{R}^{M_v} \) of the virtual error signals at the moving virtual locations. The estimation accuracy of the moving virtual sensing algorithm is again defined by the virtual output errors \( \epsilon_v(n) \in \mathbb{R}^{M_v} \), which are given by

\[
\epsilon_v(n) = e_v(n) - \hat{e}_v(n).
\]

(5.32)

In Sections 5.4.2 and 5.4.3, the previously proposed spatially fixed virtual sensing algorithms called the adaptive LMS virtual microphone technique [14] and the remote microphone technique [104, 112] are first modified to allow for a virtual location that is moving through the sound field rather than being spatially fixed. The structures adopted in these spatially fixed virtual sensing algorithms, which have been introduced in Chapter 3, are not changed. Given the assumed structure, these spatially fixed virtual sensing algorithms are modified to allow for virtual locations that move through the sound field. Once the structure has been chosen, optimal solutions for the unknown parameters of the moving virtual sensing algorithms can be computed by minimising a time-variant cost function \( J_\epsilon(n) \) defined as the mean-square value of the virtual output errors, such that

\[
J_\epsilon(n) = \text{tr} \left( \mathbf{E} \left[ \epsilon_v(n) \epsilon_v(n)^T \right] \right).
\]

(5.33)

This cost function is time-variant due to the movement of the virtual locations through the sound field. The question now arises as to whether there is an optimal structure that can be used to solve the moving virtual sensing for active noise control problem, which again amounts to a linear estimation problem. It is well-known that the Kalman filter provides an optimal structure for solving linear estimation problems [60], even when the state-space matrices that describe the system are time-variant as in Eq. (5.31). An optimal solution to the moving virtual sensing for active noise control problem is therefore derived in Section 5.4.4 using Kalman filtering theory.
In the following sections, the developed moving virtual sensing algorithms are derived by first assuming that the time-variant virtual primary and secondary transfer paths defined in Eq. (5.13) are known, or have been identified offline, for every location the moving virtual sensors occupy at the samples \( n = 1, \ldots, \infty \). In other words, it is first assumed that the state-space matrices \( C_v(n), D_{vu}(n), \) and \( D_{vs}(n) \) in Eq. (5.31) are known at every sample \( n \). The algorithms derived under this assumption are referred to in the following sections as the optimal implementation.

Practical implementation

In practice, the time-variant virtual primary and secondary transfer paths defined in Eq. (5.13) are generally not known for every sample \( n \), because the moving virtual locations \( X_v(n) \) are not known a priori. Furthermore, even if this is the case, an impractically large number of virtual transfer path models would need to be estimated and stored to describe the time-variant virtual transfer paths, i.e. one for each possible moving virtual location. Therefore, more practical implementations of the introduced moving virtual sensing algorithms are discussed in each section, which are based on spatial interpolation. The algorithms derived using this method are referred to as the practical implementation. In the following sections, the adaptive LMS virtual microphone technique [14], the remote microphone technique [104, 112], and the Kalman filter based virtual sensing technique introduced in Chapter 3 are extended to moving virtual sensing algorithms.

5.4.2 Adaptive LMS moving virtual microphone technique

In this section, the adaptive LMS virtual microphone technique [14] is extended to a moving virtual sensing algorithm, which will be called the adaptive LMS moving virtual microphone technique. This technique is derived here without altering the structure of the adaptive LMS virtual microphone technique, which has been illustrated in Fig. 3.3. The block diagram from this figure is modified in Fig. 5.6 to allow for a virtual location that is moving through the sound field rather than being spatially fixed. Similar to Fig. 3.3, the estimate of the virtual error signals is still computed as a weighted summation of the physical primary and secondary disturbances at the physical sensors. These disturbances are again obtained by separating the physical error signals into their primary and secondary components, as illustrated in Fig. 5.6. However, to account for a virtual location that is moving through the sound field rather than being spatially fixed, the physical sensor weights are now made time-variant in Fig. 5.6 instead of time-invariant as in Fig. 3.3. The estimate of the virtual error signals is now computed as

\[
\hat{e}_v(n) = H_s(n)^T e_p(n) + (H_u(n) - H_s(n))^T G_{pu} u(n),
\]

(5.34)
where $H_s(n) \in \mathbb{R}^{M_p \times M_v}$ is the matrix of time-variant physical sensor weights for the primary sound field, and $H_u(n) \in \mathbb{R}^{M_p \times M_v}$ the matrix of time-variant physical sensor weights for the secondary sound field. Note that when the optimal weights for the primary and secondary sound fields are equal, the estimate of the virtual error signals can simply be calculated as

$$\hat{e}_v(n) = H(n)^T e_p(n), \quad (5.35)$$

where $H(n) \in \mathbb{R}^{M_p \times M_v}$ is the matrix of time-variant physical sensor weights directly applied to the physical error signals. Following the discussions presented in Section 3.3, this is similar to the implementation used in previous research for the case of spatially fixed virtual locations [80], where one set of weights was directly applied to the virtual error signals without separating them into primary and secondary components.

Optimal solution for time-variant weights

In this section, a time-variant optimal solution for the physical sensor weights is derived given the state-space model of the active noise control system defined in Eq. (5.31). By setting the disturbance source signals $s(n) = 0$, time-variant optimal weights $H_{u_0}(n) \in \mathbb{R}^{M_p \times M_v}$ for the secondary sound field can be derived. Similarly, time-variant optimal weights $H_{u_0}(n) \in \mathbb{R}^{M_p \times M_v}$ for the primary sound field can be derived by setting the control signals $u(n) = 0$. Here, the disturbance source signals are set to zero, and Eq. (5.31) can now be written as

$$z(n+1) = Az(n) + B_u u(n)$$
$$y_p(n) = C_p z(n) + D_{pu} u(n) + v_p(n) \quad n \geq 0 \quad (5.36)$$
$$y_v(n) = C_v z(n) + D_{vu}(n) u(n) + v_v(n).$$

An estimate of the virtual secondary disturbances $y_v(n)$ is now computed as

$$\hat{y}_v(n) = H_u(n)^T y_p(n). \quad (5.37)$$
From Eq. (5.36), a state-space system that models the estimated virtual primary disturbances $\hat{y}_v(n)$ is given by

$$
\begin{align*}
  z(n + 1) &= A z(n) + B_u u(n) \\
  \hat{y}_v(n) &= H_u(n)^T C_p z(n) + H_u(n)^T D_{pu} u(n) + H_u(n)^T v_p(n).
\end{align*}
$$

(5.38)

The virtual output errors $\epsilon_v(n)$ defined in Eq. (5.32) are now given by

$$
\epsilon_v(n) = y_v(n) - \hat{y}_v(n).
$$

(5.39)

Using Eqs (5.36) and (5.38), a state-space system that models the virtual output errors can be written as

$$
\begin{align*}
  z(n + 1) &= A z(n) + B_u u(n) \\
  \epsilon_v(n) &= C_v(n) z(n) + D_{eu}(n) u(n) + v_v(n) - H_u(n)^T v_p(n),
\end{align*}
$$

(5.40)

where the time-variant state-space matrices $C_v(n) \in \mathbb{R}^{M_v \times N}$ and $D_{eu}(n) \in \mathbb{R}^{M_v \times L}$ are defined as

$$
\begin{align*}
  C_v(n) &= C_v(n) - H_u(n)^T C_p, \\
  D_{eu}(n) &= D_{eu}(n) - H_u(n)^T D_{pu}.
\end{align*}
$$

(5.41)

The time-variant optimal weights $H_{uo}(n) \in \mathbb{R}^{M_v \times M_p}$ are defined as the weights that minimise the time-variant cost function $J_e(n)$ defined in Eq. (5.33) at every sample $n$. As discussed in Section 3.3, it is assumed that the control signals $u(n)$ are zero-mean white and stationary random process when identifying the optimal weights, which has been the case in previous research [80]. Furthermore, the measurement noise signals $v_p(n)$ and $v_v(n)$ are assumed to be zero-mean white and stationary random processes that are uncorrelated to $u(n)$, such that the following covariance matrices can be defined

$$
E \begin{bmatrix} u(n) \\ v_p(n) \\ v_v(n) \\ z(0) \end{bmatrix} E \begin{bmatrix} u(k) \\ v_p(k) \\ v_v(k) \\ z(0) \end{bmatrix}^T = \begin{bmatrix} Q_u & 0 & 0 & 0 \\ 0 & R_p & R_{pv} & 0 \\ 0 & R_{pv}^T & R_p & 0 \\ 0 & 0 & 0 & \Pi_0 \end{bmatrix} \delta_{nk}.
$$

(5.42)

Following the discussions presented in Section 3.3.4 on page 85, where the time-invariant cost function $J_e$ defined in Eq. (3.61) has been derived for the case of spatially fixed virtual locations, it can then be shown that the time-variant cost function in Eq. (5.33) can be written in expanded form as

$$
J_e(n) = \text{tr} \left( C_v(n) \Pi_u C_v(n)^T + D_{eu}(n) Q_u D_{eu}(n)^T + H_u(n)^T R_p H_u(n) - 2 H_u(n)^T R_{pv} - R_p \right),
$$

(5.43)
where \( \Pi_u > 0 \) is the solution to the discrete-time Lyapunov equation [130]

\[
\Pi_u = A \Pi_u A^T + B_u Q_u B_u^T.
\]  

(5.44)

Note that the time-variant cost function defined in Eq. (5.43) can be found from the cost function derived in Eq. (3.61) for the case of fixed virtual primary disturbances, simply by making the time-invariant matrices \( C_p, D_{pv} \) and \( H_u \) time-variant.

A time-variant optimal solution \( H_{uo}(n) \) for the physical sensor weights in Fig. 5.6 can now be derived by minimising the cost function in Eq. (5.43) at every sample \( n \). Substituting the state-space matrices defined in Eq. (5.41) into this cost function gives

\[
J_c(n) = \text{tr} \left( H_u(n)^T R_u H_u(n) - 2H_u(n)^T P_u(n) + \sigma_{vu}(n) \right),
\]  

(5.45)

where \( R_u \in \mathbb{R}^{M_p \times M_p} \) is again the \textit{time-invariant} covariance matrix of the physical primary disturbances, \( P_u(n) \in \mathbb{R}^{M_p \times M_p} \) the \textit{time-variant} cross-covariance matrix between the physical and virtual primary disturbances, and \( \sigma_{vu}(n) \in \mathbb{R}^{M_v \times M_v} \) the \textit{time-variant} covariance matrix of the virtual secondary disturbances, which are defined as

\[
R_u = C_p \Pi_u C_p^T + D_{ps} Q_u D_{ps}^T + R_p,
\]

\[
P_u(n) = C_p \Pi_u C_v(n)^T + D_{ps} Q_u D_{vu}(n)^T + R_p,
\]

\[
\sigma_{vu}(n) = C_v(n) \Pi_v C_v(n)^T + D_{vu}(n) Q_u D_{vu}(n)^T + R_v.
\]  

(5.46)

Note that the covariance matrix \( R_u \) is time-invariant because the physical sensors are assumed \textit{spatially fixed} within a \textit{stationary} sound field. The time-variant optimal weights can be found by differentiating the cost function \( J_c(n) \) in Eq. (5.45) with respect to the weights \( H_u(n) \), and setting all of the resulting derivatives to zero at every sample \( n \) [25], such that

\[
\frac{\partial J_c}{\partial H_u}(n) = 2[R_u H_u(n) - P_u(n)] = 0.
\]  

(5.47)

Solving Eq. (5.47) with respect to the weights \( H_u(n) \) results in

\[
H_{uo}(n) = R_u^{-1} P_u(n).
\]  

(5.48)

Similarly, time-variant optimal weights for the \textit{primary sound field} can be computed as

\[
H_{so}(n) = R_s^{-1} P_s(n),
\]  

(5.49)

where \( R_s \in \mathbb{R}^{M_p \times M_p} \) is again the \textit{time-invariant} covariance matrix of the physical primary disturbances, and \( P_s(n) \in \mathbb{R}^{M_p \times M_p} \) the \textit{time-variant} cross-covariance matrix between the physical and virtual primary disturbances, which are defined as

\[
R_s = C_p \Pi_s C_p^T + D_{ps} Q_s D_{ps}^T + R_p,
\]

\[
P_s(n) = C_p \Pi_s C_v(n)^T + D_{ps} Q_s D_{vs}(n)^T + R_p,
\]  

(5.50)
with $\Pi_s > 0$ the solution to the discrete-time Lyapunov equation [130]

$$\Pi_s = A\Pi_s A^T + B_s Q_s B_s^T.$$  \hfill (5.51)

The above discussions are now summarised in the following theorem.

**Theorem 5.1** (Time-varying optimal weights from state-space model). Given the state-space model in Eq. (5.31), the time-variant optimal weights $H_{uo}(n) \in \mathbb{R}^{M_p \times M_v}$ for the secondary sound field are given by

$$H_{uo}(n) = R_u^{-1}P_u(n),$$  \hfill (5.52)

with $R_u$ and $P_u(n)$ the covariance matrices defined in Eq. (5.46). These weights minimise the cost function

$$J_v(n) = \text{tr} \left( E \left[ \varepsilon_v(n)\varepsilon_v(n)^T \right] \right),$$  \hfill (5.53)

where the virtual output errors $\varepsilon_v(n)$ are given by

$$\varepsilon_v(n) = y_v(n) - H_{uo}(n)^T y_p(n),$$  \hfill (5.54)

with $y_p(n)$ and $y_v(n)$ the outputs of the state-space model in Eq. (5.31) while setting the disturbance source signals $s(n) = 0$. Similarly, the time-variant optimal weights $H_{so}(n) \in \mathbb{R}^{M_p \times M_v}$ for the primary sound field are given by

$$H_{so}(n) = R_s^{-1}P_s(n),$$  \hfill (5.55)

with $R_s$ and $P_s(n)$ the covariance matrices defined in Eq. (5.50). These weights minimise the cost function defined in Eq. (5.53), where the virtual output errors are now given by

$$\varepsilon_v(n) = d_v(n) - H_{so}(n)^T d_p(n),$$  \hfill (5.56)

with $d_p(n)$ and $d_v(n)$ the outputs of the state-space model in Eq. (5.31) while setting the control signals $u(n) = 0$.

Theorem 5.1 can be used to compute time-variant optimal solutions for the physical sensor weights provided that the time-variant virtual primary and secondary transfer paths defined in Eq. (5.13) are known at every sample $n$. As discussed previously, this will generally not be the case in a practical situation because the moving virtual locations will not be known a priori.
5.4 Moving virtual sensing algorithms

**Practical implementation of adaptive LMS moving virtual microphone technique**

The assumption introduced in Section 5.2, i.e. that the moving virtual locations are confined to relatively small regions of the sound field called target zones, is now used to develop a more practical implementation of the adaptive LMS moving virtual microphone technique, which is illustrated in Fig. 5.7.

![Block diagram of the practical implementation of the adaptive LMS moving virtual microphone technique](image)

Figure 5.7: Block diagram of the practical implementation of the adaptive LMS moving virtual microphone technique, with \( L \) control sources, \( M_p \) physical sensors, \( \bar{M}_v \) spatially fixed virtual sensors suitably positioned throughout the target zones at \( \bar{X}_v \), and \( M_v \) moving virtual sensors that track the moving virtual locations \( X_v(n) \).

Fig. 5.7 shows that the practical implementation is again based on using spatial interpolation. This approach has also been used in Section 5.3 to generate estimates of the virtual filtered-reference signals, with the estimates computed using spatial interpolation between the virtual filtered-reference signals generated for the spatially fixed virtual locations \( \bar{X}_v \in \mathbb{R}^{3 \times \bar{M}_v} \) defined in Eq. (5.20). These spatially fixed virtual locations are suitably positioned throughout the target zones. Because the locations \( \bar{X}_v \) are spatially fixed, estimates \( \bar{e}_v(n) \in \mathbb{R}^{\bar{M}_v} \) of the virtual error signals at these locations can be computed as described in Section 3.3 and illustrated in Fig. 5.7, such that

\[
\bar{e}_v(n) = \hat{H}_{so}^T e_p(n) + (\hat{H}_{uo} - \hat{H}_{so})^T G_{pu} u(n),
\]

where the optimal weights \( \hat{H}_{so} \in \mathbb{R}^{M_p \times \bar{M}_v} \) and \( \hat{H}_{uo} \in \mathbb{R}^{M_p \times \bar{M}_v} \) can be calculated as defined in Theorem 3.1 given the state-space models

\[
\hat{G}_{vos} \sim \begin{bmatrix} A & B_v \\ \bar{C}_v & \bar{D}_{vos} \end{bmatrix}, \quad \hat{G}_{vou} \sim \begin{bmatrix} A & B_u \\ \bar{C}_v & \bar{D}_{vou} \end{bmatrix},
\]

with \( \hat{G}_{vos} \in \mathcal{RH}_\infty^{M_v \times S} \) the virtual primary transfer path matrix between the disturbance source signals \( s(n) \) and the virtual primary disturbances \( \bar{d}_v(n) \) at the spatially fixed virtual locations \( \bar{X}_v \), and \( \hat{G}_{vou} \in \mathcal{RH}_\infty^{M_v \times S} \) the virtual secondary transfer path matrix between the control signals \( u(n) \) and the virtual secondary disturbances \( \bar{y}_v(n) \) at the spatially fixed virtual locations \( \bar{X}_v \).
As illustrated in Fig. 5.7, an estimate $\hat{e}_v(n)$ of the virtual error signals at the moving virtual locations $X_v(n)$ is now computed by spatially interpolating between the estimated virtual error signals $\hat{\bar{e}}_v(n)$ at the spatially fixed virtual locations $\bar{X}_v$, with

$$\hat{e}_v(n) = \begin{bmatrix} \hat{\bar{e}}_{v1}(n) & \hat{\bar{e}}_{v2}(n) & \ldots & \hat{\bar{e}}_{vM_v}(n) \end{bmatrix}^T. \quad (5.59)$$

Note again that when the optimal weights for the primary and secondary sound fields are equal in Eq. (5.57), the estimate $\hat{\bar{e}}_v(n)$ of the virtual error signals at the spatially fixed locations $\bar{X}_v$ can simply be calculated as

$$\hat{\bar{e}}_v(n) = \bar{H}_o^T e_p(n), \quad (5.60)$$

where $\bar{H}_o \in \mathbb{R}^{M_p \times \bar{M}_v}$ are the optimal physical sensor weights directly applied to the physical error signals. Following the discussions presented in Section 3.3, this is similar to the implementation used in previous research for the case of spatially fixed virtual locations [80], where one set of weights was directly applied to the virtual error signals without separating them into primary and secondary components.

The proposed method will be implemented on an acoustic duct arrangement in Chapter 10, where a moving zone of quiet at a moving virtual sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances. In these experiments, the estimated virtual error signal is adaptively minimised as illustrated in Fig. 5.7, with the estimate computed using the practical implementation of the adaptive LMS moving virtual microphone technique shown in Fig. 5.7.

### 5.4.3 Remote moving microphone technique

In this section, the remote microphone technique [104, 112] is extended to a moving virtual sensing algorithm, which will be called the remote moving microphone technique. This technique is derived here without altering the structure of the remote microphone technique, which has been illustrated in Fig. 3.6. The block diagram from this figure is modified in Fig. 5.8 to allow for a virtual location that is moving through the sound field rather than being spatially fixed. Similarly to Fig. 3.6, the estimate of the virtual error signals is computed using the physical and virtual secondary transfer path matrices, and a filter that computes an estimate $\hat{d}_v(n)$ of the virtual primary disturbances given the physical primary disturbances $d_p(n)$. However, to account for a virtual location that is moving through the sound field rather than being spatially fixed, the virtual secondary transfer path matrix $G_{vu}(n)$ and the filter $H(n)$ are now time-variant in Fig. 5.8 instead of time-invariant as in Fig. 3.6. The estimate of the virtual error signals is now computed as

$$\hat{e}_v(n) = G_{RMT}(n) \begin{bmatrix} e_p(n) \\ u(n) \end{bmatrix} = \begin{bmatrix} H(n) & G_{vu}(n) - H(n)G_{pu} \end{bmatrix} \begin{bmatrix} e_p(n) \\ u(n) \end{bmatrix}, \quad (5.61)$$
where $G_{RMT}(n) \in \mathbb{R}^{M_v \times (L+M_p)}$ is the time-variant transfer function matrix that describes the input-output behaviour of the remote moving microphone technique. The time-variant virtual secondary transfer path $G_{vu}(n)$ has been defined in Eq. (5.13).

Optimal time-variant solution for filter $H(n)$

To implement the remote moving microphone technique illustrated in Fig. 5.8, an optimal solution for the time-variant filter $H(n)$ needs to be derived. In Chapter 3, an optimal solution for the case of virtual locations that are spatially fixed within the sound field was derived in Theorem 3.2 by solving the filter problem shown in Fig. 3.7. The block diagram from this figure is modified in Fig. 5.9 to the case of virtual locations that are moving through the sound field rather than being spatially fixed.

The filter problem illustrated in Fig. 5.9 is to find a time-variant optimal filter $H_v(n)$ that minimises the time-variant cost function defined in Eq. (5.33) as

$$J_\epsilon(n) = \text{tr} \left( E \left[ \epsilon_v(n) \epsilon_v(n)^T \right] \right),$$  

(5.62)
with \( \epsilon_v(n) \) the virtual output errors given by
\[
\epsilon_v(n) = d_v(n) - \hat{d}_v(n) = (G_{vs}(n) - H(n)G_{ps}) s(n).
\] (5.63)

In Theorem 3.2, a time-invariant optimal solution \( H_0 \) has been defined for the case of virtual locations that are spatially fixed within the sound field. This solution has been derived by solving the filter problem illustrated in Fig. 3.7, and has been defined in Theorem 3.2 as
\[
H_0 = \begin{bmatrix} G_{vs} G_{ps,ci}^* + G_{ps,co}^* 
\end{bmatrix}.
\] (5.64)

The question now arises as to whether the optimal solution to the filter problem shown in Fig. 5.9 is simply given by
\[
H_0(n) = \begin{bmatrix} G_{vs}(n) G_{ps,ci}^* + G_{ps,co}^* 
\end{bmatrix}.
\] (5.65)

where the time-invariant virtual primary transfer path matrix \( G_{vs} \) in Eq. (5.64) has simply been replaced with the time-variant virtual primary transfer path matrix \( G_{vs}(n) \) defined in Eq. (5.13). Whether Eq. (5.65) is the optimal solution to the filter problem shown in Fig. 5.9 is not immediately obvious because the optimal solution \( H_0 \) defined in Eq. (5.64) has been derived in Theorem 3.2 assuming that the virtual primary transfer path matrix is time-invariant. Fortunately, it will be shown in Section 5.4.4 that Eq. (5.65) indeed defines the time-variant optimal solution to the filter problem shown in Fig. 5.9. In that section, an optimal solution to the moving virtual sensing problem is derived using Kalman filter theory.

**Practical implementation of remote moving microphone technique**

As stated previously, the time-variant virtual primary and secondary transfer paths defined in Eq. (5.13) are generally not known for every sample \( n \), because the moving virtual locations are generally not known a priori. Therefore, the assumption introduced in Section 5.2, i.e. that the moving virtual locations are confined to relatively small regions called target zones, is used to develop a more practical implementation of the remote moving microphone technique, which is illustrated in Fig. 5.10. The practical implementation is again based on using spatial interpolation, where a number of spatially fixed virtual locations \( \tilde{X}_v \) are suitably positioned throughout the target zones. Because these locations are spatially fixed, an estimate \( \hat{e}_v(n) \in \mathbb{R}^{M_v} \) of the virtual error signals at these locations can be computed as described in Section 3.4 and illustrated in Fig. 5.10, such that
\[
\hat{e}_v(n) = \begin{bmatrix} \hat{H}_0 & \tilde{G}_{vu} - \hat{H}_0 \tilde{G}_{pu} 
\end{bmatrix} \begin{bmatrix} e_p(n) \\ u(n) \end{bmatrix},
\] (5.66)
5.4 Moving virtual sensing algorithms

Figure 5.10: Block diagram of the practical implementation of the remote moving microphone technique, with \( L \) control sources, \( M_p \) physical sensors, \( M_v \) spatially fixed virtual sensors suitably positioned throughout the target zones at \( \bar{X}_v \), and \( M_v \) moving virtual sensors that track the moving virtual locations \( \bar{X}_v(n) \).

with \( \tilde{G}_{vu} \in \mathcal{RH}_{\infty}^{M_v \times S} \) the virtual secondary transfer path matrix between the control signals \( u(n) \) and the virtual primary disturbances \( \bar{y}_v(n) \) at the spatially fixed virtual locations \( \bar{X}_v \), which has been defined in Eq. (5.58). The optimal filter \( \bar{H}_o \in \mathcal{RH}_{\infty}^{M_v \times M_p} \) in Eq. (5.66) can be calculated as defined in Theorem 3.2, such that

\[
\bar{H}_o = \left[ \tilde{G}_{vs} G^*_{ps,cl} \right] + G^+_{ps,co} \quad \text{(5.67)}
\]

with \( G_{ps} \) the physical primary transfer path matrix, and \( \tilde{G}_{vs} \in \mathcal{RH}_{\infty}^{M_v \times S} \) the virtual primary transfer path matrix between the disturbance source signals \( s(n) \) and the virtual primary disturbances \( \hat{d}_v(n) \) at the spatially fixed virtual locations \( \bar{X}_v \), which has been defined in Eq. (5.58). As illustrated in Fig. 5.10, an estimate \( \hat{e}_v(n) \) of the virtual error signals at the moving virtual locations \( \bar{X}_v(n) \) is now computed by spatially interpolating between the estimated virtual error signals \( \hat{\bar{e}}_v(n) \), which are computed in Eq. (5.66) for the spatially fixed virtual locations \( \bar{X}_v \).

5.4.4 Kalman filter based moving virtual sensing algorithm

In this section, an optimal solution to the moving virtual sensing for active noise control problem introduced in Section 5.2 is derived using Kalman filtering theory [60]. This approach was also used in Section 3.7 to derive an optimal solution to the spatially fixed virtual sensing for active noise control problem. For this case, the active noise control system was described by the state-space model defined in Eq. (3.1), where the state-space matrices are all time-invariant. In the moving virtual sensing problem
Chapter 5  Active noise control at moving physical and virtual sensors

considered here, the active noise control system is described by the state-space model defined in Eq. (5.31), where the state-space matrices \( C_v(n) \), \( D_{vu}(n) \), and \( D_{vs}(n) \) are time-variant due to the movement of the virtual locations through the sound field.

The advantage of the state-space based approach that was adopted in Section 3.7 to solve the spatially fixed virtual sensing problem is that the extension to time-variant systems can now be made, quoting Kailath et al. [60], ‘just by the stroke of a pen’. In other words, an optimal solution to the moving virtual sensing for active noise control problem can simply be found by replacing the time-invariant state-space matrices \( C_v \), \( D_{vu} \), and \( D_{vs} \) with their time-variant counterparts \( C_v(n) \), \( D_{vu}(n) \), and \( D_{vs}(n) \) in the derivations presented in Section 3.7. The block diagram of the optimal spatially fixed virtual sensing algorithm illustrated in Fig. 3.11 can thus be modified to an optimal moving virtual sensing algorithm as shown in Fig. 5.11.

![Block diagram of the Kalman filter based moving virtual sensing algorithm](image)

Figure 5.11: Block diagram of the Kalman filter based moving virtual sensing algorithm, with \( L \) control sources, \( M_p \) physical sensors, and \( M_v \) moving virtual sensors that track the moving virtual locations \( X_v(n) \).

Fig. 5.11 illustrates that the input signals into the Kalman filter based moving virtual sensing algorithm are the physical error signals, or observations, \( e_p(n) \) and the deterministic control signals \( u(n) \). The prediction form of the Kalman filter, which has been introduced in Section 3.7, is then used to compute the innovations \( e_p(n) \) and a predicted state estimate \( \hat{z}(n|n-1) \). As shown in Fig. 5.11, these output signals from the Kalman filter plus the deterministic control signals \( u(n) \) are then used to compute a current estimate \( \hat{e}_v(n|n) \) of the virtual error signals given the time-variant state-space matrices \( C_v(n) \) and \( D_{vu}(n) \) of the state-space model defined in Eq. (5.31), and the time-variant virtual gain matrix \( M_{vs}(n) \), which is defined in the following theorem.

**Optimal state-space solution**

From Theorem 3.4, a state-space model of the Kalman filter based moving virtual sensing algorithm illustrated in Fig. 5.11 is defined by the following theorem.
Theorem 5.2 (Kalman filter based moving virtual sensing algorithm).

Let a state-space realisation of the plant be given by Eq. (5.4), and let the covariance matrices $\bar{Q}_s$, $\bar{S}_{ps}$, $\bar{R}_p$, and $\bar{R}_{pv}(n)$ be defined as in Eqs (5.5)–(5.11). Furthermore, let

- the pair $(C_p, A)$ be detectable;
- $\bar{R}_p > 0$, $\bar{Q}_s - \bar{S}_{ps}^T \bar{R}_p^{-1} \bar{S}_{ps} \geq 0$;
- $(A - \bar{S}_{ps}^T \bar{R}_p^{-1} C_p, \bar{Q}_s - \bar{S}_{ps}^T \bar{R}_p^{-1} \bar{S}_{ps})$ has no uncontrollable modes on the unit circle.

Then a state-space realisation of the moving virtual sensing algorithm that gives optimal current estimates $\hat{e}_c(n|n)$ of the virtual error signals, given observations $e_p(i)$ of the physical error signals up to $i = n$, is defined as

$$
\begin{bmatrix}
\hat{z}(n+1|n) \\
\hat{e}_c(n|n)
\end{bmatrix} =
\begin{bmatrix}
A - K_{ps} C_p & B_u - K_{ps} D_{ps} \\
C_v(n) - M_{vs}(n) C_p & D_{vu}(n) - M_{vs}(n) D_{ps}
\end{bmatrix}
\begin{bmatrix}
\hat{z}(n|n-1) \\
\hat{u}(n)
\end{bmatrix},
$$

(5.68)

where the Kalman gain matrix $K_{ps}$, and the time-variant virtual gain matrix $M_{vs}(n)$ are given by

$$
K_{ps} = (A P_{ps} C_p^T + \bar{S}_{ps}^T) R_{pe}^{-1},
$$

$$
M_{vs}(n) = (C_v(n) P_{ps} C_p^T + R_{ps}(n))^T R_{pe}^{-1},
$$

(5.69)

with $P_{ps} = P_{ps}^T > 0$ the unique stabilising solution to the DARE given by

$$
P_{ps} = A P_{ps} A^T - (A P_{ps} C_p^T + \bar{S}_{ps}^T)(C_p P_{ps} C_p^T + R_p)^{-1}(A P_{ps} C_p^T + \bar{S}_{ps}^T)^T + Q_s,
$$

(5.70)

and where $R_{pe} \in \mathbb{R}^{M_p \times M_p}$ is the covariance matrix of the white innovation signals $\epsilon_p(n)$ given by

$$
R_{pe} = \text{E}[\epsilon_p(n)\epsilon_p(n)^T] = C_p P_{ps} C_p^T + \bar{R}_p,
$$

(5.71)

with $\epsilon_p(n)$ defined in Eq. (3.142).

Note that the state-space solution defined in Theorem 5.2 is simply found by replacing the time-invariant state-space matrices $C_v$, $D_{vu}$, and $D_{vs}$ in the optimal state-space solution to the spatially fixed virtual sensing problem defined in Theorem 3.4 with their time-variant counterparts $C_v(n)$, $D_{vu}(n)$, and $D_{vs}(n)$. Also note that the Kalman gain matrix $K_{ps}$ remains time-invariant as in in Theorem 3.4 because the physical sensors are assumed spatially fixed within a stationary sound field.

State-space solution for the optimal time-variant filter $H_v(n)$

In Section 5.4.3, the remote moving microphone technique was introduced and a optimal time-variant transfer function solution $H_v(n)$ for the filter problem illustrated in Fig. 5.9 was defined in Eq. (5.65) as

$$
H_v(n) = \begin{bmatrix} G_{vs}(n) G_{ps,cl}^* \end{bmatrix} G_{ps,co}^+.
$$

(5.72)
The optimal time-variant solution in Eq. (5.72) was found by replacing the \textit{time-invariant} virtual primary transfer path matrix $G_{vs}$ with the \textit{time-variant} virtual primary transfer path matrix $G_{vs}(n)$ in the optimal solution for the \textit{spatially fixed} virtual sensing problem, which was defined in Theorem 3.2 as

$$H_o = \left[ G_{vs}G_{ps,ci}^* \right] + G_{ps,co}^\dagger. \quad (5.73)$$

Whether Eq. (5.72) is the optimal time-variant solution to the filter problem shown in Fig. 5.9 is not immediately obvious because the optimal solution $H_o$ defined in Eq. (5.73) was derived in Theorem 3.2 assuming that the virtual primary transfer path matrix $G_{vs}$ is \textit{time-invariant}. In Appendix B, it is shown that a state-space solution for the optimal filter $H_o$ defined in Eq. (5.73) is given by

$$H_o \sim \left[ \begin{array}{cc} A - K_{ps}C_p & K_{ps} \\ C_v - M_{vs}(n)C_p & M_{vs}(n) \end{array} \right]. \quad (5.74)$$

with the Kalman gain matrix $K_{ps}$ and the virtual gain matrix $M_{vs}$ defined in Theorem 3.4. Using the derivations presented in Appendix B, a state-space solution for the optimal time-variant filter $H_o(n)$ defined in Eq. (5.72) can now be computed for every sample $n$ as

$$H_o(n) \sim \left[ \begin{array}{cc} A - K_{ps}C_p & K_{ps} \\ C_v(n) - M_{vs}(n)C_p & M_{vs}(n) \end{array} \right]. \quad (5.75)$$

Eq. (5.75) shows that the optimal time-variant transfer function solution $H_o(n)$ defined in Eq. (5.72) is indeed the optimal solution to the filter problem illustrated in Fig. 5.9. This can be derived by setting the control signals $u(n) = 0$ in Eq. (5.68), which then reduces to

$$\begin{bmatrix} \hat{z}(n+1|n) \\ \hat{d}(n|n) \end{bmatrix} = \begin{bmatrix} A - K_{ps}C_p & K_{ps} \\ C_v(n) - M_{vs}(n)C_p & M_{vs}(n) \end{bmatrix} \begin{bmatrix} \hat{z}(n|n-1) \\ \hat{d}(n|n-1) \end{bmatrix}. \quad (5.76)$$

Eq. (5.76) defines a state-space model that is equivalent to the state-space solution defined in Eq. (5.75). This solution has been derived from the optimal time-variant transfer function solution defined in Eq. (5.72), which thus defines the optimal solution to the filter problem illustrated in Fig. 5.9.

**Practical implementation of Kalman filter based moving virtual sensing algorithm**

As discussed previously, the time-variant state-space matrices $C_v(n)$, $D_{vu}(n)$, and virtual gain matrix $M_{vs}(n)$ shown in Fig. 5.11 are generally not known for every sample $n$, because the moving virtual locations are not known a priori. The assumption introduced in Section 5.2, i.e. that the moving virtual locations are confined to relatively small
5.4 Moving virtual sensing algorithms

Figure 5.12: Block diagram of the practical implementation of the Kalman filter based moving virtual sensing algorithm, with $L$ control sources, $M_p$ physical sensors, $M_v$ spatially fixed virtual sensors suitably positioned throughout the target zones at $\bar{X}_v$, and $M_o$ moving virtual sensors that track the moving virtual locations $X_v(n)$.

regions called target zones, is therefore used to arrive at a more practical implementation of the Kalman filter based moving virtual sensing algorithms, which is illustrated in Fig. 5.12. The practical implementation is again based on using spatial interpolation, where a number of spatially fixed virtual locations $\bar{X}_v$ are suitably positioned throughout the target zones. Because these locations are spatially fixed, an estimate $\hat{e}_v(n) \in \mathbb{R}^{M_v}$ of the virtual error signals at these locations can be computed as defined in Theorem 3.4 and illustrated in Fig. 3.11, such that

$$\begin{bmatrix} \hat{z}(n+1|n) \\ \hat{e}_v(n|n) \end{bmatrix} = \begin{bmatrix} A - K_{ps} C_p & B_{vu} - K_{ps} D_{pu} \\ \bar{C}_v - \bar{M}_{vs} C_p & \bar{D}_{vu} - \bar{M}_{vs} D_{pu} - \bar{M}_{vs} \end{bmatrix} \begin{bmatrix} \hat{z}(n|n-1) \\ u(n) \\ e_p(n) \end{bmatrix}, \quad (5.77)$$

with $\bar{C}_v$ and $\bar{D}_{vu}$ the state-space matrices of the virtual secondary transfer path matrix $\bar{G}_{vu}$ between the control signals $u(n)$ and the virtual primary disturbances $\bar{y}_v(n)$ at the spatially fixed virtual locations $\bar{X}_v$, which has been defined in Eq. (5.58). The virtual gain matrix $\bar{M}_{vs} \in \mathbb{R}^{M_v \times M_p}$ in Fig. 5.12 can be calculated as defined in Theorem 3.4, such that

$$\bar{M}_{vs} = (\bar{C}_v P_{ps} C_p^T + \bar{R}_{ps}) R_{pe}^{-1}, \quad (5.78)$$

with $P_{ps} = P_{ps}^T > 0$ the unique stabilising solution to the DARE defined in Theorem 3.4, $R_{pe}$ the covariance matrix of the white innovation signals $e_p(n)$ defined in Theorem 3.4, and $\bar{R}_{ps}$ the covariance matrix between the auxiliary measurement noises on the physical sensors and virtual sensors spatially fixed at $\bar{X}_v$, which is defined similarly to Eq. (3.134). As illustrated in Fig. 5.12, an estimate $\hat{e}_v(n)$ of the virtual error signals at the moving virtual locations $X_v(n)$ is now computed by spatially interpolating between the
estimated virtual error signals \( \hat{e}_v(n) \), which are calculated in Eq. (5.77) for the spatially fixed virtual locations \( \bar{X}_v \).

The proposed method will be implemented on an acoustic duct arrangement in Chapter 10, where a moving zone of quiet at a moving virtual sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances. In these experiments, the estimate of the virtual error signal is adaptively minimised as illustrated in Fig. 5.5, with the estimate computed using the practical implementation of the Kalman filter based moving virtual sensing algorithm shown in Fig. 5.12.

5.5 Conclusion

The local active noise control algorithms introduced in Chapter 2 and the spatially fixed virtual sensing algorithms presented in Chapter 3 can be combined as described in Chapter 4 to create zones of quiet at a number of virtual locations that are spatially fixed within the sound field. Because an observer is very likely to move their head, the desired locations of the zones of quiet are generally moving through the sound field rather than being spatially fixed. The performance of a local active noise control system can thus be improved by creating moving zones of quiet that track the observer’s ears. In this chapter, algorithms have therefore been proposed that can be used to create moving zones of quiet that track the desired locations of maximum attenuation. These algorithms have been developed assuming that an exact measurement of these locations is available.

The multi-channel filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms, which have been introduced in Chapter 2 for the case of virtual error signals that are directly measured during real-time control by physical sensors positioned at the spatially fixed virtual locations, have been modified to allow for virtual locations that are moving through the sound field. The proposed adaptive feedforward control method can be used to attenuate the non-stationary primary disturbances at a number of moving physical sensors that track the desired locations of maximum attenuation. The proposed method will be implemented on an acoustic duct arrangement in Chapter 9, where a moving zone of quiet at a moving physical sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances.

The developed adaptive feedforward control method has been modified to the case where the true virtual error signals are not directly measured during real-time control by moving physical sensors. In the proposed method, an estimate of the virtual error signals is adaptively minimised rather than the true virtual error signals. To obtain this estimate, a moving virtual sensing algorithm has been developed that is able to compute
an estimate of the virtual error signals at virtual locations that are moving through the sound field. The previously proposed spatially fixed virtual sensing algorithms introduced in Chapter 3 have been extended to allow for a virtual location that is moving through the sound field rather than being spatially fixed. It has been shown that an optimal solution to the moving virtual sensing problem can be derived using Kalman filtering theory [60]. The proposed method will be implemented on an acoustic duct arrangement in Chapter 10, where a moving zone of quiet at a moving virtual sensor that tracks the desired moving location of maximum attenuation is successfully created inside an acoustic duct arrangement for narrowband disturbances. The acoustic duct arrangement used in these experiments is introduced in the next chapter.
Part II

Experimental validation of algorithms
Chapter 6

Acoustic duct arrangement

6.1 Introduction

This chapter introduces the rigidly terminated, rectangular acoustic duct arrangement that will be used in the real-time experiments presented in Chapters 7–10, where the active noise control and virtual sensing algorithms developed in Chapters 2–5 are implemented in real-time. A schematic diagram of the acoustic duct arrangement is introduced in Section 6.2, and the hardware and software that is used in the real-time experiments is briefly discussed. In Section 6.3, two acoustic models of a rectangular acoustic duct of finite length are presented, which are called a modal model [88] and a travelling wave model [129]. A more succinct expression for the travelling wave model introduced by Zander and Hansen [129] is derived as well. The presented acoustic duct models are used in Chapters 7–8 to compute numerical results that can be compared with the obtained experimental results. To implement the active noise control and virtual sensing algorithms presented in Chapters 2–5 in real-time, models of the transfer paths that define the input-output behaviour of the acoustic duct arrangement need to be estimated in a preliminary identification stage. In Section 6.4, subspace model identification techniques [50, 124] are therefore briefly introduced, which are used here to estimate an innovations model of the acoustic duct arrangement in a preliminary identification stage.

6.2 Acoustic duct arrangement

A schematic diagram of the rigidly terminated, rectangular acoustic duct arrangement is shown in Fig. 6.1. The acoustic duct is of length $L_x = 4.83$ m, width $L_y = 0.205$ m, and height $L_z = 0.205$ m, and is made from 20 mm thick high density fibreboard. The coupling between the sound field inside the acoustic duct and the duct walls is therefore assumed to be negligible.
Figure 6.1: Schematic diagram of the experimental acoustic duct arrangement, with $S = 1$ disturbance source, $L = 1$ control source, $M_p = 5$ physical sensors, and $M_v = 1$ virtual sensor. The developed algorithms are implemented in real-time using a host-target software program called xPC Target.
6.2 Acoustic duct arrangement

Actuator and sensor configurations

A 4” loudspeaker located at \( x_p = 4.73 \text{ m} \) is used as a primary disturbance source. This primary loudspeaker is excited by the disturbance source signal \( s(n) \). Before exciting the primary loudspeaker, this signal is sent through a 16-bit PCI-DDA08/16 analog output board from Measurement Computing, which is used as an 8-channel DA-converter, and an amplifier. Another 4” loudspeaker located at \( x_s = 0.5 \text{ m} \) is used as a control source. This control loudspeaker is excited by the control signal \( u(n) \), which is first sent through the 16-bit DA-converter and the amplifier before exciting the control loudspeaker. The sound field inside the duct is measured by five electret microphones located at

\[
x_p = [1.4250 \ 1.4375 \ 1.4500 \ 1.4675 \ 1.4750] \text{ m},
\]

which can be used as physical sensors that measure the physical error signals \( e_p(n) \) in the real-time experiments. These physical microphones are connected to a power supply which is also used as an amplifier. The amplified physical error signals are then passed through a 16-bit PCI-DAS1602/16 multifunction analog & digital I/O Board from Measurement Computing, which is used as a 16-channel AD-converter.

The aim of the real-time experiments presented in Chapters 7–10 is to attenuate the virtual primary disturbance at the virtual sensor shown in Fig. 6.1, which is either spatially fixed inside or moving through the acoustic duct arrangement. This virtual sensor is positioned at the desired location of maximum attenuation, which is defined by the virtual location \( x_v(n) \) given by

\[
x_v(n) = 1.4750 + v(n),
\]

where \( v(n) \) is the distance to the closest physical sensor located at \( x_{p5} = 1.4750 \text{ m} \). The distance \( v(n) \) will be called the virtual distance in the following, and is time-variant when the virtual location is moving through the acoustic duct arrangement. When the virtual location is spatially fixed inside the acoustic duct arrangement, the virtual distance is time-invariant and will be denoted by \( v \) in the following.

Traversing microphone

Also located inside the acoustic duct arrangement, but not included in Fig. 6.1, is a traversing microphone mounted on a cable wrapped around a pulley at each end of the duct. This cable is wound onto a spool with circumference 0.150 m which is mounted on the shaft of a DC servo-motor encoder unit. This encoder provides dual track TTL output signals of 500 pulses per revolution, such that a resolution of 0.3 mm is obtained for the measurement of the traversing microphone position. This enables accurate position control of the traversing microphone using a PI-controller. The traversing microphone is connected to the microphone power supply which also amplifies the
measured signal. The amplified signal is then passed through the 16-bit AD-converter. The traversing microphone can be placed at the desired spatially fixed virtual location \( x_v \) in a preliminary system identification stage that is required when implementing the various algorithms developed in Chapters 2–5. The traversing microphone can also be used to analyse the primary and controlled sound pressure distributions inside the acoustic duct arrangement. This is very useful when analysing whether the zone of quiet has effectively been moved away from the physical microphones to the intended spatially fixed virtual location. Furthermore, the traversing microphone can be position controlled such that it tracks a virtual location \( x_v(n) \) that is moving through the sound field. This is very useful when analysing the control performance that is obtained when the virtual location is moving through the acoustic duct arrangement.

**xPC Target**

To implement the developed algorithms in real-time, a host-target software program called xPC Target is used [75], as illustrated in Fig. 6.1. xPC Target is an environment that uses a target PC, which is connected to a separate host PC via a TCP/IP network connection, for running real-time applications. A desktop PC or laptop with MATLAB and SIMULINK installed can be used as a host PC for creating models using SIMULINK blocks. SIMULINK provides an environment in which the physical system and the developed algorithms can be modelled as a block diagram, which can then be analysed in nonreal-time. For real-time applications, the physical system model is then replaced by I/O blocks that are connected to the sensors and actuators, which in the case of the acoustic duct arrangement considered here are microphones and loudspeakers, respectively. After adding these I/O blocks to the model, the host PC can be used together with REAL-TIME WORKSHOP and a C compiler to create executable code from the created SIMULINK block diagram. This code is then downloaded from the host PC to the target PC that runs the xPC TARGET real-time kernel, after which the target application can be run and tested in real-time. The target application can be started and stopped with the host PC from the MATLAB command line or from the SIMULINK block diagram if it is running in external mode. Real-time data can be logged, displayed, and analysed using MATLAB on the host PC, and parameters can be changed from the host PC while the target application is running in real-time.

**6.3 Numerical modelling**

In this section, two acoustic models of a rectangular acoustic duct of finite length are presented, namely a modal model [88] and a travelling wave model [129]. Both models describe the sound pressure distribution within an acoustic duct due to a point
monopole source located inside the duct. Fig. 6.2 shows a schematic diagram of a rectangular acoustic duct of length $L_x$, width $L_y$, height $L_z$, and with arbitrary termination conditions characterised by $\Phi_1$ and $\Phi_2$.

![Schematic diagram of a rectangular acoustic duct](image)

Figure 6.2: Schematic diagram of a rectangular acoustic duct of dimensions $(L_x, L_y, L_z)$, with arbitrary termination conditions characterised by $\Phi_1$ and $\Phi_2$. The complex transfer impedance $Z_{pq}$ relates the complex pressure $p(r)$ at $(\odot)$ to the complex source strength $q(r_s)$ of a point monopole source located at $(\bullet)$.

The acoustic models introduced in this section can be used to model the complex acoustic transfer impedance $Z_{pq}$ illustrated in Fig. 6.2, which relates the complex pressure $p(r)$ at a point $r = (x, y, z)$ arising from a point monopole source of complex volume velocity $q(r_s)$ located at $r_s = (x_s, y_s, z_s)$, such that [88]

$$Z_{pq} = \frac{p(r)}{q(r_s)}. \quad (6.3)$$

For an angular excitation frequency $\omega$, the complex transfer impedance thus relates the magnitude and phase of the sound pressure $p(r)$ to the magnitude and phase of the complex volume velocity $q$ of a point monopole source located at $r_s$. In the following sections, two acoustic duct models are presented that model these acoustic transfer impedances.

### 6.3.1 Travelling wave model

In this section, the travelling wave model of a rectangular acoustic duct originally introduced by Zander and Hansen [129] is presented, and a more succinct expression for this model is derived. It is assumed that the arbitrary termination conditions in Fig. 6.2 are characterised by the termination phasors $\Phi_1 \in \mathbb{C}$ and $\Phi_2 \in \mathbb{C}$, which are defined as

$$\Phi_i = \pi \alpha_i - j \pi \beta_i, \quad i = 1, 2. \quad (6.4)$$
Chapter 6  Acoustic duct arrangement

The termination phasor \( \Phi_i \) in Eq. (6.4) is related to the reflection coefficient \( R_i = e^{-2\Phi_i} \). The coefficients \( \alpha_i \in \mathbb{R}^+ \) and \( \beta_i \in \mathbb{R}^+ \) are thus determined by the acoustic properties of the material at each end of the duct. A rigid, totally reflective termination is achieved for \( \alpha_i = \beta_i = 0 \), while an anechoic termination is obtained for \( \alpha_i = \infty \).

In the travelling wave model, the complex acoustic pressure \( p(r) \) at a point \( r = (x, y, z) \) in the duct due to a point monopole source of complex volume velocity \( q \) located at \( r_s = (x_s, y_s, z_s) \), is given by

\[
p(r) = \sum_{n=0}^{N-1} q A_n(r, r_s) \left( e^{-jk_n|x_s-x|} + e^{-jk_n|x_s+x|-2\Phi_1} + e^{jk_n|x_s-x|-2\Phi_1} + e^{jk_n|x_s+x|-2\Phi_1} \right) T_n, \tag{6.5}
\]

where \( A_n(r, r_s) \) is the modal amplitude of the \( n \)th mode defined as

\[
A_n(r, r_s) = \frac{\rho \omega \phi_n(r_s) \phi_n(r)}{2S \Lambda_n k_n}, \tag{6.6}
\]

and with \( T_n \in \mathbb{C} \) the modal reverberation factor of the duct given by

\[
T_n = \frac{1}{1 - e^{-2k_nL_x-2\Phi_1-2\Phi_2}}, \tag{6.7}
\]

with \( n = (n_y, n_z) \) the modal indices, and \( N = (N_y, N_z) \) the number of modes considered in the travelling wave model. In the above equations, the modal wave numbers \( k_n \in \mathbb{C} \) are defined as

\[
k_n^2 = k^2 - k_{n_y}^2 - k_{n_z}^2 = \left( \frac{\omega}{c} \right)^2 - \left( \frac{n_y \pi}{L_y} \right)^2 - \left( \frac{n_z \pi}{L_z} \right)^2, \tag{6.8}
\]

the mode shapes or eigenfunctions \( \phi_n(r) \in \mathbb{R} \) of the duct are given by

\[
\phi_n(r) = \cos(k_{n_y}y) \cos(k_{n_z}z), \tag{6.9}
\]

and the modal normalisation factors \( \Lambda_n \) are defined as

\[
\Lambda_n = \frac{1}{S} \int_S \phi_n^2 dS = \frac{1}{2 \varepsilon_y \varepsilon_z}, \tag{6.10}
\]

where \( \varepsilon_i \) is the unit step function for integers which can be expressed as

\[
\varepsilon_i = \begin{cases} 
0 & n_i = 0 \\
1 & n_i > 0.
\end{cases} \tag{6.11}
\]

Furthermore, \( \rho \) is the density of air, \( c \) the speed of sound, \( S = L_y L_z \) the cross sectional area of the duct, and \( k \) the acoustic wave number.
6.3 Numerical modelling

Simplification of Travelling Wave Model

The travelling wave model defined in Eq. (6.5) can be simplified, assuming \( 0 \leq x \leq x_s \), by writing the part containing the exponential terms as

\[
\phi(x) = e^{-\beta_n (x - x_s)} + e^{-\beta_n (x + x_s)} - 2\beta_1 - q e^{\beta_n (x - x_s)} - 2\beta_2 - q e^{\beta_n (x + x_s)} - 2\Phi_1 - 2\Phi_2 =
\]

A similar simplification can be found for \( x_s \leq x \leq L_x \). The final expression for the sound pressure \( p(r) \) in the duct due to a point monopole source of complex volume velocity \( q \) located at \( r_s \) can now be expressed as

\[
p(r) = \sum_{n=0}^{N-1} \frac{2q A_i (r, r_s) \cos(k_n (L_x - x_s) - j \Phi_2) \cos(k_n x - j \Phi_1)}{j \sin(k_n L_x - j(\Phi_1 + \Phi_2))}, \quad 0 \leq x \leq x_s \quad (6.12)
\]

\[
p(r) = \sum_{n=0}^{N-1} \frac{2q A_i (r, r_s) \cos(k_n x_s - j \Phi_1) \cos(k_n (L_x - x) - j \Phi_2)}{j \sin(k_n L_x - j(\Phi_1 + \Phi_2))}, \quad x_s \leq x \leq L_x. \quad (6.13)
\]

These equations provide more succinct expressions for the travelling wave model presented in Eq. (6.5), which defines the expression originally derived by Zander and Hansen [129].

Neglecting higher-order modes

An acoustic duct is an enclosure for which the dimension in one direction is generally much greater than the dimensions in the other two directions, such that \( L_x \gg (L_y, L_z) \) in Fig. 6.2. For this case, a cut-on frequency can be defined below which no higher order modes will propagate away from the source that excites the duct. The cut-on frequency \( f_c \) can be defined, assuming \( L_y \geq L_z \), as [25]

\[
f_c = \frac{c}{2L_y}. \quad (6.14)
\]

The cut-on frequency of the acoustic duct arrangement that is described in Section 6.2 is therefore given by \( f_c \approx 837 \text{ Hz} \), where it is assumed that the speed of sound \( c = 343 \text{ m/s} \). In the experiments that will be presented in Chapters 7–10, the frequencies of interest are below 500 Hz, which is well below the cut-on frequency. In the numerical modelling, it is therefore assumed that no higher-order modes are present in the acoustic duct by setting \( (N_y, N_z) = (0, 0) \) in Eqs (6.12) and (6.13). It can be shown that
Eqs (6.12) and (6.13) can then be written more succinctly for the plane waves only case as

\[
p(x) = \frac{\rho c q \cos(k(L_x - x_s) - j\Phi_2) \cos(kx - j\Phi_1)}{j S \sin(kL_x - j(\Phi_1 + \Phi_2))}, \quad 0 \leq x \leq x_s \tag{6.15}
\]

\[
p(x) = \frac{\rho c q \cos(kx_s - j\Phi_1) \cos(k(L_x - x) - j\Phi_2)}{j S \sin(kL_x - j(\Phi_1 + \Phi_2))}, \quad x_s \leq x \leq L_x. \tag{6.16}
\]

For totally reflective termination conditions \((\Phi_{1,2} = 0)\), Eqs (6.15) and (6.16) reduce to the well-known expressions discussed by Nelson and Elliott [88] for the case of plane waves inside a rigidly terminated duct.

### 6.3.2 Modal model

In this section, a modal model [88] of a rectangular acoustic duct is presented. It is assumed that the termination conditions in Fig. 6.2 are characterised by \(\Phi_i = 0\), such that rigid, totally reflective terminations are assumed. The pressure \(p(r)\) at a point \(r\) in the duct due to a point monopole source of complex volume velocity \(q\) located at \(r_s\) is given by

\[
p(r) = \sum_{n=0}^{N-1} \lambda_n(k_n^2 + 2j\zeta_n k_n k - k^2), \tag{6.17}
\]

where \(\omega\) is the angular excitation frequency, \(\rho\) the density of the air, \(k\) the acoustic wave number, \(c\) the speed of sound, \(\zeta_n\) are the damping ratios of the modes, \(n = (n_x, n_y, n_z)\) the modal indices, and \(N = (N_x, N_y, N_z)\) the number of modes considered in each direction. The modal wave numbers \(k_n = \omega_n / c\), with \(\omega_n\) the modal eigenfrequencies, can be expressed in terms of the enclosure dimensions and the modal indices as

\[
k_n^2 = k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2 = \left(\frac{n_x \pi}{L_x}\right)^2 + \left(\frac{n_y \pi}{L_y}\right)^2 + \left(\frac{n_z \pi}{L_z}\right)^2. \tag{6.18}
\]

In Eq. (6.17), the eigenfunctions or mode shapes \(\phi_n(r)\) are defined as

\[
\phi_n(r) = \cos(k_{n_x} x) \cos(k_{n_y} y) \cos(k_{n_z} z). \tag{6.19}
\]

The modal volume \(\Lambda_n\) in Eq. (6.17), defined as the volume integration of the square of the mode shape function \(\phi_n(r)\), is given by

\[
\Lambda_n = \int_V \phi_n^2(r) \, dV = \frac{V}{2^n \epsilon_i \epsilon_s}, \tag{6.20}
\]

where \(V\) is the volume of the acoustic duct enclosure, and \(\epsilon_i\) is the unit step function for integers defined in Eq. (6.11).
Neglecting higher-order modes

As discussed previously, in the experiments that will be presented in Chapters 7–10, the frequencies of interest are well below the cut-on frequency of the acoustic duct arrangement. In the numerical modal modelling, it is therefore assumed that no higher-order modes are present in the acoustic duct by setting \((N_y, N_z) = (0, 0)\) in Eq. (6.17). It can be shown that the modal model then reduces to

\[
p(x) = \sum_{n=0}^{N_x-1} \frac{j\omega p q \cos(k_n x_s) \cos(k_n x)}{\Lambda_n s(k_n^2 + 2\zeta_n k_n k - k^2)}.
\] (6.21)

In the numerical modal modelling, the first \(N_x = 25\) modes are included in Eq. (6.21), such that the highest modal frequency included is \(f_{24} = 852\) Hz. Furthermore, a quality factor \(Q = 1/2\zeta = 50\) is used for all modes, such that all modes are assumed to have a damping ratio \(\zeta = 0.01\).

6.3.3 Comparison of models

Assuming that no higher-order modes are excited, the rectangular acoustic duct can either be modelled using the travelling wave model defined in Eqs (6.15) and (6.16) or the modal model defined in Eq. (6.21). Unfortunately, a proof that these models are equivalent under certain conditions could not be found in the literature, which could be the subject of further investigations as suggested in the future work section of Chapter 11. However, a few observations can be made when comparing the two models. In the modal model defined in Eq. (6.21), damping is introduced by setting the damping ratios \(\zeta_n \neq 0\) for each mode, and the termination conditions of the duct were characterised as rigid and totally reflective, which resulted in mode shapes as defined in Eq. (6.19). The resulting pressure distribution over the length of the duct, and consequently the resulting pattern of nodes and anti-nodes, are determined by these mode shape functions. To realise a similar pattern of nodes and anti-nodes when using a travelling wave model, the imaginary parts of the termination phasors \(\Phi_i\) need to be set to zero in the travelling wave model, such that \(\beta_i = 0\) in Eq. (6.4). Damping can be added to the travelling wave model by setting \(\alpha_i \neq 0\). By trial and error, it has been found that setting \(\alpha_i = \varepsilon k\), with \(\varepsilon \in \mathbb{R}^+\) a positive scalar constant and \(k\) the acoustic wave number, resulted in similar pressure distributions over the length of the duct as predicted by the modal model with a constant damping ratio \(\zeta\) assumed for all modes, such that \(\zeta_n = \zeta\) in Eq. (6.21). As discussed previously, in the numerical modelling, a modal model with a constant damping ratio \(\zeta = 0.01\) for all modes is used. By trial and error, it has been found that setting the termination phasors \(\Phi_i = 0.025k\) in Eq. (6.4) results in nearly identical pressure distributions over the length of the duct as predicted with the modal model.

One may wonder why two models of a rectangular acoustic duct are presented when one would be sufficient. The reason for this was explained by Cazzolato et al.
Chapter 6  Acoustic duct arrangement

[17], who showed that convergence of the modal model solution for the sound field inside a lightly damped acoustic duct may require a large number of modes, especially under conditions when a high level of control is achieved. However, when a large number of modes is used, the quadratic optimisation techniques that are often used in a numerical analysis to compute the narrowband primary and controlled sound pressure distributions inside the duct can be shown to give flawed results for certain configurations of control sources and error sensors, due to numerical noise.\(^1\) The advantage of the travelling wave model over the modal model is the computational efficiency in which it may be used for arbitrary termination conditions, unlike the modal solution, which may require the summation of a very large number of modes for a lightly damped duct. Unfortunately, the travelling wave model also has a disadvantage compared to the model model. Unlike the modal model, which can be used to construct one compact state-space model that defines the broadband response of the acoustic duct to a broadband input, the travelling wave model can only provide the complex transfer impedances for a number of frequencies within the considered broadband frequency range. The response of the acoustic duct to a broadband input, which for instance needs to be known when analysing broadband active noise control, can therefore not be modelled using the travelling wave model. This is the reason a modal model of an acoustic duct has been presented as well.

6.4 System identification

To implement the active noise control and virtual sensing algorithms introduced in Chapters 2–5 in real-time on the acoustic duct arrangement, a preliminary system identification stage is required in which a model of the acoustic duct arrangement is estimated. A number of different model structures can be chosen to represent the acoustic duct arrangement, i.e. a matrix of FIR filters, a matrix of IIR filters, or a state-space structure. If the transfer paths are modelled as FIR filters, an adaptive system identification technique [51] can be used to estimate the filter coefficients. If the transfer paths are modelled as IIR filters, prediction error modelling [69] can be used to estimate the coefficients of the IIR filters. A state-space model of the acoustic duct arrangement in innovations form [60] can be estimated using subspace model identification techniques [50, 124]. These techniques have successfully been used in real-time active noise and vibration control implementations [34, 77, 90, 125], and have been shown to provide more accurate models than prediction error modelling techniques [34].

\(^1\) It was shown by Cazzolato et al. [17] that this occurs when the control source and the primary source are located on the same side of the error sensor, which is fortunately different from the configuration used here.
Subspace model identification techniques are therefore used here to obtain an accurate state-space model of the acoustic duct arrangement.

### 6.4.1 Subspace model identification

Subspace model identification techniques [50, 124] can be used to estimate an *innovations model* of the acoustic duct arrangement, which has been defined in Eq. (3.201) as

\[
\begin{align*}
\hat{z}(n+1|n) &= A\hat{z}(n|n-1) + B_u u(n) + \tilde{K}_{ps} \varepsilon_p(n) + \tilde{K}_{vs} \varepsilon_v(n) \\
e_p(n) &= C_p \hat{z}(n|n-1) + D_{pu} u(n) + \varepsilon_p(n) \\
e_v(n) &= C_v \hat{z}(n|n-1) + D_{vu} u(n) + \varepsilon_v(n),
\end{align*}
\]

(6.22)

with \(e_p(n)\) the physical error signals measured by the physical microphones in Fig. 6.1, \(e_v(n)\) the virtual error signals measured by the traversing microphone that can temporarily be located at the virtual locations of interest, and \(u(n)\) the signal that excites the control loudspeaker during identification. Note that in Eq. (3.201), the matrices \(\tilde{K}_{ps}\) and \(\tilde{K}_{vs}\) defined in Eq. (6.22) are contained in the Kalman gain matrix \(K_s\), such that

\[
K_s = \begin{bmatrix} \tilde{K}_{ps} & \tilde{K}_{vs} \end{bmatrix}.
\]

(6.23)

In Eq. (6.22), the white innovation signals \(\varepsilon_p(n)\) and \(\varepsilon_v(n)\) are defined, following the discussions presented in Section 3.7, as

\[
\varepsilon(n) = \begin{bmatrix} \varepsilon_p(n) \\ \varepsilon_v(n) \end{bmatrix} = \begin{bmatrix} e_p(n) - \hat{e}_p(n|n-1) \\ e_v(n) - \hat{e}_v(n|n-1) \end{bmatrix}.
\]

(6.24)

The objective of subspace model identification can now be formulated as follows. Given measured input-output data

\[
\{u(n), \begin{bmatrix} e_p(n) \\ e_v(n) \end{bmatrix} \}_{n=1}^{N_s}
\]

(6.25)

the objective of subspace model identification is to estimate the state-space matrices \((A, B_u, C_p, C_v, D_{pu}, D_{vu})\), and the Kalman gain matrix \(K_s\) of the innovations model defined in Eq. (6.22) up to a similarity transformation, and the covariance matrix \(R_\varepsilon\) of the innovation signals which was defined in Eq. (3.206) as

\[
R_\varepsilon = E \begin{bmatrix} \varepsilon_p(n) \\ \varepsilon_v(n) \end{bmatrix} \begin{bmatrix} \varepsilon_p(n) \\ \varepsilon_v(n) \end{bmatrix}^T = \begin{bmatrix} R_{\varepsilon_{pp}} & R_{\varepsilon_{pv}} \\ R_{\varepsilon_{vp}} & R_{\varepsilon_{vv}} \end{bmatrix}.
\]

(6.26)

A minimum-phase relationship between the innovation signals and the error signals is guaranteed by constraining the eigenvalues of the matrix

\[
A - \begin{bmatrix} \tilde{K}_{ps} & \tilde{K}_{vs} \end{bmatrix} \begin{bmatrix} C_p \\ C_v \end{bmatrix},
\]

(6.27)
to be inside the unit circle. Several subspace identification algorithms have been proposed \cite{50, 55, 73, 124}, which are all based on numerically reliable algorithms such as the QR-factorisation and singular value decomposition. Here, the PO-MOESP \cite{50} and SSARX \cite{55} subspace model identification methods are used.

### 6.4.2 Identification example

In the real-time experiments that will be discussed in Chapters 7–10, a two-step identification procedure is used to obtain an innovations model of the acoustic duct arrangement, which is defined in Eq. (6.22). In the first step, the deterministic part of the innovations model is estimated by setting the input signal into the primary loudspeaker equal to zero, such that \( s(n) = 0 \). Effectively, a deterministic state-space model of the physical and virtual secondary transfer paths \( G_{pu} \) and \( G_{vu} \), respectively, is therefore estimated in the first step. In the second step, the stochastic part of the innovations model of the acoustic duct arrangement is estimated by setting the input signal into the control loudspeaker equal to zero, such that \( u(n) = 0 \). A one step approach in which a full innovations model of the acoustic duct arrangement is estimated at once could also be used, but a two-step approach usually results in a more accurate model of the true plant \cite{34}. In this section, a typical example of the two-step identification procedure is presented. In this example, the physical microphone located at \( x_{p5} = 1.475 \text{ m} \) is used as a physical sensor, and the traversing microphone is temporarily located at a spatially fixed virtual location \( x_v = 1.575 \text{ m} \), such that the virtual distance \( v = 0.100 \text{ m} \) in Fig. 6.1. A sample frequency of \( f_s = 1.6 \text{ kHz} \) is used in the presented identification example.

#### Deterministic part of the innovations model

In the first step, a deterministic state-space model of the physical and virtual secondary transfer paths is estimated using the PO-MOESP \cite{50} subspace model identification method, with the deterministic state-space model defined as

\[
\begin{align*}
\mathbf{z}_1(n+1) &= \mathbf{A}_1\mathbf{z}_1(n) + \mathbf{B}_u u(n) \\
y_p(n) &= \mathbf{C}_{p1}\mathbf{z}_1(n) + D_{pu} u(n) \\
y_v(n) &= \mathbf{C}_{v1}\mathbf{z}_1(n) + D_{vu} u(n).
\end{align*}
\]

(6.28)

The control loudspeaker is excited by a band-limited white noise signal \( u(n) \) in the frequency range of 50–500 Hz, and an input/output data-set

\[
\{u(n), \begin{bmatrix} y_p(n) \\ y_v(n) \end{bmatrix}\}_{n=1}^{32000}
\]

(6.29)

is recorded, with \( y_p(n) \) the physical secondary disturbance measured by the physical microphone located at \( x_{p5} = 1.475 \text{ m} \), and \( y_v(n) \) the virtual secondary disturbance
measured by the traversing microphone that is temporarily located at a spatially fixed virtual location \( x_v = 1.575 \text{ m} \). The recorded data-set is divided into a training data-set and a validation data-set each 16000 samples long. The accuracy of the estimated model is expressed by the variance-accounted-for (VAF) value, which is defined as \[ VAF = \frac{1}{2} \sum_{m=1}^{2} \max \left( 1 - \frac{\text{var}[Y_m - \hat{Y}_m]}{\text{var}[Y_m]}, 0 \right) \times 100\% \] (6.30)

where \( \text{var}[\cdot] \) is the variance of the data sequence between brackets, \( Y \in \mathbb{R}^{16000 \times 2} \) the output validation data-set, and \( \hat{Y} \in \mathbb{R}^{16000 \times 2} \) the data-set that results from filtering the input validation data-set with the estimated state-space model. The VAF value in Eq. (6.30) is 100% if the matrices \( Y \) and \( \hat{Y} \) are identical, and decreases as the difference in these matrices becomes greater. In the example, a state-space model of the deterministic part of the plant of order \( N = 32 \) is estimated, which results in a VAF = 99.9% on the validation data-set. This indicates that a very accurate state-space model of the physical and virtual secondary transfer paths has been obtained. This is illustrated in Fig. 6.3, where Bode diagrams of the estimated state-space models of the physical and virtual secondary transfer paths \( G_{pu} \) and \( G_{vu} \), respectively, are compared with the frequency response functions between the control signal and the measured physical and secondary virtual disturbances, which are calculated from the validation input/output data-set. This figure shows that an excellent fit on the validation data is obtained over the frequency band of interest between 50–500 Hz. The high magnitudes of the computed frequency response functions at frequencies outside this band occur because the computer generated input signal \( u(n) \) has very little energy at these frequencies while there still is some measurement noise at these frequencies.

Stochastic part of the innovations model

In the second step, the control signal \( u(n) \) is set to zero, and an estimate of the stochastic part of the innovations model defined in Eq. (6.22) is computed using the SSARX subspace identification algorithm [55], with the stochastic part given by

\[
\begin{align*}
\dot{z}_2(n+1|n) &= A_2 \dot{z}_2(n|n-1) + \hat{K}_{ps} \epsilon_p(n) + \hat{K}_{sv} \epsilon_v(n) \\
d_p(n) &= C_p \dot{z}_2(n|n-1) + \epsilon_p(n) \\
d_v(n) &= C_v \dot{z}_2(n|n-1) + \epsilon_v(n),
\end{align*}
\] (6.31)

with \( \epsilon_p(n) \) and \( \epsilon_v(n) \) the white innovation signals defined as

\[
\begin{bmatrix}
\epsilon_p(n) \\
\epsilon_v(n)
\end{bmatrix} = 
\begin{bmatrix}
d_p(n) - \hat{d}_p(n|n-1) \\
d_v(n) - \hat{d}_v(n|n-1)
\end{bmatrix}.
\] (6.32)
Note that in the second step, the minimum-phase spectral factors of the physical and virtual primary transfer paths $G_{ps}$ and $G_{vs}$ are determined because the disturbance source signal $s(n)$ is assumed unknown, which is usually the case in a practical situation. Therefore, the primary loudspeaker is excited by a band-limited white noise signal $s(n)$ in the frequency range of 50–500 Hz, and only an output data-set is recorded, with $d_p(n)$ the physical primary disturbance measured by the physical microphone located at $x_{ps} = 1.475$ m, and $d_v(n)$ the virtual primary disturbance measured by the traversing microphone that is temporarily located at a spatially fixed virtual location $x_v = 1.575$ m. Again, the recorded data-set is divided into a training data-set and a validation data-set each 16000 samples long. The accuracy of the estimated innovations model is evaluated by calculating the VAF value defined in Eq. (6.30) based on the validation output data-set, and the predicted estimates $\hat{d}_p(n|n-
1) and $\hat{d}_v(n|n-1)$ of the physical and virtual primary disturbances, respectively. These predicted estimates are calculated from Eqs (6.31) and (6.32) as

$$
\begin{align*}
\hat{z}_2(n+1|n) &= (\hat{A}_2 - \hat{K}_{ps}\hat{C}_{p2} - \hat{K}_{vs}\hat{C}_{v2})\hat{z}_2(n|n-1) + \hat{K}_{ps}\hat{d}_p(n) + \hat{K}_{vs}\hat{d}_v(n) \\
\hat{d}_p(n|n-1) &= \hat{C}_{p2}\hat{z}_2(n|n-1) \\
\hat{d}_v(n|n-1) &= \hat{C}_{v2}\hat{z}_2(n|n-1),
\end{align*}
$$

with $\hat{A}_2$, $\hat{C}_{p2}$, $\hat{C}_{v2}$, $\hat{K}_{ps}$ and $\hat{K}_{vs}$ the estimated state-space matrices that define the stochastic part of the innovations model defined in Eq. (6.31). In the final experiments, a 40th order innovations model was estimated on the training data-set. For the physical primary disturbance $d_p(n)$, a VAF = 99.9% was obtained on the validation data-set, while for the virtual primary disturbance $d_v(n)$, a VAF = 99.2% was obtained on the validation data-set. These VAF values indicate that an accurate innovations model of the stochastic part of the acoustic duct arrangement is estimated. This is also illustrated in Fig. 6.4, where power spectral density plots of the measured physical and virtual primary disturbances, their predicted estimates computed as defined in Eq. (6.34), and the innovation signals are shown.

![Power spectral density plots](image)

Figure 6.4: Power spectral density plots of (a) the measured physical primary disturbance $d_p(n)$, the predicted estimate $\hat{d}_p(n|n-1)$, and the innovation signal $\epsilon_p(n)$; and (b) the measured virtual primary disturbance $d_v(n)$, the predicted estimate $\hat{d}_v(n|n-1)$, and the innovation signal $\epsilon_v(n)$.

Fig. 6.4 illustrates that the power spectral density plots of the measured primary disturbances and their predicted estimates are very similar, which shows that an accurate stochastic model of the measured physical and virtual primary disturbances has been identified. In other words, accurate minimum-phase spectral factors of the physical and virtual primary transfer paths $G_{ps}$ and $G_{vs}$ have been determined. Fig. 6.4 also illustrates that the innovation signals defined in Eq. (6.32) approximate white noise.
processes because the power spectral density plots of these signals are reasonably flat. This is another indicator that a stochastic innovations model of the measured physical and virtual primary disturbances has successfully been identified.

6.5 Conclusion

In this chapter, the rigidly terminated, rectangular acoustic duct arrangement that will be used in the real-time experiments presented in Chapters 7–10 has been introduced. A modal model and a travelling wave model of a rectangular acoustic duct of finite length have been presented, and a more succinct expression for the travelling wave model derived by Zander and Hansen [129] has been presented. Finally, subspace model identification techniques [50, 124] have been discussed and an identification example has been presented.
Chapter 7

Adaptive LMS virtual microphone technique

7.1 Introduction

In this chapter, the adaptive LMS virtual microphone technique [14] is implemented on the acoustic duct arrangement introduced in Chapter 6. This technique is used to estimate the virtual error signal at a virtual location that is spatially fixed inside the acoustic duct arrangement. The estimated virtual error signal is then minimised using the filtered-x LMS algorithm as described in Chapter 4. The estimation and control performance that is obtained at a spatially fixed virtual location inside the acoustic duct arrangement is analysed for narrowband and broadband disturbances. The real-time experimental results are compared to numerical results which are computed using the acoustic duct models introduced in Chapter 6.

The adaptive LMS virtual microphone technique has been analysed previously [80] for the acoustic duct case, and some insights are presented here that explain the disagreement between numerical and experimental results reported in this previous research. Furthermore, the broadband adaptive feedforward control performance that can be obtained at a spatially fixed virtual location inside the acoustic duct is examined, which was not done in the previous research.

In Section 7.2, the narrowband estimation performance of the adaptive LMS virtual microphone technique is analysed. An analytical optimal solution for the physical sensor weights is derived for the case of single tone disturbances inside the acoustic duct arrangement. This analytical optimal solution is derived using the travelling wave model of an acoustic duct introduced in Chapter 6. It is also shown that this analytical optimal solution can be derived using the principle of travelling wave decomposition of the sound field inside an acoustic duct. This method decomposes the sound field
into forwards and backwards travelling waves, and is commonly known as the two-microphone method or the transfer function method [21, 22]. Parts of the analysis presented here have been published in Petersen et al. [103].

In Section 7.3, the broadband estimation performance of the adaptive LMS virtual microphone technique is analysed. Numerical and experimental results are obtained for three different physical sensor configurations, which have also been used in previous research [80] on the adaptive LMS virtual microphone technique. Optimal solutions for the physical sensor weights are derived using the theory presented in Chapter 3, and some insights are presented that explain the disagreement between the numerical and experimental results reported in the previous research [80]. The overall estimation performance is analysed for a number of spatially fixed virtual locations, and a spectral analysis of the broadband estimation performance is presented.

In Section 7.4, the narrowband control performance that is obtained at a spatially fixed virtual location inside the acoustic duct arrangement is analysed. In the numerical analysis, the narrowband control performance is computed for a range of frequencies and spatially fixed virtual locations using the quadratic optimisation techniques introduced in Chapters 2 and 4. In the real-time control experiments, the estimated virtual error signal is minimised using the filtered-x LMS algorithm as described in Chapter 4. The real-time narrowband control performance is analysed for three excitation frequencies that correspond to the sixth, seventh, and eighth resonance frequencies of the acoustic duct arrangement, and for a number of virtual distances. In each experiment, the primary and controlled sound pressure distributions inside the acoustic duct arrangement are measured with the traversing microphone. This is done to verify whether the zone of quiet has effectively been moved away from the physical sensors to the desired spatially fixed virtual location.

In Section 7.5, the broadband feedforward control performance that is obtained at a spatially fixed virtual location inside the acoustic duct arrangement is analysed. In the numerical analysis, the broadband feedforward control performance is computed for a number of spatially fixed virtual locations using the causal Wiener filter techniques introduced in Chapters 2–4. In the real-time experiments, the estimated virtual error signal is minimised using the filtered-x LMS algorithm as described in Chapter 4. The real-time broadband feedforward control performance is analysed for a number of spatially fixed virtual locations and is compared with the numerical results. In each experiment, the overall primary and controlled sound pressure distributions inside the acoustic duct arrangement are measured with the traversing microphone. Again, this is done to verify whether the zone of quiet has effectively been moved away from the physical sensors to the desired spatially fixed virtual location.

In Section 7.6, forward difference prediction weights [13] are compared with the analytical optimal solutions for the weights for the case of single tone disturbances, which are
7.2 Narrowband estimation

In this section, the narrowband estimation performance of the adaptive LMS virtual microphone technique is analysed for the acoustic duct arrangement shown in Fig. 7.1.

Figure 7.1: Schematic diagram of an acoustic duct of length $L_x$, with a primary source located at $x_p$, a control source at $x_s$, two physical sensors at $x_{p1}$ and $x_{p2}$, one spatially fixed virtual sensor at $x_v$, and with arbitrary termination conditions characterised by the complex termination phasors $\Phi_1$ and $\Phi_2$.

The arrangement illustrated in Fig. 7.1 is equivalent to the acoustic duct arrangement that is used in the real-time experiments, which has been introduced in Chapter 6. A primary disturbance source is located at $x_p$, a control source at $x_s$, and the termination conditions are characterised by the complex termination phasors $\Phi_1$ and $\Phi_2$, which have been defined in Eq. (6.4). Two physical sensors are located at $x_{p1}$ and $x_{p2}$, with $\Delta = x_{p2} - x_{p1}$ defined as the physical sensor separation distance. A virtual sensor is located at $x_v = x_{p2} + v$, where $v$ is the spatially fixed virtual distance as shown in Fig. 7.1. As discussed in Section 3.3, for the case of single tone disturbances, only two physical sensors are required to compute a perfect estimate of the virtual error signal $e_v(n)$ at the spatially fixed virtual location $x_v$, such that the virtual output error $\epsilon_v(n) = e_v(n) - \hat{e}_v(n)$ is equal to zero. From the block diagram of the adaptive LMS
virtual microphone technique illustrated in Fig. 3.3, the estimate of the virtual error signal $e_v(n)$ at the spatially fixed virtual location is computed as

$$e_v(n) = h_{so}^T d_p(n) + h_{uo}^T y_p(n), \quad (7.1)$$

with $h_{so} \in \mathbb{R}^2$ the optimal physical sensor weights for the primary sound field, and $h_{uo} \in \mathbb{R}^2$ the optimal physical sensor weights for the secondary sound field. The optimal weights for the secondary sound field have been defined in Eq. (3.35) for the single tone case as

$$h_{uo} = \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \left[ -\frac{Y_v}{Y_{p1}} \sin(\theta_{p2} - \theta_v) \right. \left. \frac{Y_v}{Y_{p2}} \sin(\theta_{p1} - \theta_v) \right], \quad (7.2)$$

with $Y_{pi}$ and $\theta_{pi}$ the magnitudes and phases, respectively, of the physical secondary disturbances, and $Y_v$ and $\theta_v$ the magnitude and phase, respectively, of the virtual secondary disturbance, with respect to some arbitrary reference. The optimal weights for the primary sound field are defined in a similar way. In the next section, the travelling wave model introduced in Section 6.3 is used to derive more succinct expressions for the optimal weights for the acoustic duct configuration shown in Fig. 7.1.

### 7.2.1 Analytical solution from travelling wave model

First, the optimal weights $h_{so}$ for the primary sound field are calculated for the perfect estimation of the complex virtual primary disturbance $d_v$, such that

$$d_v = d_v = h_{so}^T d_p = \begin{bmatrix} h_{so1} & h_{so2} \end{bmatrix} \begin{bmatrix} d_{p1} \\ d_{p2} \end{bmatrix}, \quad (7.3)$$

with $d_p$ the two complex physical primary disturbances. Using the travelling wave model in Eq. (6.16), this can be written as

$$\cos(kx_v - j\Phi) = \begin{bmatrix} h_{so1} & h_{so2} \end{bmatrix} \begin{bmatrix} \cos(kx_{p1} - j\Phi) \\ \cos(kx_{p2} - j\Phi) \end{bmatrix}, \quad (7.4)$$

where it is assumed for convenience that the termination phasors $\Phi_1 = \Phi_2 = \Phi$, and noting that this will not make a difference in the final result. By using the fact that $\cos \theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$, and by combining the terms in $e^{j\Phi}$ and $e^{-j\Phi}$, Eq. (7.4) can be transformed into the matrix equation

$$A h_{so} = b, \quad (7.5)$$
where the matrix $A \in \mathbb{C}^{2 \times 2}$ and the vector $b \in \mathbb{C}^2$ are given by

$$A = \begin{bmatrix} e^{jkx_p1} & e^{jkx_p2} \\ e^{-jkx_p1} & e^{-jkx_p2} \end{bmatrix}, \quad b = \begin{bmatrix} e^{jkx_o} \\ e^{-jkx_o} \end{bmatrix}. \quad (7.6)$$

The optimal weights for the primary sound field can now be calculated from Eq. (7.5) as

$$h_{uo} = A^{-1}b = \frac{1}{\sin k\Delta} \begin{bmatrix} -\sin kv \\ \sin k(v + \Delta) \end{bmatrix}, \quad (7.7)$$

where use has been made of the fact that

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \quad (7.8)$$

Note that the optimal weights in Eq. (7.7) are real-valued as is required in the adaptive LMS virtual microphone technique. In a similar way, it can be shown that the two optimal weights $h_{uo}$ for the secondary sound field are given by Eq. (7.7). This means that these optimal weights can be applied directly to the two physical error signals $e_{p}(n)$, such that a perfect estimate of the virtual error signal is computed as

$$e_v(n) = \frac{1}{\sin k\Delta} \begin{bmatrix} -\sin kv \\ \sin k(v + \Delta) \end{bmatrix}^T \begin{bmatrix} e_{p1}(n) \\ e_{p2}(n) \end{bmatrix}. \quad (7.9)$$

Eq. (7.7) indicates that the optimal solution for the weights depends on the wave number $k$, the physical sensor separation distance $\Delta$, and the virtual distance $v$. Also, the optimal weights are independent of the source locations, the physical and virtual sensor locations, and the termination conditions $\Phi_1$ and $\Phi_2$. Furthermore, it can be seen that no solution exists for the optimal weights if $k\Delta = m\pi$, with $m$ an integer. In this instance, an integer multiple $m$ of half an acoustic wavelength $\lambda = 2\pi/k$ exactly fits within the physical sensor separation distance $\Delta$. Using the travelling wave model in Eqs (6.15) and (6.16), it can be shown that for this case, the two disturbances at the physical sensor locations $x_{p1}$ and $x_{p2} = x_{p1} + \Delta$ are such that, for the primary sound field, $d_{p1} = -d_{p2}$. As a result, the matrix $A$ in Eq. (7.6) becomes singular, and the inverse in Eq. (7.7) does not exist. This means that the estimation problem is ill-conditioned. To avoid this problem, the physical sensor separation distance should be such that $\Delta \neq \lambda/2$.

### 7.2.2 Travelling wave decomposition

The optimal solution in Eq. (7.9) can also be derived using the principle of *travelling wave decomposition* of the sound field inside an acoustic duct. This method decomposes the sound field into forwards and backwards travelling waves, and is commonly known...
as the two-microphone method or the transfer function method. It is used in impedance tube testing for calculating the impedance and absorption of material samples and other acoustic systems [21, 22]. Fig. 7.2 illustrates the decomposition of the sound field into a forwards travelling wave \( w_+ \) and a backwards travelling wave \( w_- \).

![Diagram of sound field decomposition](image)

**Figure 7.2**: Travelling wave decomposition of the sound field inside an acoustic duct with \( w_+ \) the forwards travelling wave and \( w_- \) the backwards travelling wave.

In Fig. 7.2, the complex physical error signal \( e_{p1} \) at the physical sensor located at \( x_{p1} \) is assumed to be the superposition of the forwards and backwards travelling waves, such that

\[
e_{p1} = w_+ + w_- .
\]  

(7.10)

The complex secondary disturbance \( e_{p2} \) at the physical sensor located at \( x_{p2} = x_{p1} + \Delta \) can now be expressed, using delayed versions of the forwards and backwards travelling waves at \( x_{p1} \), as

\[
e_{p2} = w_+ e^{-jk\Delta} + w_- e^{jk\Delta},
\]  

(7.11)

where the forwards travelling wave \( w_+ \) at \( x_{p1} \) has been delayed with a positive time delay \( \tau = \Delta/c \), with \( c \) the speed of sound, and the backwards travelling wave \( w_- \) at \( x_{p1} \) has been delayed with a negative time delay \(-\tau\). The forwards and backwards travelling waves can now be expressed in terms of the physical error signals at the two physical sensors as

\[
\begin{bmatrix} w_+ \\ w_- \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{-jk\Delta} & e^{jk\Delta} \end{bmatrix}^{-1} \begin{bmatrix} e_{p1} \\ e_{p2} \end{bmatrix}.
\]  

(7.12)

The complex primary disturbance \( e_v \) at the virtual sensor located at \( x_v = x_{p1} + \Delta + \nu \) can be expressed, using delayed versions of the forwards and backwards travelling waves at \( x_{p1} \), as

\[
d_v = w_+ e^{-jk(\Delta + \nu)} + w_- e^{jk(\Delta + \nu)},
\]  

(7.13)

where the forwards travelling wave \( w_+ \) at \( x_{p1} \) has been delayed with a positive time delay \( \tau = (\Delta + \nu)/c \), and the backwards travelling wave \( w_- \) at \( x_{p1} \) has been delayed...
with a negative time delay $-\tau$. Using Eq. (7.12), the complex virtual error signal $e_v$ in Eq. (7.13) can be written as

$$e_v = \left[ e^{-jk(\Delta + v)} \quad e^{jk(\Delta + v)} \right] \left[ \begin{array}{c} 1 \\ e^{-jk\Delta} \end{array} \right]^{-1} \left[ \begin{array}{c} e_p1 \\ e_p2 \end{array} \right],$$

(7.14)

which can also be expressed, using Eq. (7.8), as

$$e_v = \frac{1}{\sin k\Delta} \left[ \begin{array}{c} -\sin kv \\ \sin k(v + \Delta) \end{array} \right] \left[ \begin{array}{c} e_p1 \\ e_p2 \end{array} \right].$$

(7.15)

Eq. (7.15) again indicates that the optimal weights derived in Eq. (7.9) for the primary sound field are also the optimal weights for the secondary sound field. These weights can thus be applied directly to the physical error signals as defined in Eq. (7.9) and derived in Eq. (7.15).

### 7.3 Broadband estimation performance

In this section, the broadband estimation performance of the adaptive LMS virtual microphone technique [14] is analysed for the acoustic duct arrangement introduced in Chapter 6. A schematic diagram of the implementation of the adaptive LMS virtual microphone technique that is analysed in this section is shown in Fig. 7.3.

![Schematic diagram of the acoustic duct implementation of the adaptive LMS virtual microphone technique](image)

Figure 7.3: Schematic diagram of the acoustic duct implementation of the adaptive LMS virtual microphone technique, with a primary source located at $x_p = 4.730$ m, a control source at $x_s = 0.100$ m, five physical sensors located at $x_p$, and one spatially fixed virtual sensor located at $x_v = x_p + v$, with $v$ the virtual distance.
The acoustic duct arrangement illustrated in Fig. 7.3, which has been discussed in more detail in Chapter 6, has a length \( L_x = 4.830 \) m, a primary source located at \( x_p = 4.730 \) m, and a control source located at \( x_c = 0.100 \) m. A physical sensor array consisting of five electret microphones is located at

\[
x_p = [1.4250 \ 1.4375 \ 1.4500 \ 1.4675 \ 1.4750] \text{ m.}
\] (7.16)

The broadband estimation performance is analysed using three different physical sensor configurations. In the first configuration, the two physical sensors located at 1.4250 m and 1.4750 m are used to compute an estimate \( \hat{e}_v(n) \) of the virtual error signal. In the second configuration, the three physical sensors located at 1.4250 m, 1.4500 m, and 1.4750 m are used to compute an estimate of the virtual error signal. In the third configuration, all the physical sensors are used to compute an estimate of the virtual error signal. These configurations, which will be referred to as the two, three, and five physical sensor configurations in the following, have also been investigated by previous researchers [62, 80]. The aim here is to provide additional insights that explain some of the results that have been reported by these researchers.

From the block diagram of the adaptive LMS virtual microphone technique illustrated in Fig. 3.3, the estimate of the virtual error signal \( e_v(n) \) at the spatially fixed virtual location is computed in Fig. 7.3 as

\[
e_v(n) = h_{so}^T d_p(n) + h_{uo}^T y_p(n),
\] (7.17)

with \( h_{so} \in \mathbb{R}^{M_p} \) the optimal physical sensor weights for the primary sound field, and \( h_{uo} \in \mathbb{R}^{M_p} \) the optimal physical sensor weights for the secondary sound field. In the next sections, numerical and experimental solutions for the optimal physical sensor weights are computed for a range of virtual distances \( v \) for both the primary and secondary sound fields. These weights are then used to analyse the broadband estimation performance that is obtained at the considered virtual distances.

### 7.3.1 Numerical broadband estimation performance

In the numerical analysis, a discrete-time state-space model of the acoustic duct arrangement is calculated using the modal modelling technique described in Section 6.3. Measurement noise is not included in the numerical analysis because the aim is to analyse the theoretical limits on the broadband estimation performance. A plane wave situation is assumed by only including the first \( N_x = 25 \) modes in the direction of the duct with length \( L_x = 4.83 \) m. A quality factor of \( Q = 50 \) is used for all modes, such that all modes are assumed to have a damping ratio \( \zeta = 0.01 \). The discrete-time state-space model is computed using a sample frequency \( f_s = 4 \) kHz and a zero-order hold discretisation method [35]. This sample frequency is also used in the real-time experiments.
Numerical optimal physical sensor weights

To derive optimal physical sensor weights $h_{so}$ for the primary sound field, it is assumed that the primary disturbance source in Fig. 7.3 is excited by a white disturbance source signal $s(n)$ filtered with an 8th order Butterworth filter with a pass-band between 50–500 Hz. This assumption is made because the experimental optimal weights for the primary sound field are identified while exciting the primary loudspeaker with such a band-limited white noise signal. The optimal weights $h_{uo}$ for the secondary sound field are derived in a similar way by assuming that the control source in Fig. 7.3 is excited by a band-limited control identification signal $u(n)$ in the frequency range of 50–500 Hz.

From the state-space model of the acoustic duct arrangement, which is computed using the modal modelling technique described previously, optimal physical sensor weights for the primary and secondary sound fields can be computed using Theorem 3.1 on page 88. The results of the described numerical analysis are shown in Fig. 7.4, where the optimal weights $h_{so}$ and $h_{uo}$ for the primary and secondary sound fields, respectively, are plotted against virtual distance $v$ for the three considered physical sensor configurations. The virtual distance ranges from 0.000–0.150 m in increments of 0.001 m, such that numerical weights are computed for 151 virtual locations. The optimal weights $h_{so}$ for the primary sound field are denoted by black lines, and the optimal weights $h_{uo}$ for the secondary sound field by grey lines.

The results in Fig. 7.4 show that for all considered physical sensor configurations, the optimal physical sensor weights for the primary and secondary sound fields are almost identical for all of the virtual distances examined. This already indicates why previous researchers [80] obtained good results by applying the optimal weights for the secondary sound field directly to the physical error signals when computing an estimate of the virtual error signal.

Numerical overall estimation performance versus virtual distance

The overall estimation performance that is obtained at the examined virtual distances can be analysed by computing the normalised mean-square virtual output error. For the primary sound field, the normalised mean-square virtual output error is calculated as

$$\frac{E[\varepsilon_v(n)^2]}{E[d_v(n)^2]}, \quad (7.18)$$

with the virtual output error for the primary sound field given by

$$\varepsilon_v(n) = d_v(n) - h_{so}^T d_p(n). \quad (7.19)$$

The normalised mean-square virtual output error for the secondary sound field is calculated in a similar way, with the virtual output error $\varepsilon_v(n)$ given by

$$\varepsilon_v(n) = y_v(n) - h_{uo}^T y_p(n). \quad (7.20)$$
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Figure 7.4: Numerical optimal physical sensor weights plotted against virtual distance $v$ for the two, three, and five physical sensor configurations, with $h_{so} — h_{wo}$.

In Fig. 7.5, the normalised mean-square virtual output error in dB has been plotted against virtual distance. The normalised mean-square virtual output error (NMSV) in dB is calculated, for the primary sound field, as

$$NMSV = -10 \log_{10} \left( \frac{E[\varepsilon_v(n)^2]}{E[d_v(n)^2]} \right) \text{ dB,} \quad (7.21)$$

such that a larger positive value in dB indicates better estimation performance. The black lines in Fig. 7.5 denote the primary sound field results, the grey lines the secondary sound field results, the solid lines the two physical sensors configuration, the dashed lines the three physical sensor configuration, and the dash-dotted lines the five physical sensor configuration.

The results in Fig. 7.5 show that the normalised mean-square virtual output error for the primary and secondary sound fields are almost identical for all examined virtual
Figure 7.5: Numerical normalised mean-square virtual output error plotted against virtual distance $v$ for primary sound field $\dashed$, and secondary sound field $\cdashed$, for $M_p = 2$ $\cdashed - \cdashed - M_p = 3$ $\cdashed - \cdashed - M_p = 5$.

It can also be seen that for all of the physical sensor configurations considered, the overall estimation performance decreases with an increase in the virtual distance $v$ between the virtual sensor and the physical sensor located at 1.4750 m. The best estimation performance is in theory achieved for the five physical sensor configuration. However, as can be seen in Figs 7.4(b) and 7.4(c), the physical sensor weights for the three and five physical sensor configurations have very large values, indicating that the conditioning of the estimation problem should also be taken into consideration.

**Conditioning of estimation problem**

An explanation for the large values of the physical sensor weights for the three and five physical sensor configurations can be found by computing the condition numbers of the covariance matrices $R_s$ and $R_u$ of the primary and secondary physical disturbances, respectively. These matrices are inverted when calculating the optimal weights $h_{so}$ and $h_{uo}$ for the primary and secondary sound fields, and are defined in Theorem 3.1, without including measurement noise, as

$$
R_s = C_p \Pi_s C_p^T + D_{ps} Q_s D_{ps}^T,
$$

$$
R_u = C_p \Pi_u C_p^T + D_{ps} Q_s D_{ps}^T.
$$

(7.22)

Table 7.1 gives the condition numbers for the three considered physical sensor configurations. A large condition number $\kappa(R_s)$ indicates that the matrix $R_s$ is ill-conditioned, and that it is close to being singular. In this case, the inverse $R_s^{-1}$ can have some very large terms [25]. The condition numbers given in Table 7.1 show that this issue is causing the quite large values for the optimal weights for the three and five physical sensor configurations.
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<table>
<thead>
<tr>
<th></th>
<th>$\kappa(R_s)$</th>
<th>$\kappa(R_u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>105</td>
<td>101</td>
</tr>
<tr>
<td>3</td>
<td>$1.5 \cdot 10^5$</td>
<td>$1.4 \cdot 10^5$</td>
</tr>
<tr>
<td>5</td>
<td>$3.2 \cdot 10^{12}$</td>
<td>$3.0 \cdot 10^{12}$</td>
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Table 7.1: Numerical condition numbers of the covariance matrices $R_s$ and $R_u$.

In the real-time experiments, measurement noise on the physical sensors is generally unavoidable, and the covariance matrices in Eq. (7.22) are then given by

$$
R_s = C_p \Pi_s C_p^T + D_{ps} Q_s D_{ps}^T + R_p,
$$

$$
R_u = C_p \Pi_u C_p^T + D_{ps} Q_s D_{ps}^T + R_p,
$$

(7.23)

with $R_p$ the covariance matrix of the measurement noise on the physical sensors. Because the covariance matrices $R_s$ and $R_u$ defined in Eq. (7.22) are ill-conditioned for the measurement noise free case for the three and five physical sensor configurations, the physical sensor weights calculated with measurement noise included, with the covariance matrices $R_s$ and $R_u$ then defined by Eq. (7.23), can be very different for these configurations even for a large signal-to-noise ratio. Thus, although the numerical results shown in Fig. 7.5, which are computed without including measurement noise, predict that the five physical sensor configuration provides by far the best overall estimation performance, the large difference in the normalised mean-square virtual output error will most likely not be observed in the real-time experiments presented in the next section, where measurement noise is generally unavoidable.

As discussed in Section 3.3, the optimal physical sensor weights were determined in previous research [14, 80] using the LMS algorithm as shown in Fig. 3.2 on page 76. For the secondary sound field, the convergence speed of the LMS algorithm depends heavily on the condition number, or eigenvalue spread, of the covariance matrix $R_u$, with a very large condition number typically resulting in very slow convergence [51, 127]. The large condition numbers for the three and five physical sensor configurations given in Table 7.1 are thus expected to result in very slow convergence to the optimal physical sensor weights. This explains the discrepancy between the numerical and experimental weights that was reported in previous research [80].

**Numerical spectral estimation performance for $v = 0.100$ m**

In this section, the spectral estimation performance is analysed for a virtual distance $v = 0.100$ m, which corresponds to a spatially fixed virtual location $x_v = 1.5750$ m inside the acoustic duct arrangement. Fig. 7.6(a) shows power spectral density plots
of the virtual primary disturbance and the virtual output error $\varepsilon(n) = d_v(n) - \hat{d}_v(n)$ for the primary sound field for the three considered physical sensor configurations. The solid black line indicates the virtual primary disturbance, the solid grey line the virtual output error for the two physical sensor configuration, the dashed line the virtual output error for the three physical sensor configuration, and the dash-dotted line the virtual output error for the five physical sensor configuration. Results are also shown for the secondary sound field in Fig. 7.6(b), with the virtual output error given by $\varepsilon(n) = y_v(n) - \hat{y}_v(n)$.

The results in Fig. 7.6 again indicate that the best estimation performance is in theory achieved for the five physical sensor configuration, for both the primary and secondary sound fields. However, as discussed previously, the estimation problem is ill-conditioned for this configuration, and the large difference in the estimation performance between the five and two physical sensor configurations will most likely not be observed in the real-time experiments presented in the next section.

The spectral estimation performance can be further analysed by examining the complex transfer impedance $T_d$ between the virtual primary disturbance and its estimate, and the complex transfer impedance $T_y$ between the virtual secondary disturbance and its estimate, which are defined as

$$T_y = \frac{y_v}{\hat{y}_v}, \quad T_d = \frac{d_v}{\hat{d}_v}. \quad (7.24)$$

In Fig. 7.7, the magnitudes and phases of these transfer impedances are plotted against frequency for a virtual distance $v = 0.100\,\text{m}$, for all of the three physical sensor configurations considered. The solid grey lines indicate the two physical sensor configuration,
the dashed lines the three physical sensor configuration, and the dash-dotted lines the five physical sensor configuration. The vertical dotted lines indicate the frequencies of deep anti-resonances in the virtual primary and secondary transfer paths, which occur at lightly damped zeros. In Fig. 7.6, these anti-resonance frequencies correspond to the deep troughs in the power spectral density plots of the virtual primary and secondary disturbances.

![Figure 7.7: Numerical magnitude and phase plots of the transfer impedance (a) $T_d$ between the virtual primary disturbance and its estimate; (b) $T_y$ between the virtual secondary disturbance and its estimate, for the two — , three ---, and five ----- physical sensor configurations, for a virtual distance $v = 0.100$ m. The vertical dotted lines indicate the anti-resonance frequencies.](image)

The results in Fig. 7.7 indicate that the estimation performance is poorest at the deep anti-resonance frequencies in the virtual primary and secondary transfer paths. This result is intuitive because the estimate is expected to be least accurate at frequencies where the virtual disturbances have the least amount of energy, which is at the anti-resonance frequencies indicated by the vertical dotted lines in Fig. 7.7.

### 7.3.2 Experimental broadband estimation performance

In this section, the experimental broadband estimation performance of the adaptive LMS virtual microphone technique is analysed for the acoustic duct arrangement introduced in Chapter 6. The experimental results are compared to the numerical results presented in the previous section. The experimental analysis is performed for sixteen virtual locations $x_v = x_p^2 + v$, with $v$ the virtual distance illustrated in Fig. 7.3. The sixteen virtual distances examined are contained in a vector $v$ that is given by

$$v = \begin{bmatrix} 0.000 & 0.010 & 0.020 & \ldots & 0.150 \end{bmatrix} \text{ m.} \quad (7.25)$$
Optimal weights $h_{so}$ for the primary sound field are identified by exciting the primary disturbance source with a white disturbance source signal $s(n)$ filtered by an 8th order Butterworth filter with a pass-band between 50–500 Hz. Similarly, optimal weights $h_{uo}$ for the secondary sound field are identified by exciting the control source with a white control identification signal $u(n)$ that is band-limited in the frequency range of 50–500 Hz. For each of the virtual distances defined in Eq. (7.25), the covariance matrices $R_s$ and $R_u$ of the physical primary and secondary disturbances, respectively, and the cross-covariance matrices $p_s$ and $p_u$ between the physical and virtual disturbances are calculated based on 30 s of data measured by the physical microphones and the traversing microphone. This traversing microphone is positioned at the virtual distance of interest for each measurement, and measures the virtual primary and secondary disturbances. Using the measured covariance and cross-covariance matrices, the optimal weights $h_{so}$ and $h_{uo}$ for the primary and secondary sound fields, respectively, are then computed as defined in Theorem 3.1.

**Experimental optimal physical sensor weights**

The results of the described identification procedure are shown in Fig. 7.8, where the experimental optimal weights have been plotted against virtual distance for the two, three, and five physical sensor configurations. The solid black lines indicate the optimal weights $h_{so}$ for the primary sound field, and the solid grey lines the optimal weights $h_{uo}$ for the secondary sound field.

The experimental weights shown in Fig. 7.8 can be compared to the numerical weights shown in Fig. 7.4. For the two physical sensor configuration, the experimental and numerical optimal weights are very similar. However, for the three and five physical sensor configurations, the numerical and experimental results are very different. This was already anticipated from the discussions presented in Section 7.3.1, where it was explained that using the three or five physical sensor configuration results in an ill-conditioned estimation problem. The optimal weights are thus very sensitive to the unavoidable measurement noise on the physical sensors for these configurations, which results in experimental weights that are very different to the numerical ones.

As discussed previously, the numerical results in Fig. 7.4 show that for all of the three considered physical sensor configurations, the numerical weights for the primary and secondary sound fields are nearly identical for all considered virtual distances. In contrast, the results in Fig. 7.8 show that the experimental weights for the primary and secondary sound fields are similar for all considered virtual distances, especially for the two physical sensor configuration, but not as similar as the numerical weights. This discrepancy between the numerical and experimental results is most likely caused by small relative sensitivity and phase mismatches and position errors between the microphones in the array, which have not been accounted for in the numerical
modelling of the acoustic duct arrangement and which are generally unavoidable. Because the conditioning of the estimation problem is better for the two physical sensor configuration than for the three and five physical sensor configurations, the two physical sensor configuration is less sensitive to these relative mismatches and position errors. The numerical and experimental results are therefore in closer agreement for this configuration.

As discussed before, an estimate of the virtual error signal was computed in previous research \[80\] by directly applying the optimal weights for the secondary sound field to the physical error signals. Because the experimental optimal weights for the primary and secondary sound fields are most similar for the two physical sensor configuration, as shown in Fig. 7.8, this gives another explanation why previous research found that in real-time experiments, the best narrowband control performance was obtained using this configuration \[80\].

Figure 7.8: Experimental optimal physical sensor weights plotted against virtual distance \(v\), for the two, three, and five physical sensor configurations, with \(-h_{10} - h_{10}\).
Experimental overall estimation performance versus virtual distance

The overall estimation performance that is obtained at the various virtual distances can again be analysed by computing the normalised mean-square virtual output error for both the primary and secondary sound fields. In Fig. 7.9, the normalised mean-square virtual output error in dB, which has been defined in Eq. (7.18) for the primary sound field, has been plotted against the virtual distance \( v \), with the black lines denoting the results for the primary sound field, and the grey lines the results for the secondary sound field. The solid lines indicate the two physical sensor configuration, the dashed lines the three physical sensor configuration, and the dash-dotted lines the five physical sensor configuration.

![Graph showing the normalised mean-square virtual output error versus virtual distance](image)

Figure 7.9: Experimental normalised mean-square virtual output error plotted against virtual distance \( v \) for the primary sound field ---, and the secondary sound field ---, for --- \( M_p = 2 \) --- \( M_p = 3 \) --- \( M_p = 5 \).

The results in Fig. 7.9 show that for each of the physical sensor configurations considered, a similar estimation performance is obtained for the primary and secondary sound fields. For virtual distances \( v < 0.05 \) m, the estimation performance is relatively constant for the three physical sensor configurations considered, with the five physical sensor configuration providing about 4 dB additional estimation performance over the two physical sensor configuration. For larger virtual distances, the estimation performance decreases and becomes almost equal for all of the considered physical sensor configurations. Thus, although the numerical results presented in Section 7.3.1 predict that the five physical sensor configuration gives by far the best overall estimation performance, this expected large difference is not observed in the real-time experiments due to the fact that the estimation problem is ill-conditioned for this configuration.
Conditioning of estimation problem

Table 7.2 shows the condition numbers of the covariance matrices $R_s$ and $R_u$ of the measured physical primary and secondary disturbances, respectively. These matrices are computed in the identification procedure of the optimal weights described at the start of this section.

<table>
<thead>
<tr>
<th>$M_p$</th>
<th>$\kappa(R_s)$</th>
<th>$\kappa(R_u)$</th>
</tr>
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<td>75.4</td>
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<td>$5.8 \cdot 10^3$</td>
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</tbody>
</table>

Table 7.2: Experimental condition numbers of the covariance matrices $R_s$ and $R_u$.

The experimental condition numbers in Table 7.2 are all smaller than the numerical condition numbers given in Table 7.1. As explained in Section 7.3.1, this is due to the unavoidable measurement noise on the physical microphones, which has not been included in the numerical modelling of the acoustic duct arrangement. Thus, while the numerical condition numbers have been calculated with the covariance matrices $R_s$ and $R_u$ given by Eq. (7.22), the experimental condition numbers are effectively calculated with the covariance matrix $R_p$ of the physical measurement noise added to these matrices, as defined in Eq. (7.23). This causes the experimental condition numbers to be smaller than the numerical condition numbers. However, the condition numbers in Table 7.2 for the three and five physical sensor configurations are still quite large. This indicates that the estimation performance of these configurations will be more sensitive to small changes in the covariance matrices $R_s$ and $R_u$, and thus to small changes in the acoustic environment or the relative phase, sensitivity, and position mismatches in the physical sensor array, than the two physical sensor configuration.

Experimental spectral estimation performance for $v = 0.100$ m

In this section, the experimental spectral estimation performance is analysed for a virtual distance $v = 0.100$ m, which corresponds to a spatially fixed virtual location $x_v = 1.5750$ m inside the acoustic duct arrangement. Fig. 7.10(a) shows a power spectral density plot of the virtual primary disturbance and the virtual output error $\varepsilon_v(n) = d_v(n) - \hat{d}_v(n)$ for the primary sound field. Results are also shown for the secondary sound field in Fig. 7.10(b), with the virtual output error given by $\varepsilon_v(n) = y_v(n) - \hat{y}_v(n)$. In these figures, the solid black lines indicate the virtual primary and secondary disturbances, the solid grey lines the virtual output error for the two physical sensor configuration, the dashed lines the virtual output error for the three physical sensor configuration.
configuration, and the dash-dotted lines the virtual output error for the five physical sensor configuration.

![Figure 7.10: Experimental power spectral density plots of the virtual primary and secondary disturbances, and the virtual output error for the two, three, and five physical sensor configurations, for a virtual distance \( v = 0.100 \text{ m} \).](image)

The experimental results shown in Fig. 7.10 can be compared to the numerical results shown in Fig. 7.6. Again, it can be seen that although the numerical results predict that the five physical sensor configuration gives far better estimation performance than the two physical sensor configuration, this predicted large difference is not observed in the real-time experiments because the estimation problem is ill-conditioned for this configuration. The results presented in Fig. 7.9 show that for \( v = 0.100 \text{ m} \), the five physical sensor configuration only gives an additional 1 dB estimation performance over the two physical sensor configuration. Fig. 7.10 indicates that this additional performance is mainly obtained at frequencies below 250 Hz.

The experimental spectral estimation performance can further be analysed by computing the frequency response function \( T_d \) between the virtual primary disturbance and its estimate, and the frequency response function \( T_y \) between the virtual secondary disturbance and its estimate, which have been defined in Eq. (7.24). In Fig. 7.11, the measured frequency response functions are plotted for a virtual distance \( v = 0.100 \text{ m} \), for all of the three physical sensor configurations considered. The solid grey lines indicate the two physical sensor configuration, the dashed lines the three physical sensor configuration, and the dash-dotted the five physical sensor configuration.

The experimental results shown in Fig. 7.11 can be compared to the numerical results shown in Fig. 7.7, where it has been observed that the numerical estimation performance is poorest at the lightly damped anti-resonance frequencies in the primary and virtual secondary transfer paths. This can also be observed in the experimental
results shown in Fig. 7.11, where the estimation performance is poorest at the deep troughs in the power spectral density plots of the virtual primary and secondary disturbances shown in Fig. 7.10. Furthermore, the results in Fig. 7.11 again indicate that the small additional overall estimation performance of the five physical sensor configuration over the two physical sensor configuration for $v = 0.100$ m is mainly obtained at frequencies below 250 Hz. Above this frequency, the overall estimation performance of the considered physical sensor configurations is nearly identical for $v = 0.100$ m.

### 7.4 Narrowband adaptive feedforward control

In this section, the narrowband control performance that is obtained at a spatially fixed virtual location inside the acoustic duct arrangement when using the adaptive LMS virtual microphone technique is analysed. A schematic diagram of the implementation that is analysed in this section is shown in Fig. 7.12. The aim of this implementation is to minimise the estimate $\hat{e}_v(n)$ of the virtual error signal at the spatially fixed virtual location $x_0 = x_{p5} + v$, with $v$ the virtual distance and $x_{p5} = 1.4750$ m the location of the closest physical sensor. The estimate of the virtual error signal is computed using the adaptive LMS virtual microphone technique. The broadband estimation performance of this technique has been analysed in Section 7.3, where it has been shown that using the three or five physical sensor configuration results in an ill-conditioned estimation problem. The two physical sensor configuration is therefore
used here to compute an estimate of the virtual error signal, with the two physical sensors located at \(x_{p1} = 1.4250\) m and \(x_{p5} = 1.4750\) m. Numerical values for the physical sensor weights, which are used to compute the estimate \(\hat{e}_v(n)\) of the virtual error signal, have been calculated for 151 spatially fixed virtual locations \(x_v\). As shown in Fig. 7.4, these numerical weights are nearly identical for the primary and secondary sound fields for the two physical sensor configuration. Experimental values for the physical sensor weights have been calculated in the identification procedure described in Section 7.3.2 for the sixteen virtual locations defined in Eq. (7.25). As discussed in that section, the experimental weights shown in Fig. 7.8 are very similar for the primary and secondary sound fields for the two physical sensor configuration. The adaptive LMS virtual microphone technique is therefore implemented to compute an estimate of the virtual error signal, as illustrated in Fig. 7.12, as

\[
\hat{e}_v(n) = h_{uo}^T \mathbf{e}_p(n),
\]

with \(h_{uo}\) the optimal weights for the secondary sound field. These weights have been determined assuming that the control source is excited by band-limited white noise in the frequency range of 50–500 Hz. These weights are thus directly applied to the
physical error signals $e_p(n)$ when computing an estimate of the virtual error signal. This implementation was successfully used in previous research [80], and some insights are presented here that explain the reported real-time narrowband control results.

In the numerical analysis, the narrowband control performance that can in theory be obtained at a spatially fixed virtual location $x_v$ is analysed by minimising the estimate $\hat{e}_v(n)$ of the virtual error signal using the quadratic optimisation technique described in Chapter 4. The numerical results are again computed using a modal model of the acoustic duct arrangement without including measurement noise. In the experimental analysis, the estimate of the virtual error signal is minimised as illustrated in Fig. 7.12 using the filtered-x LMS algorithm as described in Section 4.3. The narrowband control performance is analysed for eight virtual distances given by

$$v = \begin{bmatrix} 0.000 & 0.020 & \ldots & 0.140 \end{bmatrix} \text{ m}, \quad (7.27)$$

and for three excitation frequencies given by 213 Hz, 249 Hz, and 284 Hz, which correspond to the sixth, seventh, and eighth resonance frequencies of the acoustic duct arrangement. In total, $3 \times 8 = 24$ real-time narrowband control experiments are therefore conducted. In each experiment, the primary and controlled sound pressure distributions inside the acoustic duct arrangement are measured with the traversing microphone.

### 7.4.1 Narrowband control performance versus frequency

In this section, the narrowband control performance is analysed for a virtual distance $v = 0.100$ m. The narrowband attenuation that can in theory be achieved for the case of one virtual sensor using one control source is given by Eq. (4.22) as

$$\eta_0 = -20 \log \left| 1 - \frac{T_y}{T_d} \right| \text{ dB}, \quad (7.28)$$

with $T_d$ the complex transfer impedance between the virtual primary disturbance $d_v(n)$ and its estimate, and $T_y$ the complex transfer impedance between the virtual secondary disturbance $y_v(n)$ and its estimate, which have been defined in Eq. (7.24). In the numerical analysis, Eq. (7.28) is used to analyse the narrowband attenuation that can in theory be obtained for a range of frequencies, with the estimate of the virtual error signal computed as defined in Eq. (7.26). In the experimental analysis, Eq. (7.28) is used to compute offline the narrowband attenuation that is expected to be achieved in the real-time narrowband control experiments. This expected real-time narrowband attenuation is computed offline using the experimental results shown in Fig. 7.11, where the measured frequency response functions $T_d$ and $T_y$ have been plotted for the two, three, and five physical sensor configurations. The numerical and experimental results of the described procedure are shown in Fig. 7.13, where the narrowband attenuation
\( \eta_0 \) has been plotted against frequency for the two, three, and five physical sensor configurations. The grey lines in Fig. 7.13 indicate the two physical sensor configuration, the dashed lines the three physical sensor configuration, and the dash-dotted lines the five physical sensor configuration. The vertical dotted lines in Fig. 7.13 indicate the resonance frequencies of the acoustic duct arrangement. For the numerical results, these resonance frequencies are derived from the modal model of the acoustic duct arrangement. For the experimental results, these resonance frequencies are derived from a state-space model of the acoustic duct arrangement obtained using the subspace model identification methods described in Section 6.4.

The results in Fig. 7.13 indicate that the best narrowband control performance is expected to be achieved at the resonance frequencies of the acoustic duct arrangement, with the estimate of the virtual error signal computed as in Eq. (7.26). Although the numerical results shown in Fig. 7.13(a) predict that for \( v = 0.100 \, \text{m} \), the five physical sensor configuration gives far better narrowband control performance than the two and three physical sensor configurations, this is not observed in the experimental results shown in Fig. 7.13(b). This discrepancy is again caused by the fact that the estimation problem is ill-conditioned for the five physical sensor configuration. The experimental results shown in Fig. 7.13(b) indicate that at frequencies above 200 Hz, the narrowband attenuation that is expected to be obtained in the real-time control experiments is almost equal for all of the considered physical sensor configurations. At frequencies below 200 Hz, the two physical sensor configuration is expected to outperform the three
and five physical sensor configurations around resonance frequencies. As discussed previously, the two physical sensor configuration is also less sensitive to changes in the acoustic environment and the relative phase, sensitivity, and position mismatches between the microphones in the array, as indicated by the experimental condition numbers in Table 7.2. These results explain why previous research has found that the two physical sensor configuration gives the best real-time narrowband control performance out of the configurations considered [80]. This is also why the two physical sensor configuration has been used in the real-time narrowband control experiments presented here, in which the estimate $\hat{e}_v(n)$ of the virtual error signals is minimised using the filtered-x LMS algorithm as illustrated in Fig. 7.12. These real-time narrowband control experiments are conducted for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, and the real-time narrowband attenuations that are achieved at these frequencies for $v = 0.100$ m are indicated by the black squares in Fig. 7.13(b). It can be seen that the obtained real-time narrowband attenuations for $v = 0.100$ m are in close agreement with the expected narrowband attenuations for the two physical sensor configuration that have been calculated offline.

As discussed previously, the numerical and experimental results shown in Fig. 7.13 indicate that the best narrowband control performance is expected to be achieved at the resonance frequencies of the acoustic duct arrangement. An explanation for this observation can be found by looking at the numerical and experimental results shown in Fig. 7.14, where the magnitude and phase of the ratio of the complex transfer impedances $T_d$ and $T_y$ have been plotted against frequency, for a virtual distance $v = 0.100$ m. From Eq. (7.28), it is the ratio of these two complex transfer impedances that determines the narrowband attenuation $\eta_o$ that is expected to be obtained in the real-time experiments. The grey lines in Fig. 7.14 indicate the two physical sensor configuration, the dashed lines the three physical sensor configuration, and the dash-dotted lines the five physical sensor configuration. The vertical dotted lines in Fig. 7.14 again indicate the resonance frequencies of the acoustic duct arrangement.

The results in Fig. 7.14 show that the magnitude and phase of the ratio of the two complex transfer impedances are closest to 0 dB and 0°, respectively, at the resonance frequencies of the acoustic duct arrangement. From Eq. (7.28), the attenuation $\eta_o$ is thus expected to be the greatest at these resonance frequencies. Looking at the individual magnitude and phase plots of the complex transfer impedances $T_d$ and $T_y$ shown in Fig. 7.11, these observations indicate that the best narrowband attenuation is obtained at the resonance frequencies because the magnitude and phase errors in the estimated virtual primary and secondary disturbances are equal at these frequencies, and thus cancel each other out in Eq. (7.28). Furthermore, it can again be seen in Fig. 7.14 that the numerical and experimental results for the three and five physical sensor configurations
7.4 Narrowband adaptive feedforward control

Figure 7.14: Magnitude and phase of the ratio of the complex transfer impedances $T_d$ and $T_y$ plotted against frequency for the two —, three —–, and five —— physical sensor configurations, for $v = 0.100$ m. The vertical dotted lines indicate the resonance frequencies of the acoustic duct arrangement.

are very different due to the fact that the estimation problem for these configurations is ill-conditioned.

7.4.2 Narrowband control performance versus virtual distance

In this section, the narrowband control performance is analysed for excitation frequencies of 213 Hz, 249 Hz and 284 Hz, and for the eight virtual distances defined in Eq. (7.27). The narrowband control performance that is obtained when minimising the estimate $\hat{e}_v(n)$ of the virtual error signal at the virtual locations for which $v > 0$ can then be compared with the narrowband control performance that is obtained at these locations when the physical error signal $e_{p5}(n)$ at the physical sensor located at $v = 0.000$ m is minimised. The two physical sensor configuration is used to compute an estimate $\hat{e}_v(n)$ of the virtual error signal as defined in Eq. (7.26). In the numerical analysis, the estimate of the virtual error signal is minimised using the quadratic optimisation technique described in Chapter 4. In the real-time control experiments, the filtered-x LMS algorithm described in Section 4.3 is used to minimise the estimate of the virtual error signal as illustrated in Fig. 7.12. This algorithm is implemented using a convergence coefficient $\mu = 5 \cdot 10^{-6}$ and $I = 2$ control filter coefficients. The tonal disturbance source signal $s(n)$ that excites the primary loudspeaker is also used as a feedforward reference signal $x(n)$ in the filtered-x LMS algorithm. In the real-time control experiments, the primary and controlled sound pressure distributions inside the acoustic duct arrangement are measured using the traversing microphone. The results of the described numerical and experimental analyses are shown in Fig. 7.15,
where the primary and controlled sound pressure distributions inside the acoustic duct are plotted over a 0.500 m section located between a virtual distance of $-0.200$ m and 0.300 m. For the experimental results, the primary and controlled sound pressure distributions are measured at increments of 0.020 m. The sound pressure level (SPL) at the excitation frequency is computed by averaging 40 power spectra. For the controlled sound pressure distributions, additional measurements are taken up to 0.020 m away from the intended virtual location in increments of 0.005 m. This is done in order to analyse more accurately if the zone of quiet is effectively moved away from the physical sensors to the desired virtual location. The black squares in Fig. 7.15 indicate the desired locations of maximum attenuation, i.e. the intended virtual locations. The dash-dotted lines indicate the primary sound pressure distributions, the dashed lines the controlled sound pressure distributions when minimising the physical error signal $e_p(n)$ at the physical sensor located at $v = 0.000$ m, and the solid lines of different grey scale the controlled sound pressure distributions when minimising the estimate $\hat{e}_v(n)$ of the virtual error signal at the considered virtual distances.

The numerical results shown in Fig. 7.15 indicate that for all of the excitation frequencies considered, the zone of quiet is centred at the desired virtual location for the smaller virtual distances $v \leq 0.040$ m. For larger virtual distances, the zone of quiet is located slightly to the left of the desired virtual location, with the distance between the desired and actual location of the zone of quiet increasing for larger virtual distances. These trends are also observed in the experimental results shown in Fig. 7.15, with the exception of one discrepancy between the numerical and experimental results. This discrepancy is that in the experimental results, the zone of quiet for smaller virtual distances $v \leq 0.040$ m is not centred at the desired virtual location, but about 0.005 m to the left of this location, for all of the excitation frequencies considered. However, the experimental results show that for all of the considered virtual distances, a higher narrowband attenuation is obtained at the desired virtual location when minimising the estimate of the virtual error signal instead of the physical error signal measured by the physical sensor located at $v_0 = 0.000$ m. This illustrates the potential benefits of employing a virtual sensing method over a conventional sensing method.

In Fig. 7.16, the numerical and experimental narrowband attenuations that are obtained at the desired spatially fixed virtual locations are plotted against virtual distance, for each of the considered excitation frequencies.

The numerical results in Fig. 7.16(a) indicate that for excitation frequencies of 213 Hz and 249 Hz, the attenuation decreases with increasing virtual distance. For an excitation frequency of 284 Hz, the numerical narrowband attenuation is smallest at a virtual distance $v = 0.040$ m. This observation can be explained by examining the primary sound pressure distribution inside the acoustic duct arrangement for this frequency which is shown in Fig. 7.15(e). It can be seen that for this frequency, the primary sound
Figure 7.15: Numerical and experimental sound pressure distribution plotted against virtual distance \( v \) for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with \( \cdots \) primary sound pressure distribution; controlled sound pressure distribution while minimising the \( \cdots \) physical error signal \( e_p(n) \) at the physical sensor located at \( v = 0.000 \) m; \( \cdots \) estimate \( \hat{e}_v(n) \) of the virtual error signal at the considered virtual distance \( v \). \( \diamond \) desired spatially fixed locations of the zone of quiet.
field has a node in the vicinity of a virtual distance \( v = 0.040 \) m. Because the location of the nodes are determined by the length of the acoustic duct, and not by the location of the sources, the secondary sound field has a node at this location as well. This means that when the numerical optimal weights for the secondary sound field are determined as described in Section 7.3.1, the virtual secondary disturbance \( y_v(n) \) has little energy at 284 Hz in comparison to other resonance frequencies. As a result, the estimate of the virtual error signal at this location will be less accurate for a frequency of 284 Hz in comparison to the other excitation frequencies. The numerical narrowband attenuation is thus expected to be smaller at frequencies for which the virtual sensor is located on a node in the sound field.

The experimental results in Fig. 7.16(b) confirm that for excitation frequencies of 213 Hz and 249 Hz, the narrowband attenuation decreases with increasing virtual distance. Also, the experimental attenuation is smaller than the numerical attenuation for all excitation frequencies, especially at smaller virtual distances. This can be attributed to the measurement noise on the physical sensors, which is unavoidable in the real-time experiments and which limits the amount of narrowband attenuation that can be achieved in real-time. For an excitation frequency of 284 Hz, the narrowband attenuation that is obtained in the real-time experiments is smallest at a virtual distance \( v = 0.020 \) m. As discussed before, this can be explained by noting that for this frequency, the primary sound field has a node in the vicinity of this location as can be seen in Fig. 7.15(f). For the largest virtual distance \( v = 0.140 \) m, Fig. 7.16(b) shows that narrowband attenuations of 45 dB, 28 dB, and 22 dB are still obtained in the real-time control experiments for the excitation frequencies considered.
Surface plot of numerical narrowband attenuation

Fig. 7.17 shows a surface plot of the numerical narrowband attenuation $\eta_o$ defined in Eq. (7.28) plotted against excitation frequency and virtual distance $v$.

![Surface plot of numerical narrowband attenuation](image)

Figure 7.17: Surface plot of the numerical narrowband attenuation $\eta_o$ plotted against frequency $f$ and virtual distance $v$.

The numerical results in Fig. 7.17 again illustrate that the best narrowband control performance is expected to be obtained at the resonance frequencies of the acoustic duct arrangement, which has been confirmed by the numerical and experimental results presented here and in previous research [80]. Furthermore, it can be seen that the narrowband attenuation $\eta_o$ is expected to decrease for increasing virtual distance $v$. An exception to this rule occurs at frequencies for which the virtual sensor is located on a node in the sound field inside the acoustic duct. For this case, the narrowband attenuation will generally be smaller for this virtual distance than for larger virtual distances, as was confirmed by the numerical and experimental results shown in Fig. 7.16 for 284 Hz. The influence of a node in the sound field can also be observed in Fig. 7.17 for an excitation frequency of 391 Hz, for which the sound field has a node around $v = 0.060$ m.
Chapter 7  Adaptive LMS virtual microphone technique

7.5 Broadband adaptive feedforward control

In this section, the broadband adaptive feedforward control performance that can be obtained at a spatially fixed virtual sensor located inside the acoustic duct arrangement is analysed. The implementation illustrated in Fig. 7.12 is again used to minimise the estimate \( \hat{e}_v(n) \) of the virtual error signal. The two physical sensors located at \( x_{p1} = 1.425 \) m and \( x_{p5} = 1.475 \) m are again used to compute an estimate \( \hat{e}_v(n) \) of the virtual error signal as defined in Eq. (7.26). This implementation has successfully been used in the real-time narrowband control experiments described in Section 7.4, and was also used in real-time narrowband control experiments reported in previous research [80]. In the numerical analysis presented in the following, the optimal broadband feedforward control performance that can in theory be obtained at a spatially fixed virtual location \( x_v \) is analysed by minimising the estimate \( \hat{e}_v(n) \) of the virtual error signal using the causal Wiener filter techniques introduced in Chapters 2 and 4. In the real-time experiments, the estimate \( \hat{e}_v(n) \) of the virtual error signal is minimised in Fig. 7.12 using the filtered-x LMS algorithm as described in Section 4.3. This algorithm is implemented using a convergence coefficient \( \mu = 5 \cdot 10^{-6} \) and \( I = 450 \) control filter coefficients.

Broadband feedforward control performance for \( v = 0.100 \) m

In this section, the broadband feedforward control performance that is obtained while minimising the estimate \( \hat{e}_v(n) \) of the virtual error signal is compared with the broadband feedforward control performance that is obtained while minimising the true virtual error signal \( e_v(n) \) for a virtual distance \( v = 0.100 \) m. The true virtual error signal is directly measured by the traversing microphone that is positioned at the considered virtual distance. For both cases, the broadband feedforward control performance is measured after convergence of the filtered-x LMS algorithm. The numerical and experimental results are shown in Fig. 7.18, where power spectral density plots of the virtual primary disturbance \( d_v(n) \) and the residual virtual error signal \( e_v(n) \) at the spatially fixed virtual location are shown for both control cases.

The experimental power spectral density plots in Fig. 7.18(b) are generated by averaging 50 power spectral densities, which are computed from measurements taken by the traversing microphone located at the considered virtual distance. In the numerical results shown in Fig. 7.18(a), an overall attenuation of 20.1 dB is obtained at the virtual location while minimising the estimate \( \hat{e}_v(n) \) of the virtual error signal, compared to an overall attenuation of 36.2 dB obtained while minimising the true virtual error signal \( e_v(n) \), which is a difference of 16.1 dB. In the experimental results shown in Fig. 7.18(b), an overall attenuation of 13.5 dB is obtained at the virtual location while minimising the estimate \( \hat{e}_v(n) \) of the virtual error signal, compared to an overall attenuation of 23.8 dB obtained while minimising the virtual error signal \( e_v(n) \) directly measured...
Figure 7.18: Numerical and real-time broadband feedforward control performance at virtual location after convergence of adaptive algorithm, with — virtual primary disturbance $d_v(n)$ — residual virtual error signal while minimising $\hat{e}_v(n)$ — residual virtual error signal while minimising $e_v(n)$.

by the traversing microphone positioned at the desired virtual location, which is a difference of 10.3 dB. Because a similar difference is observed in the numerical results, this difference is predominantly determined by the broadband estimation performance of the adaptive LMS virtual microphone technique, and not by factors such as modelling errors and measurement noise. In other words, the main cause of the 10.3 dB difference in the real-time broadband feedforward control performance is that the estimate $\hat{e}_v(n)$ of the virtual error signal is not perfect for $v = 0.100 \text{ m}$, which has been observed in Section 7.3 where the broadband estimation performance of the adaptive LMS virtual microphone technique has been analysed.

The real-time broadband feedforward control performance obtained at the virtual location while minimising the estimate $\hat{e}_v(n)$ of the virtual error signal is now compared with the real-time performance obtained at this location while minimising the physical error signal $e_{p5}(n)$ at the physical sensor located at $x_{p5} = 1.4750 \text{ m}$. This physical error signal, which is directly measured during real-time control, is minimised using the standard implementation of the filtered-x LMS algorithm, which is again implemented employing a convergence coefficient of $\mu = 5 \cdot 10^{-6}$ and $I = 450$ filter coefficients. The broadband feedforward control performance that is obtained at the physical sensor is shown in Fig. 7.19(a), where power spectral density plots of the physical primary disturbance $d_{p5}(n)$ and the residual physical error signal $e_{p5}(n)$ measured after convergence of the filtered-x LMS algorithm are shown.

The power spectral density plots in Fig. 7.19(a) are computed by averaging 50 measured power spectral densities. An overall attenuation of 24.4 dB is obtained at
Figure 7.19: Experimental broadband feedforward control performance obtained at the (a) physical sensor located at \( x_{p5} = 1.4750 \) m while minimising the physical error signal \( e_{p5}(n) \), with physical primary disturbance \( d_{p5}(n) \) — residual physical error signal \( e_{p5}(n) \); and (b) virtual sensor located at \( v = 0.100 \) m, with virtual primary disturbance \( d_{v}(n) \) — residual virtual error signal \( e_{v}(n) \) while minimising \( \hat{e}_{v}(n) \) — residual virtual error signal \( e_{v}(n) \) while minimising \( e_{p5}(n) \).

the physical sensor located at \( x_{p5} = 1.4750 \) m while minimising the physical error signal \( e_{p5}(n) \). Fig. 7.19(b) shows a power spectral density plot of the virtual primary disturbance \( d_{v}(n) \) at the virtual distance \( v = 0.100 \) m, and power spectral density plots of the residual virtual error signal \( e_{v}(n) \) at this virtual distance that is obtained while minimising either the estimate \( \hat{e}_{v}(n) \) of the virtual error signal or the physical error signal \( e_{p5}(n) \) at the physical sensor located at \( x_{p5} = 1.4750 \) m. Although an overall attenuation of 24.4 dB is obtained at the physical sensor while minimising the physical error signal, the overall attenuation obtained at the virtual location for this case is only 3.3 dB. Thus, minimising the estimate \( \hat{e}_{v}(n) \) of the virtual error signal instead of the physical error signal \( e_{p5}(n) \), which results in an overall attenuation of 13.5 dB at the desired virtual distance, increases the overall attenuation that is obtained at the virtual distance \( v = 0.100 \) m by 10.2 dB.

**Broadband feedforward control performance versus virtual distance**

In this section, the broadband feedforward control performance is analysed for the eight virtual distances defined in Eq. (7.27). In the numerical analysis, the controlled sound pressure distributions inside the acoustic duct arrangement are computed using the causal Wiener filter techniques introduced in Chapters 2–4. In the real-time experiments, the primary and controlled sound pressure distributions inside the acoustic duct arrangement are measured with the traversing microphone. The measurements of the
sound pressure distributions are performed similarly to the narrowband control case described in Section 7.4. The numerical and experimental results are shown in Fig. 7.20, where the primary and controlled sound pressure distributions inside the acoustic duct are plotted over a 0.500 m section located between a virtual distance of −0.200 m and 0.300 m. The black squares in Fig. 7.20 indicate the desired locations of maximum attenuation, i.e. the desired virtual locations.

Figure 7.20: Numerical and experimental primary and controlled sound pressure distributions plotted against virtual distance $v$, with primary sound pressure distribution; controlled sound pressure distribution while minimising the physical error signal $e_p(n)$ at the physical sensor located at $v = 0.000$ m; estimate $\hat{e}_v(n)$ of the virtual error signal at the considered virtual distance $v$. ■ desired spatially fixed locations of the zone of quiet.

The numerical results in Fig. 7.20(a) show that the zone of quiet has been moved away from the physical sensor at $v = 0.000$ m to the desired virtual location for all of the virtual distances considered. It can also be seen that the overall numerical attenuation obtained at the desired virtual location decreases as the virtual distance increases. This is because the numerical broadband estimation performance of the adaptive LMS virtual microphone technique decreases for increasing virtual distance, which has been observed in the numerical results shown in Fig. 7.5. The same trend is observed in the experimental results shown in Fig. 7.20(b), where the overall attenuation also decreases for increasing virtual distance. Again, this is because the real-time broadband estimation performance of the adaptive LMS virtual microphone technique decreases for increasing virtual distance, which has been observed in the experimental results illustrated in Fig. 7.9. In the real-time experimental results shown in Fig. 7.20(b), the zone of quiet is also located slightly to the left of the desired virtual location, with the distance between the actual and desired location of the zone of quiet increasing as the virtual distance increases.
In Fig. 7.21, the numerical and experimental overall attenuations that are obtained at the considered virtual distances while minimising an estimate $\hat{e}_v(n)$ of the virtual error signal are compared with the overall attenuations obtained while minimising the physical error signal $e_{p5}(n)$ at the physical sensor located at $v = 0.000$ m.

The numerical and experimental results in Fig. 7.21 show that for each of the virtual distances considered, the overall attenuation is always larger when minimising an estimate $\hat{e}_v(n)$ of the virtual error signal instead of the physical error signal $e_{p5}(n)$. This illustrates the potential benefits of employing a virtual sensing method over a conventional sensing method.

### 7.6 Forward difference prediction techniques

Another method for determining a solution for the physical sensor weights is to use forward difference prediction techniques [13]. Forward difference prediction techniques calculate the weights by fitting a polynomial through the physical error signals $e_p(n)$ at the physical sensors located at $x_p$, and by extrapolating this polynomial to the virtual location $x_v$ in order to obtain an estimate $\hat{e}_v(n)$ of the virtual error signal. These techniques have been investigated extensively for a rigidly terminated rectangular acoustic duct [62, 80]. In this section, the forward difference prediction weights are compared to analytical optimal solutions for the weights for the case of single tone disturbances inside the acoustic duct arrangement. The analysis shows that the forward difference prediction weights approach the analytical optimal solutions for small
virtual distances and low frequencies. As a result, the narrowband attenuation that can be obtained at a virtual location inside the acoustic duct arrangement is expected to deteriorate for larger virtual distances and higher frequencies, which has been confirmed in real-time control results presented in previous research [62, 80]. The analysis presented here has been published in Petersen et al. [103].

The numerical results presented in this section are computed using a modal model of the acoustic duct arrangement, which was introduced in Chapter 6. For the numerical linear forward difference prediction results, the two physical sensor configuration introduced in Section 7.3 is used. For the numerical over-constrained linear and quadratic forward difference prediction results, the three physical sensor configuration is used. These configurations have also been analysed in previous research into the forward difference prediction techniques [63, 64, 84].

**Linear forward difference prediction**

The linear forward difference prediction method which uses two physical sensors is illustrated in Fig. 7.22.

![Figure 7.22: Linear forward difference prediction.](image)

In Fig. 7.22, a first-order polynomial is fitted through the physical error signals \( e_p(n) \), which is then extrapolated to obtain an estimate \( \hat{e}_v(n) \) of the virtual error signal. This estimate is therefore calculated as [83]

\[
\hat{e}_v(n) = \begin{bmatrix} \frac{v}{\Delta} & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{p1}(n) \\ e_{p2}(n) \end{bmatrix},
\]

(7.29)

which can also be written as

\[
\hat{e}_v(n) = \begin{bmatrix} -\frac{v}{\Delta} & \frac{v+\Delta}{\Delta} \end{bmatrix} \begin{bmatrix} e_{p1}(n) \\ e_{p2}(n) \end{bmatrix} = h_{fl}^T e_p(n),
\]

(7.30)

with \( h_{fl} \in \mathbb{R}^2 \) the linear forward difference prediction weights. These weights can be compared to the analytical optimal solution for the weights, which has been defined in
Chapter 7  Adaptive LMS virtual microphone technique

Eq. (7.7) as

\[
\mathbf{h}_o = \frac{1}{\sin k\Delta} \left[ \begin{array}{c} -\sin kv \\ \sin k(v + \Delta) \end{array} \right].
\] (7.31)

If the wave number \( k \), the virtual distance \( v \), and the microphone separation distance \( \Delta \) are such that \( kv \ll 1 \) and \( k\Delta \ll 1 \), the weights in Eq. (7.31) can be approximated by the weights defined in Eq. (7.30), where use has been made of small angle approximations, such that

\[
\sin \theta \approx \theta, \quad \theta \ll 1.
\] (7.32)

This is illustrated in Fig. 7.23 where the analytical optimal weights \( \mathbf{h}_o \) and the linear forward difference prediction weights \( \mathbf{h}_{fl} \) have been plotted against frequency and virtual distance, for a physical sensor separation distance \( \Delta = 0.05 \text{ m} \).

![Figure 7.23: Analytical optimal weights \( \mathbf{h}_o \) and linear forward difference prediction weights \( \mathbf{h}_{fl} \) plotted against (a) frequency for a virtual distance \( v = 0.100 \text{ m} \); and (b) virtual distance for a frequency \( f = 249 \text{ Hz} \), with --- \( \mathbf{h}_o \) —- \( \mathbf{h}_{fl} \).](image)

The results in Fig. 7.23(a) illustrate that for a virtual distance \( v = 0.100 \text{ m} \), the linear forward difference prediction weights approximate the analytical optimal weights at low frequencies. Fig. 7.23(b) shows that for a frequency \( f = 249 \text{ Hz} \), the linear forward difference prediction weights approximate the analytical optimal weights for small virtual distances. Both figures also indicate that the estimation accuracy of the linear forward difference prediction method is expected to deteriorate for larger virtual distances and increasing excitation frequencies, which has been confirmed in real-time narrowband control experiments presented in previous research [62, 80].
Over-constrained linear forward difference prediction

The over-constrained linear forward difference prediction method which uses three physical sensors is illustrated in Fig. 7.24.

![Diagram](image)

Figure 7.24: Over-constrained linear forward difference prediction.

In Fig. 7.24, a first-order polynomial is fitted through the physical error signals $e_p(n)$, resulting in an over-constrained problem that is solved using a least squares approximation. The estimate $\hat{e}_v(n)$ is then calculated as [82]

$$
\hat{e}_v(n) = h_{fl3}^T e_p(n),
$$

where $h_{fl3} \in \mathbb{R}^3$ are the over-constrained linear forward difference prediction weights given by

$$
h_{fl3} = \begin{bmatrix}
-\frac{3v+\Delta}{2\lambda} & \frac{1}{3} & \frac{3v+5\Delta}{6\lambda}
\end{bmatrix}^T.
$$

These weights can be compared to the analytical optimal solution $h_o \in \mathbb{R}^3$ for the three physical sensors case. This analytical optimal solution can be derived using a similar method as described in Section 7.2, where the analytical optimal solution has been calculated for the two physical sensor case using a travelling wave model of the acoustic duct. Similar to Eq. (7.5), the following matrix equation can be solved in order to find the analytical optimal solution

$$
A h_o = \begin{bmatrix}
e^{-jkx_p1+\Phi} \\
e^{-jkx_p2+\Phi} \\
e^{-jkx_p3+\Phi}
\end{bmatrix},
$$

with the matrix $A \in \mathbb{C}^{2\times3}$ given by

$$
A = \begin{bmatrix}
e^{-jkx_p1+\Phi} & e^{-jkx_p2+\Phi} & e^{-jkx_p3+\Phi} \\
e^{jkx_p1-\Phi} & e^{jkx_p2-\Phi} & e^{jkx_p3-\Phi}
\end{bmatrix}.
$$

Eq. (7.35) is an under-determined system of equations, which requires the introduction of an additional constraint in order to obtain a unique solution for the analytical optimal
weights $h_o$. A common additional constraint \cite{88} is to minimise $h_o^H h_o$ while setting the virtual output error $e_v = e_o - \hat{e}_o$ equal to zero. The analytical optimal weights are then given by

$$h_o = A^\dagger \left[ \frac{e^{-jkx_o + \Phi}}{e^{jkx_o - \Phi}} \right], \quad (7.37)$$

where the matrix $A^\dagger \in \mathbb{C}^{3 \times 2}$ is the pseudo-inverse of the matrix $A$ defined in Eq. (7.36) given by

$$A^\dagger = A^H (AA^H)^{-1}. \quad (7.38)$$

It can be shown that Eq. (7.37) can be succinctly written as

$$h_o = \begin{bmatrix} -\frac{2 \sin k(v+ \Delta) + \sin k(v- \Delta)}{3 \sin k \Delta + \sin 3k \Delta} \\
\frac{\cos k(v+ \Delta)}{2 + \cos 2k \Delta} \\
\frac{2 \sin k(v+ \Delta) + \sin k(v+3 \Delta)}{3 \sin k \Delta + \sin 3k \Delta} \end{bmatrix}. \quad (7.39)$$

For small angle approximations, it can be shown that Eq. (7.39) reduces to Eq. (7.34). This is illustrated in Fig. 7.25 where the analytical optimal weights $h_o$ and the over-constrained linear forward difference prediction weights $h_{f/3}$ have been plotted against frequency and virtual distance, for a physical sensor separation distance $\Delta = 0.025$ m.

![Figure 7.25: Analytical optimal weights $h_o$ and over-constrained linear forward difference prediction weights $h_{f/3}$ plotted against (a) frequency for a virtual distance $v = 0.100$ m; and (b) virtual distance for a frequency $f = 249$ Hz, with $h_o$ and $h_{f/3}$.](image)

The results in Fig. 7.25(a) illustrate that for a virtual distance $v = 0.100$ m, the over-constrained linear forward difference prediction weights approximate the analytical optimal weights at low frequencies. Fig. 7.25(b) shows that for a frequency $f = 249$ Hz, the over-constrained linear forward difference prediction weights approximate the
7.6 Forward difference prediction techniques

optimal weights for small virtual distances. Both figures also indicate that the estimation accuracy of the over-constrained linear forward difference prediction method is expected to deteriorate for larger virtual distances and increasing excitation frequencies, which has been confirmed in real-time narrowband control experiments presented in previous research [62, 80].

Quadratic forward difference prediction

The quadratic forward difference prediction method which uses three physical sensors is illustrated in Fig. 7.26.

\[
\hat{e}_v(n) = h_{fq}^T e_p(n), \quad (7.40)
\]

where \( h_{fq} \in \mathbb{R}^3 \) are the quadratic forward difference prediction weights given by

\[
h_{fq} = \begin{bmatrix}
\frac{v^2(3v^2 + 2\Delta^2)}{2\Delta^4} \\
\frac{v^2 + 3v\Delta + 2\Delta^2}{2\Delta^2}
\end{bmatrix}^T. \quad (7.41)
\]

These weights can be compared to the analytical optimal solution \( h_o \in \mathbb{R}^3 \) for the weights, which can be found in a similar way as described previously for the over-constrained linear forward difference prediction case. Again, the analytical optimal solution can be found from Eq. (7.35) by introducing an additional constraint. In order to make a comparison with the quadratic forward difference prediction weights \( h_{fq} \), one of the weights of \( h_o \) is constrained to be equal to the corresponding weight of \( h_{fq} \). Here, the weights \( h_{fq1} \) and \( h_{o1} \) for the physical sensor located at \( x_{p1} \) in Fig. 7.26 are constrained to be equal. Eq. (7.35) can then be modified to

\[
Ah_o = \begin{bmatrix}
\frac{v(v+\Delta)}{2\Delta^2} \\
\frac{v(v+\Delta)}{2\Delta^2} \\
e^{-jkx_o + \Phi} \\
e^{jkx_o - \Phi}
\end{bmatrix}, \quad (7.42)
\]
Chapter 7  Adaptive LMS virtual microphone technique

with the matrix $A \in \mathbb{C}^{3 \times 3}$ given by

$$A = \begin{bmatrix}
1 & 0 & 0 \\
e^{-j k x_{p1} + \Phi} & e^{-j k x_{p2} + \Phi} & e^{-j k x_{p3} + \Phi} \\
e^{j k x_{p1} - \Phi} & e^{j k x_{p2} - \Phi} & e^{j k x_{p3} - \Phi}
\end{bmatrix}. \quad (7.43)$$

Eq. (7.42) is a fully-determined system of equations which can be solved for the weights $h_o$ as

$$h_o = A^{-1} \begin{bmatrix}
v(v + \Delta) \\
e^{-j k x \nu + \Phi} \\
e^{j k x \nu - \Phi}
\end{bmatrix}, \quad (7.44)$$

which can be succinctly written as

$$h_o = \begin{bmatrix}
v(v + \Delta)
\frac{\nu(v + \Delta) \sin 2k \Delta + 2 \Delta^2 \sin k \nu}{2 \Delta^2 \sin k \Delta}
\frac{\nu(v + \Delta) \sin k \Delta + 2 \Delta^2 \sin k (v + \Delta)}{2 \Delta^2 \sin k \Delta}
\end{bmatrix}. \quad (7.45)$$

For small angle approximations, it can be shown that Eq. (7.45) reduces to Eq. (7.41). This is illustrated in Fig. 7.27 where the analytical optimal weights $h_o$ and the quadratic forward difference prediction weights $h_{fq}$ have been plotted against frequency and virtual distance $v$, for a physical sensor separation distance $\Delta = 0.025 \text{ m}$.

![Figure 7.27: Analytical optimal weights $h_o$ and quadratic forward difference prediction weights $h_{fq}$ plotted against (a) frequency for a virtual distance $v = 0.100 \text{ m}$; and (b) virtual distance for a frequency $f = 249 \text{ Hz}$, with $h_o$ --- $h_{fq}$.](image)

The results in Fig. 7.27(a) illustrate that for a virtual distance $v = 0.100 \text{ m}$, the quadratic forward difference prediction weights approximate the analytical optimal
weights at low frequencies. Fig. 7.27(b) shows that for a frequency \( f = 249 \) Hz, the quadratic forward difference prediction weights approximate the analytical optimal weights for small virtual distances. Both figures also indicate that the estimation accuracy of the quadratic forward difference prediction method is expected to deteriorate for larger virtual distances and increasing excitation frequencies, which has been confirmed in real-time narrowband control experiments presented in previous research \([62, 80]\).

**Numerical narrowband control performance**

The narrowband control performance that can in theory be achieved at a virtual location inside the acoustic duct arrangement when using the forward difference prediction techniques is now analysed using Eq. (7.28). The results are again computed using a modal model of the acoustic duct arrangement introduced in Chapter 6, with the modal model calculated as described in Section 7.3. The numerical results are shown Fig. 7.28, where the narrowband attenuation \( \eta_0 \) defined in Eq. (7.28) has been plotted against frequency for a virtual distance \( v = 0.100 \) m, and against virtual distance for a frequency \( f = 249 \) Hz. The solid black lines indicate the linear forward difference prediction method, the solid grey lines the over-constrained linear forward difference prediction method, and the dash-dotted lines the quadratic forward difference prediction method. The vertical dashed lines indicate the resonance frequencies of the acoustic duct arrangement.

![Numerical narrowband attenuation η₀ that can in theory be obtained when using forward difference prediction techniques plotted against (a) frequency for a virtual distance v = 0.100 m; and (b) virtual distance for a frequency f = 249 Hz, with — linear prediction — over-constrained linear prediction — quadratic prediction.](image)

The numerical results in Fig. 7.28(a) show that the best narrowband control performance is achieved at the resonance frequencies of the acoustic duct system. Again, it can
also be seen that the narrowband attenuation is expected to deteriorate for increasing excitation frequency. Fig. 7.28(b) shows that the narrowband attenuation decreases for increasing virtual distance. Similar trends are observed in the numerical results for different frequencies and virtual distances. Both figures illustrate that, theoretically, the quadratic prediction technique outperforms the linear and over-constrained linear prediction techniques. These two techniques give near identical results, especially for larger virtual distances as illustrated in Fig. 7.28(b). However, previous real-time narrowband control experiments [63, 64, 84] indicated that the linear prediction technique outperformed the over-constrained and quadratic prediction techniques in real-time acoustic duct experiments. A reason for this disagreement between the numerical and experimental results can be found by analysing the conditioning of the different forward difference prediction techniques.

**Sensitivity to relative positioning errors of physical sensors**

The conditioning of the forward difference prediction methods can be analysed by examining how the estimate \( \hat{e}_v(n) \) of the virtual error signal is calculated for the three forward difference prediction methods considered. For the *linear* forward difference prediction technique, the estimate of the virtual error signal is calculated as

\[
\hat{e}_v(n) = \left[ x_v, 1 \right] \left[ x_{p1}, 1 \right]^{-1} \left[ e_{p1}(n), e_{p2}(n) \right],
\]

which can easily be derived from Fig. 7.22 by finding the slope and intersection coefficients \( a \) and \( b \) of the equation \( e(n) = ax + b \), and then extrapolating the resulting function to \( x_v \). Similarly, for the *over-constrained* linear forward difference prediction technique, the estimate of the virtual error signal is computed as

\[
\hat{e}_v(n) = \left[ x_v^2, x_v, 1 \right] \left[ x_{p1}, 1 \right]^{\dagger} \left[ e_{p1}(n), e_{p2}(n), e_{p3}(n) \right],
\]

with the pseudo-inverse computed as

\[
\left[ x_{p1}, 1 \right]^{\dagger} = \left( \left[ x_{p1}, x_{p2}, x_{p3} \right] \left[ x_{p1}, 1 \right] \right)^{-1} \left[ x_{p1}, x_{p2}, x_{p3} \right].
\]
For the \textit{quadratic} forward difference prediction technique, the estimate of the virtual error signal is computed as

$$\hat{e}_v(n) = \begin{bmatrix} x_p^2 & x_p & 1 \\ x_p^2 & x_p & 1 \\ x_p^2 & x_p & 1 \end{bmatrix}^{-1} \begin{bmatrix} e_{p1}(n) \\ e_{p2}(n) \\ e_{p3}(n) \end{bmatrix}. \quad (7.49)$$

The conditioning of the forward difference prediction techniques can now be analysed by computing the condition numbers of the matrices that are inverted in Eqs (7.46)–(7.49). Table 7.3 gives these condition numbers for the three considered forward difference prediction techniques. These condition numbers are calculated for the physical sensor configurations described at the start of this section. These configurations have also been used in the previous research [63, 64, 84] that reported the disagreement between the numerical and experimental results.

<table>
<thead>
<tr>
<th>Prediction method</th>
<th>Condition number</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>124</td>
<td>0.050 m</td>
</tr>
<tr>
<td>over-constrained linear</td>
<td>$2.3 \cdot 10^4$</td>
<td>0.025 m</td>
</tr>
<tr>
<td>quadratic</td>
<td>$3.5 \cdot 10^4$</td>
<td>0.025 m</td>
</tr>
</tbody>
</table>

Table 7.3: Condition numbers for the forward difference prediction techniques when implemented on the acoustic duct arrangement for a virtual distance $v = 0.100$ m and with $x_{p1} = 1.425$ m.

The large condition numbers for the \textit{over-constrained linear} and \textit{quadratic} forward difference prediction techniques indicate that these techniques are more sensitive to small errors in the positions of the physical sensors, which are generally unavoidable in practice, than the \textit{linear} forward difference technique. This explains the disagreement between numerical and experimental results that was reported in previous research [63, 64, 84] in which the numerical results predicted that the quadratic prediction method would perform best while the linear forward difference prediction technique was found to provide the best real-time narrowband control performance.

**Comparison to optimal broadband weights**

In this section, the linear forward difference prediction weights $h_{fi}$ shown in Fig. 7.23 and the numerical optimal broadband weights for the two physical sensor configuration shown in Fig. 7.4 are compared to the analytical optimal weights defined in Eq. (7.31). A typical result of this comparison is shown in Fig. 7.29, where these weights have been plotted against frequency for a virtual distance $v = 0.100$ m, and against virtual
distance for a frequency \( f = 249 \text{ Hz} \). The solid black lines indicate the analytical optimal weights defined in Eq. (7.31), the solid grey lines the linear forward difference prediction weights \( h_{fl} \), and the dash-dotted lines the optimal broadband weights.

![Graph](image)

Figure 7.29: Analytical optimal weights, linear forward difference prediction weights, and optimal broadband weights for the two physical sensor configuration, plotted against (a) frequency for a virtual distance \( v = 0.100 \text{ m} \); and (b) virtual distance for a frequency \( f = 249 \text{ Hz} \), with — analytical optimal weights — linear forward difference prediction weights —— optimal broadband weights.

The numerical results in Fig. 7.29(a) show that for a virtual distance \( v = 0.100 \text{ m} \), the linear forward difference prediction weights approximate the analytical optimal weights at low frequencies, while the optimal broadband weights approximate the average of the analytical optimal weights over the frequency range 50–500 Hz. This result is intuitive because the optimal broadband weights are computed assuming that the acoustic duct is excited over this frequency range. Numerical results for different virtual distances showed similar trends.

The numerical results in Fig. 7.29(b) show that for a frequency \( f = 249 \text{ Hz} \), the optimal broadband weights give a better approximation of the analytical optimal weights than the linear forward difference prediction weights, especially at larger virtual distances. This is because the linear prediction weights approximate the analytical optimal weights at small virtual distances only, and because the considered frequency is almost at the centre of the frequency range between 50–500 Hz that is considered when calculating the optimal broadband weights. Similar results are not observed at low frequencies, where the linear forward difference prediction weights give a better approximation of the analytical optimal weights than the optimal broadband weights. Again, this is because the linear forward difference prediction weights approximate the analytical optimal weights at low frequencies.
7.6 Forward difference prediction techniques

In Fig. 7.30, the narrowband attenuation $\eta_o$ defined in Eq. (7.28) has been plotted against frequency for a virtual distance $v = 0.100$ m, and against virtual distance for a frequency $f = 249$ Hz. The solid grey lines indicate the linear forward difference prediction technique, and the dash-dotted lines the adaptive LMS virtual microphone technique using optimal broadband weights. The vertical dashed lines indicate the resonance frequencies of the acoustic duct system.

![Graphs showing numerical narrowband attenuation](image)

Figure 7.30: Numerical narrowband attenuation obtained using linear forward difference prediction weights or optimal broadband weights for the two physical sensor configuration plotted against (a) frequency for a virtual distance $v = 0.100$ m; and (b) virtual distance for a frequency $f = 249$ Hz, with — linear forward difference prediction weights —- optimal broadband weights.

The numerical results in Fig. 7.30(a) indicate that for a virtual distance $v = 0.100$ m, the linear forward difference prediction weights give better narrowband performance at low frequencies than the optimal broadband weights. This is because the linear forward difference prediction weights approximate the analytical optimal weights at low frequencies, while the optimal broadband weights almost approximate the average of these weights over the frequency range of 50–500 Hz, as observed in Fig. 7.29. Numerical results for different virtual distances showed similar trends.

**Final remark**

It is important to keep in mind that a numerical comparison between the forward difference prediction technique and the adaptive LMS virtual microphone technique has been presented in this section using a numerical modal model of the acoustic duct arrangement. Relative phase, sensitivity, and position mismatches between the physical sensors in the array have thus not been taken into account in the numerical analysis. It has been shown in previous research [80] that the adaptive LMS virtual microphone
technique is able to compensate for these mismatches in real-time experiments, which has proved to result in better real-time narrowband control performance [80]. This is a significant advantage over the forward difference prediction techniques.

7.7 Conclusion

The adaptive LMS virtual microphone technique [14] has been implemented on the acoustic duct arrangement introduced in Chapter 6. The estimation and control performance obtained at a spatially fixed virtual location inside the acoustic duct arrangement has been analysed for narrowband and broadband disturbances. The real-time experimental results have been compared to numerical results computed using the acoustic duct models introduced in Chapter 6.

An analytical optimal solution for the physical sensor weights has been derived for the case of single tone disturbances inside the acoustic duct arrangement. It has been shown that this analytical optimal solution can be derived using the principle of *travelling wave decomposition* of the sound field inside an acoustic duct.

The *broadband estimation performance* of the adaptive LMS virtual microphone technique has been analysed. Numerical and experimental results have been obtained for three different physical sensor configurations, which were also used in previous research [80]. Optimal solutions for the physical sensor weights have been derived, which indicated that the estimation problem is ill-conditioned for the three and five physical sensor configurations. This explained the disagreement between the numerical and experimental results reported in the previous research [80]. It has been shown that the broadband estimation performance decreases as the distance between the physical sensors and virtual sensor inside the acoustic duct arrangement increases.

The *narrowband and broadband feedforward control performance* that is obtained at a spatially fixed virtual location inside the acoustic duct arrangement has been analysed. It has been shown that the narrowband control performance is best at the resonance frequencies of the acoustic duct arrangement. It has also been shown that the narrowband and broadband feedforward control performance decreases as the distance between the physical sensors and virtual sensor inside the acoustic duct arrangement increases. This is because the estimation performance of the adaptive LMS virtual microphone technique decreases as the distance between these sensors increases. However, the zone of quiet has effectively been moved away from the physical sensors to the virtual sensor. This resulted in increased narrowband and broadband feedforward control performance at the desired location of maximum attenuation in comparison to using a conventional sensing method, which illustrated the potential benefits of using a virtual sensing method.
Forward difference prediction weights [13] have been compared with analytical optimal solutions for the physical sensor weights for the case of single tone disturbances. It has been shown that the forward difference prediction weights approach the analytical optimal solutions at small virtual distances and low frequencies. This explains why the narrowband control performance that can in theory be obtained when using forward difference prediction techniques for the acoustic duct case decreases for larger virtual distances and higher frequencies as confirmed by previous experimental research [62, 80].
Chapter 8

Kalman filter based spatially fixed virtual sensing algorithm

8.1 Introduction

In this chapter, the Kalman filter based spatially fixed virtual sensing algorithm developed in Chapter 3 is implemented on the acoustic duct arrangement introduced in Chapter 6. This algorithm is used to compute an estimate of the virtual error signal at a virtual location that is spatially fixed inside the acoustic duct arrangement. The estimated virtual error signal is then minimised using the filtered-x LMS algorithm as described in Chapter 4. The estimation and control performance that is obtained at a spatially fixed virtual location inside the acoustic duct arrangement is analysed for narrowband and broadband disturbances. The experimental results are compared to numerical results that are computed using the modal model of an acoustic duct introduced in Chapter 6.

In Section 8.2, a schematic diagram of the acoustic duct implementation that is used in the real-time experiments is introduced. An innovations model of the acoustic duct arrangement is then estimated in a preliminary identification stage using the two-step identification procedure that has been described in Chapter 6. The identified innovations model is then used to compute the Kalman filter based spatially fixed virtual sensing algorithm. In Section 8.3, this algorithm is implemented on the acoustic duct arrangement and the real-time broadband estimation performance is analysed and compared to numerical results. In Sections 8.4 and 8.5, the narrowband and broadband feedforward control performance, respectively, that is obtained at a spatially fixed virtual location inside the acoustic duct arrangement is analysed. The estimate of the virtual error signal is minimised using the filtered-x LMS algorithm as described in Chapter 4, with the estimate of the virtual error signal computed in the real-time experiments using the Kalman filter based spatially fixed virtual sensing algorithm that has been identified in the preliminary identification stage.
8.2 Acoustic duct arrangement

Fig. 8.1 shows a schematic diagram of the implementation that is used in the real-time experiments conducted on the acoustic duct arrangement, which has been introduced in Chapter 6.

Figure 8.1: Schematic diagram of the acoustic duct implementation for the remote microphone technique, with a primary source located at \( x_p = 4.730 \) m, a control source at \( x_s = 0.100 \) m, one physical sensor located at \( x_p5 = 1.475 \) m, and one spatially fixed virtual sensor located at \( x_v = x_p5 + v \), with \( v \) the virtual distance.

The acoustic duct illustrated in Fig. 8.1 is of length \( L_x = 4.830 \) m, has a primary loudspeaker located at \( x_p = 4.730 \) m that is excited by the disturbance source signal \( s(n) \), and a control loudspeaker located at \( x_s = 0.100 \) m that is excited by the control signal \( u(n) \). A microphone located at \( x_p5 = 1.475 \) m is used as a physical sensor. The aim of the implementation illustrated in Fig. 8.1 is to minimise the current estimate \( \hat{e}_v(n|n) \) of the virtual error signal at a spatially fixed virtual sensor located at \( x_v = x_p5 + v \), with \( v \) the virtual distance between the physical and virtual sensor. The real-time estimation and control performance that is obtained at the spatially fixed virtual location is measured by a traversing microphone, which can be position controlled inside the acoustic duct arrangement as discussed in Chapter 6. The current estimate \( \hat{e}_v(n|n) \) of the virtual error signal is computed using the Kalman filter based spatially fixed virtual sensing algorithm that has been developed in Chapter 3. The filtered-x LMS algorithm is then used to
8.2 Acoustic duct arrangement

minimise the current estimate of the virtual error signal as described in Chapter 4. As illustrated in Fig. 8.1, it is assumed here that the feedforward reference signal \( x(n) \), which is needed to implement the filtered-x LMS algorithm, is equal to the disturbance source signal \( s(n) \). A sample frequency \( f_s = 1.6 \text{kHz} \) is employed in the real-time experiments.

Preliminary identification stage

To implement the Kalman filter based spatially fixed virtual sensing algorithm as described in Table 3.1 on page 123, an innovations model of the acoustic duct arrangement needs to be estimated. This is done in a preliminary identification stage in which the traversing microphone is positioned at the spatially fixed virtual location. The subspace model identification techniques [50, 124] described in Chapter 6 are then used to estimate an innovations model of the acoustic duct arrangement. An example of the preliminary identification procedure was described in Chapter 6, where a two-step approach was adopted to estimate the innovations model.

In the first step, a deterministic state-space model of the physical and virtual secondary transfer paths \( G_{pu} \) and \( G_{vu} \) has been estimated. From Eq. (6.28), the estimated state-space model is given by

\[
\begin{align*}
\hat{z}_1(n+1) &= \hat{A}_1 \hat{z}_1(n) + \hat{B}_u u(n) \\
\hat{y}_p(n) &= \hat{C}_{p1} \hat{z}_1(n) + \hat{D}_{pu} u(n) \\
\hat{y}_v(n) &= \hat{C}_{v1} \hat{z}_1(n) + \hat{D}_{vu} u(n),
\end{align*}
\]

with \( \hat{A}_1, \hat{B}_u, \hat{C}_{p1}, \hat{C}_{v1}, \hat{D}_{pu}, \) and \( \hat{D}_{vu} \) the estimated state-space matrices. In the identification example presented in Chapter 6, the traversing microphone was located at a virtual distance \( v = 0.100 \text{m} \), and a deterministic state-space model of order \( N_1 = 32 \) was estimated. This resulted in a VAF = 99.9% on the validation data-set, with the VAF value defined in Eq. (6.30). This indicates that an accurate state-space model of the physical and virtual secondary transfer paths has been obtained, as was illustrated in Fig. 6.3.

In the second step, an estimate of the stochastic part of the innovations model of the acoustic duct arrangement is computed. From Eq. (6.31), this stochastic part is given by

\[
\begin{align*}
\hat{z}_2(n+1|n) &= \hat{A}_2 \hat{z}_2(n|n-1) + \hat{K}_s [\epsilon_p(n) \epsilon_v(n)]^T \\
\hat{d}_p(n) &= \hat{C}_{p2} \hat{z}_2(n|n-1) + \epsilon_p(n) \\
\hat{d}_v(n) &= \hat{C}_{v2} \hat{z}_2(n|n-1) + \epsilon_v(n),
\end{align*}
\]

with \( \hat{A}_2, \hat{C}_{p2}, \) and \( \hat{C}_{v2} \) the estimated state-space matrices, \( \hat{K}_s \) the estimated Kalman gain matrix, and \( \epsilon_p(n) \) and \( \epsilon_v(n) \) the white innovation signals. The covariance matrix of
these innovation signals is also estimated, with the estimate defined from Eq. (3.224) as

$$\hat{R}_e = \begin{bmatrix} \hat{R}_{pe} & \hat{R}_{pve} \\ \hat{R}_{pve} & \hat{R}_{ve} \end{bmatrix}. \quad (8.3)$$

In the final experiments, a stochastic innovations model of order $N_2 = 40$ was estimated on the training data-set. For the physical primary disturbance $d_p(n)$, a VAF = 99.9% was obtained on the validation data-set, while for the virtual primary disturbance $d_v(n)$, a VAF = 99.2% was obtained on the validation data-set, with the VAF values calculated as described in Section 6.4. These VAF values indicate that an accurate innovations model of the stochastic part of the acoustic duct arrangement has been obtained, as illustrated in Fig. 6.4.

**Kalman filter based spatially fixed virtual sensing algorithm**

Using the innovations model of the acoustic duct arrangement, which has been estimated in the described two-step identification procedure, the Kalman filter based spatially fixed virtual sensing algorithm can be computed as defined in step 3 of Table 3.1. The resulting state-space model computes a current estimate $\hat{e}_v(n|n)$ of the virtual error signal given the current and previous observations of the physical error signal $e_p(n)$ and the deterministic control signal $u(n)$, and is given by

$$\begin{bmatrix} z_1(n+1) \\ z_2(n+1|n) \\ \hat{e}_v(n|n) \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ -K_{ps}\hat{C}_{p1} & \hat{A}_2 - K_{ps}\hat{C}_{p2} & \hat{B}_u \\ \hat{C}_{v1} - M_{vs}\hat{C}_{p1} & \hat{C}_{v2} - M_{vs}\hat{C}_{p2} & \hat{D}_{vu} \end{bmatrix} \begin{bmatrix} \hat{z}_1(n) \\ \hat{z}_2(n|n-1) \\ \hat{u}(n) \end{bmatrix},$$

where the estimated Kalman gain matrix $\hat{K}_{ps} \in \mathbb{R}^{N_2}$ and the estimated virtual gain matrix $\hat{M}_{vs}$ are computed as

$$\hat{K}_{ps} = \left( \hat{A}_2X_s\hat{C}^T_{p2} + \hat{K}_s \begin{bmatrix} \hat{R}_{pe} \\ \hat{R}_{pve} \end{bmatrix} \right) \left( \hat{C}_{p2}X_s\hat{C}^T_{p2} + \hat{R}_{pe} \right)^{-1}, \quad (8.5)$$

$$\hat{M}_{vs} = \left( \hat{C}_{v2}X_s\hat{C}^T_{p2} + \hat{R}_{pve}^T \right) \left( \hat{C}_{p2}X_s\hat{C}^T_{p2} + \hat{R}_{pe} \right)^{-1}, \quad (8.6)$$

with $X_s = X_s^T > 0$ the unique stabilising solution to the DARE given by

$$X_s = \hat{A}_2X_s\hat{A}_2^T - \hat{K}_{ps}\left( \hat{C}_{p2}X_s\hat{C}^T_{p2} + \hat{R}_{pe} \right)^{-1}\hat{K}_{ps}^T + \hat{K}_s\hat{R}_s\hat{K}_s^T. \quad (8.7)$$

Note that the Kalman gain and virtual gain matrices are computed in Eqs (8.5) and (8.6) using the estimated innovations model of the stochastic part of the acoustic duct arrangement defined in Eq. (8.2), which was computed in the second step of the two-step identification procedure.
8.3 Broadband estimation performance

In this section, the broadband estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm is analysed on the acoustic duct arrangement using the implementation illustrated in Fig. 8.1. In the numerical analysis, a discrete-time state-space model of the acoustic duct arrangement is calculated using the modal modelling technique described in Chapter 6. A plane wave situation is assumed by only including the first \( N_x = 25 \) modes in the direction of the duct with length \( L_x = 4.83 \) m. A quality factor of \( Q = 50 \) is used for all modes, such that all modes are assumed to have a damping ratio \( \zeta = 0.01 \). The discrete-time state-space model is computed using a sample frequency \( f_s = 1.6 \) kHz and a zero-order hold discretisation method \[35\].

The primary source located at \( x_p = 4.730 \) m is assumed to be excited by a white disturbance source signal \( s(n) \) filtered with an 8th order Butterworth filter with a pass-band between 50–500 Hz. This assumption is made because the experimental broadband estimation performance is measured while exciting the primary loudspeaker inside the acoustic duct arrangement with such a band-limited white noise signal. In the real-time experimental analysis, the Kalman filter based spatially fixed virtual sensing algorithm is computed in the preliminary identification procedure described in Section 8.2, and then implemented in real-time on the acoustic duct arrangement. The numerical and real-time experimental broadband estimation performance is analysed for sixteen virtual locations \( x_v = x_{ps} + v \), with the considered virtual distances given by

\[
\mathbf{v} = \begin{bmatrix} 0.000 & 0.010 & 0.020 & \ldots & 0.150 \end{bmatrix} \text{ m}.
\] (8.8)

The estimation performance of the implemented virtual sensing algorithm is analysed with the controller switched off. For this case, the virtual output error \( \epsilon_v(n) \) is defined as

\[
\epsilon_v(n) = d_v(n) - \hat{d}_v(n|n),
\] (8.9)

where \( d_v(n) \) is the stochastic virtual primary disturbance, and \( \hat{d}_v(n|n) \) the current estimate of the virtual primary disturbance given the current and previous values of the stochastic physical primary disturbance \( d_p(n) \), which is measured by the physical microphone located at \( x_{p5} = 1.4750 \) m. The current estimate \( \hat{d}_v(n|n) \) of the virtual primary disturbance is calculated by setting the control signal \( u(n) = 0 \) in Eq. (8.4), such that

\[
\begin{bmatrix}
\hat{z}_2(n + 1|n) \\
\hat{d}_v(n|n)
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_2 - \hat{K}_{ps} \hat{C}_{p2} \\
\hat{C}_{v2} - \hat{M}_{vs} \hat{C}_{p2}
\end{bmatrix}
\begin{bmatrix}
\hat{K}_{ps} \\
\hat{M}_{vs}
\end{bmatrix}
\begin{bmatrix}
\hat{z}_2(n|n - 1) \\
\hat{d}_p(n)
\end{bmatrix},
\] (8.10)

where the state-space matrices \( \hat{A}_2, \hat{C}_{p2}, \hat{C}_{v2} \), the Kalman gain matrix \( \hat{K}_{ps} \), and the virtual gain \( \hat{M}_{vs} \) were computed in the preliminary identification stage. Thus, only
the stochastic part of the estimation problem is considered in this section, because the deterministic part of the virtual error signal, which is the virtual secondary disturbance \( y_v(n) \), is effectively estimated in Eq. (8.4) by filtering the deterministic control signal \( u(n) \) with the estimated state-space model of the virtual secondary transfer path given by

\[
\hat{G}_{vu} \sim \begin{bmatrix} \hat{A}_1 & \hat{B}_u \\ \hat{C}_{v1} & \hat{D}_{vu} \end{bmatrix},
\]

where the state-space matrices \( \hat{A}_1, \hat{B}_u, \hat{C}_{v1}, \) and \( \hat{D}_{vu} \) were derived in the first step of the two-step identification procedure described in the previous section.

**Spectral estimation performance for \( v = 0.100 \text{ m} \)**

The broadband spectral estimation performance is now analysed for a virtual distance \( v = 0.100 \text{ m} \). The numerical and experimental results are illustrated in Fig. 8.2, where power spectral density plots of the virtual primary disturbance \( d_v(n) \), the current estimate \( \hat{d}_v(n|n) \) of the virtual primary disturbance, and the virtual output error \( \epsilon_v(n) \) defined in Eq. (8.9) are shown. The numerical results illustrated in Fig. 8.2(a) are computed from the modal model of the acoustic duct arrangement. The experimental results illustrated in Fig. 8.2(b) are generated by averaging over 50 power spectral densities, which are computed using the measurements provided by the traversing microphone that is positioned at the considered virtual distance \( v = 0.100 \text{ m} \).

![Frequency (Hz)](image)

**Figure 8.2:** Numerical and experimental estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm for \( v = 0.100 \text{ m} \), with — virtual primary disturbance \( d_v(n) \), — current estimate \( \hat{d}_v(n|n) \) of virtual primary disturbance, — virtual output error \( \epsilon_v(n) \).

In the numerical results, a normalised mean-square virtual output error of 37.0 dB is obtained for \( v = 0.100 \text{ m} \), with the normalised mean-square virtual output error...
Broadband estimation performance (NMSV) in dB defined as
\[
\text{NMSV} = -10 \log_{10} \left( \frac{E[\varepsilon_v(n)^2]}{E[d_v(n)^2]} \right) \text{ dB},
\] (8.12)
such that a larger positive value in dB indicates better estimation performance. The power spectral density plot of the numerical virtual output error shown in Fig. 8.2(a) indicates that an accurate numerical current estimate of the virtual primary disturbance \(d_v(n)\) is obtained over the entire frequency band of interest between 50–500 Hz. In the real-time experiments, a normalised mean-square virtual output error of 23.9 dB is obtained for \(v = 0.100 \text{ m}\). The power spectral density plot of the real-time virtual output error shown in Fig. 8.2(b) indicates that an accurate real-time current estimate of the virtual primary disturbance \(d_v(n)\) is obtained over the entire frequency band of interest between 50–500 Hz. The real-time experimental estimation performance is less than the numerical estimation performance, which is most likely due to the fact that measurement noise has not been included in the modal modelling of the acoustic duct arrangement. Another reason is that there are small and generally unavoidable errors in the innovations model of the acoustic duct arrangement that is estimated in the preliminary identification stage.

Overall estimation performance versus virtual distance

The broadband estimation performance is now analysed for all of the virtual distances \(v\) defined in Eq. (8.8). For each of the virtual distances considered, an innovations model of the acoustic duct arrangement is estimated using the two-step identification procedure discussed in Section 8.2, with the traversing microphone positioned at each virtual distance \(v\). The estimated innovations model is then used to compute the Kalman filter based spatially fixed virtual sensing algorithm as described in Section 8.2, which calculates a current estimate \(\hat{e}_v(n|n)\) of the virtual error signal at the considered virtual distance. The estimation performance is again analysed with the controller switched off such that the control signal \(u(n) = 0\). The numerical and experimental results are plotted in Fig. 8.3, where the normalised mean-square virtual output error defined in Eq. (8.12) is plotted against virtual distance \(v\).

The numerical results illustrated in Fig. 8.3(a) show that the normalised mean-square virtual output error decreases from 57.0 dB at a virtual distance \(v = 0.010 \text{ m}\) to 33.3 dB at a virtual distance \(v = 0.150 \text{ m}\). The real-time experimental results illustrated in Fig. 8.3(b) show that the real-time normalised mean-square virtual output error decreases from 35.6 dB at a virtual distance \(v = 0.010 \text{ m}\) to 21.8 dB at a virtual distance \(v = 0.150 \text{ m}\). The numerical and experimental results thus both indicate that the overall estimation performance decreases as the virtual distance \(v\) between the physical and virtual sensors increases. The reason for this is that the relationship between the
stochastic primary disturbances at the physical and virtual sensors becomes increasingly 
non-causal as the distance between these two sensors increases. Because only that part 
of the virtual primary disturbance $d_v(n)$ that is causally related to the physical primary 
disturbance $d_p(n)$ can be causally estimated from the current and previous observations 
of the physical primary disturbance, the estimation performance is expected to decrease 
for an increasing virtual distance $v$. The obtained real-time estimation performance 
is again less than the numerical estimation performance. This is most likely observed 
because measurement noise has not been included in the modal modelling of the 
acoustic duct arrangement and because there are small and generally unavoidable 
errors in the estimated innovations model of the acoustic duct arrangement.

8.4 Narrowband adaptive feedforward control

In this section, the narrowband adaptive feedforward control performance that is 
obtained at a spatially fixed virtual sensor located inside the acoustic duct arrangement 
illustrated in Fig. 8.1 is analysed. The aim of the implementation illustrated in this figure 
is to minimise the current estimate $\hat{e}_v(n \mid n)$ of the virtual error signal at the spatially 
fixed virtual location $x_v = x_p5 + v$, with $v$ the virtual distance and $x_p5 = 1.4750 \text{ m}$ the 
location of the physical sensor. The current estimate $\hat{e}_v(n \mid n)$ of the virtual error signal 
is computed using the Kalman filter based spatially fixed virtual sensing algorithm, 
which is computed in the preliminary identification procedure described in Section 8.2. 
The numerical and experimental broadband estimation performance of the Kalman 
filter based spatially fixed virtual sensing algorithm has been analysed in Section 8.3.
In the numerical analysis, the narrowband control performance that can in theory
be obtained at a spatially fixed virtual location is analysed by minimising the current
estimate \( \hat{e}_v(n|n) \) of the virtual error signal using the quadratic optimisation technique
described in Chapter 4. The numerical results are again computed using a modal
model of the acoustic duct arrangement without including measurement noise. In
the real-time experiments, the current estimate \( \hat{e}_v(n|n) \) of the virtual error signal is
minimised using the filtered-x LMS algorithm as described in Section 4.3. This algorithm
is implemented using a convergence coefficient \( \mu = 5 \cdot 10^{-6} \) and \( I = 2 \) control filter
coefficients. The virtual filtered-reference signal that is needed to implement the filtered-
x LMS algorithm is generated by filtering the reference signal \( x(n) \) with the estimated
state-space model \( \hat{G}_{vu} \) of the virtual secondary transfer path defined in Eq. (8.11). The
narrowband control performance is analysed for eight virtual distances \( v \) given by
\[
\mathbf{v} = \begin{bmatrix}
0.000 & 0.020 & \ldots & 0.140
\end{bmatrix} \text{ m},
\] (8.13)
and for three excitation frequencies given by 213 Hz, 249 Hz, and 284 Hz, which corre-
spond to the sixth, seventh, and eighth resonance frequencies of the acoustic duct ar-
rangement. In total, \( 3 \times 8 = 24 \) real-time narrowband control experiments are therefore
conducted. In each experiment, the primary and controlled sound pressure distributions
inside the acoustic duct arrangement are measured with the traversing microphone. The
narrowband control performance that is obtained when minimising the current
estimate \( \hat{e}_v(n|n) \) of the virtual error signal at the virtual locations for which \( v > 0 \) is thus
compared with the narrowband control performance that is obtained at these locations
when the physical error signal \( e_p(n) \) at the physical sensor located at \( v = 0.000 \text{ m} \) is
minimised.

**Primary and controlled sound pressure distributions**

In Fig. 8.4, the numerical and experimental primary and controlled sound pressure
distributions inside the acoustic duct are plotted over a 0.500 m section located between
a virtual distance \( v \) of \(-0.200 \text{ m}\) and \(0.300 \text{ m}\). In the real-time experiments, the primary
and controlled sound pressure distributions are measured at increments of \(0.020 \text{ m}\),
with the sound pressure level (SPL) at the excitation frequency of interest obtained by
averaging over 50 power spectra. To compute the real-time controlled sound pressure
distributions shown in Fig. 8.4, additional measurements are taken up to \(0.020 \text{ m}\) away
from the considered virtual location in increments of \(0.005 \text{ m}\). This is done to analyse
more accurately whether the zone of quiet is moved away from the physical sensor
located at \( x_{p5} = 1.4750 \text{ m} \) to the virtual sensor located at the spatially fixed virtual
location \( x_v = x_{p5} + v \). The black squares in Fig. 8.4 indicate the desired locations
of maximum attenuation, i.e. the intended virtual locations. The dash-dotted lines
indicate the primary sound pressure distributions, the dashed lines the controlled sound pressure distributions when minimising the physical error signal $e_p(n)$ at the physical sensor located at $v = 0.000\text{ m}$, and the solid lines of different grey scale the controlled sound pressure distributions when minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal at the considered virtual distances.

The numerical results shown in Fig. 8.4 illustrate that the zone of quiet is effectively moved away from the physical sensor located at $v = 0.000\text{ m}$ to the desired virtual location for all the considered excitation frequencies and virtual distances. The experimental results shown in Figs. 8.4(b) and 8.4(d) indicate that this is also achieved in the real-time experiments for excitation frequencies of 213 Hz and 249 Hz, with the exception that for an excitation frequency of 213 Hz and a virtual distance $v = 0.040\text{ m}$, the zone of quiet is 0.01 m to the left of the desired virtual location. A more significant difference between the numerical and experimental results is observed for an excitation frequency of 284 Hz. The real-time experimental results shown in Fig. 8.4(f) indicate that for larger virtual distances, the zone of quiet is located slightly to the left of the desired virtual location in the real-time experiments, with the zone of quiet located further to the left as the virtual distance increases. A likely reason for the observed discrepancy between the numerical and experimental results can be found by examining the real-time primary sound pressure distribution inside the acoustic duct arrangement for an excitation frequency of 284 Hz, which is indicated by the dash-dotted line in Fig. 8.4(f). It can be seen in this figure that for this frequency, the physical sensor located at $v = 0.000\text{ m}$ is positioned very close to a node in the primary sound field. This can also be observed in Fig. 6.4(b), where a power spectral density plot of the physical primary disturbance $d_p(n)$ was shown. It can be seen in this figure that the physical primary disturbance has little energy at a frequency of 284 Hz because the physical sensor is located in the vicinity of a node in the primary sound field for this frequency. When determining an innovations model of the stochastic part of the acoustic duct arrangement in the second step of the two-step identification procedure described in Section 8.2, the signal-to-noise-ratio (SNR) will thus be lower at this frequency than the SNR at the other two considered frequencies of 213 Hz and 249 Hz. As a result, the estimated innovations model will be less accurate at 284 Hz than at the other two frequencies considered. Furthermore, the node in the primary sound field shown in Fig. 8.4(f) also occurs in the secondary sound field at a frequency of 284 Hz because the location of this node is determined by the geometry of the acoustic duct and not by the location of the sources. In the first step of the two-step identification procedure described in Section 8.2, the estimated state-space model $\hat{G}_{pu}$ of the physical secondary transfer path will therefore also be less accurate at 284 Hz than at the other two considered frequencies of 213 Hz and 249 Hz. Consequently, the current estimate $\hat{e}_v(n|n)$ of the virtual error signal computed by the Kalman filter based spatially fixed virtual
8.4 Narrowband adaptive feedforward control

Figure 8.4: Numerical and experimental sound pressure distributions plotted against virtual distance \( v \) for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with \( \cdots \) primary sound pressure distribution; controlled sound pressure distribution while minimising the \( \cdots \) physical error signal \( e_p(n) \) at the physical sensor located at \( v = 0.000 \) m; \( \cdots \) current estimate \( \hat{e}_v(n|n) \) of the virtual error signal at the considered virtual distance \( v \). \( \blacksquare \) desired spatially fixed locations of the zone of quiet.
sensing algorithm, which is implemented using the state-space models estimated in the two-step identification procedure, is therefore less accurate at 284 Hz than at the other two frequencies considered. Because modelling errors and measurement noise are not included in the numerical results, the presented discussion is a likely explanation as to why the discrepancy between the numerical and experimental results is more significant for a frequency 284 Hz than for the other two frequencies.

To summarise, Fig. 8.4 shows that the numerical and experimental narrowband control performance obtained when minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal at the virtual locations for which $v > 0$ is always larger than the narrowband control performance obtained at these locations when the physical error signal $e_p(n)$ at the physical sensor located at $v = 0.000$ m is minimised. This indicates the potential benefits of employing the Kalman filter based spatially fixed virtual sensing method developed in Chapter 3 over a conventional sensing method.

**Narrowband control performance versus virtual distance**

In Fig. 8.5, the numerical and experimental narrowband attenuations that are obtained at the desired spatially fixed virtual locations indicated by the black squares in Fig. 8.4 are plotted against virtual distance.

<table>
<thead>
<tr>
<th>virtual distance (m)</th>
<th>attenuation (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>80</td>
</tr>
<tr>
<td>0.02</td>
<td>70</td>
</tr>
<tr>
<td>0.04</td>
<td>60</td>
</tr>
<tr>
<td>0.06</td>
<td>50</td>
</tr>
<tr>
<td>0.08</td>
<td>40</td>
</tr>
<tr>
<td>0.1</td>
<td>30</td>
</tr>
<tr>
<td>0.12</td>
<td>20</td>
</tr>
<tr>
<td>0.14</td>
<td>10</td>
</tr>
</tbody>
</table>

(a) Numerical results

(b) Experimental results

Figure 8.5: Numerical and experimental narrowband attenuation plotted against virtual distance $v$ for the three excitation frequencies considered, with $f = 213$ Hz $f = 249$ Hz $f = 284$ Hz.

For excitation frequencies of 213 Hz and 249 Hz, the numerical and experimental results in Fig. 8.5 indicate that the obtained narrowband attenuation decreases as the virtual distance $v$ between the physical and virtual sensors increases. The reason for this is that the estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm decreases as the virtual distance increases. As discussed in Section 8.3,
this is because the relationship between the stochastic primary disturbances at the physical and virtual sensors becomes increasingly non-causal as the distance between these two sensors increases. As a result, the current estimate $\hat{e}_v(n|n)$ of the virtual error signal becomes less accurate as the virtual distance $v$ between the physical and virtual sensors increases. This explains why the narrowband attenuation decreases in Fig. 8.5 for increasing virtual distance. However, this is not observed in the numerical results for an excitation frequency of 284 Hz shown in Fig. 8.5(a), where the narrowband attenuation is lowest at a virtual distance $v = 0.040$ m. It can be observed in Fig. 8.4(e) that for this frequency, the primary sound field indicated by the dash-dotted line has a node in the vicinity of this virtual distance. Thus, although the numerical results in Fig. 8.4(e) indicate that for an excitation frequency of 284 Hz, the numerical controlled sound pressure at the desired virtual location indicated by the black squares increases as the virtual distance increases, the numerical narrowband attenuation obtained at this frequency does not simultaneously decrease in Fig. 8.5(a) due to the node in the primary sound field. Note that for excitation frequencies of 213 Hz and 249 Hz, the numerical controlled sound pressure at the desired virtual locations indicated by the black squares in Figs 8.4(a) and 8.4(c) increases for increasing virtual distance in a similar way as for a frequency of 284 Hz. However, as can be seen in these figures, for these two frequencies there is no node in the primary sound fields at any of the considered virtual distances. As a result, the numerical narrowband attenuations in Fig. 8.5(a) decrease for 213 Hz and 249 Hz as the virtual distance increases. To summarise, even though the controlled sound pressure at the desired virtual locations increases for increasing virtual distance because the estimation performance of the virtual sensing algorithm decreases, the shape of the primary sound pressure distribution determines whether this is also observed in the narrowband attenuation.

Finally, it can be seen in Fig. 8.5 that the difference between the numerical and experimental narrowband attenuations is most significant for an excitation frequency of 284 Hz. This can again be explained by noting that for this frequency, the physical sensor located at $v = 0.000$ m is positioned very close to a node in the primary sound field, which is illustrated by the dash-dotted line in Fig. 8.4(e). As discussed previously, the innovations model of the acoustic duct arrangement that is estimated in the two-step identification procedure is therefore less accurate at this frequency than at the other two considered frequencies of 213 Hz and 249 Hz. Consequently, the current estimate $\hat{e}_v(n|n)$ of the virtual error signal computed by the Kalman filter based spatially fixed virtual sensing algorithm is less accurate at 284 Hz than at the other two considered frequencies. Because modelling errors and measurement noise are not included in the numerical results, this most likely causes the discrepancy between the numerical and experimental results shown in Fig. 8.5 to be more significant for an excitation frequency of 284 Hz than for the other two frequencies considered.
8.5 Broadband adaptive feedforward control

In this section, the broadband adaptive feedforward control performance that is obtained at a spatially fixed virtual sensor located inside the acoustic duct arrangement illustrated in Fig. 8.1 is analysed. The aim of the implementation illustrated in this figure is to minimise the current estimate $\hat{e}_v(n|n)$ of the virtual error signal at the spatially fixed virtual location $x_v = x_{p5} + v$, with $v$ the virtual distance and $x_{p5} = 1.4750$ m the location of the physical sensor. The current estimate $\hat{e}_v(n|n)$ of the virtual error signal is computed using the Kalman filter based spatially fixed virtual sensing algorithm, which is calculated in the preliminary identification procedure described in Section 8.2. The numerical and real-time experimental broadband estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm have been analysed in Section 8.3.

In the numerical analysis, the optimal broadband feedforward control performance that can in theory be obtained at a spatially fixed virtual location $x_v$ is analysed by minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal using the causal Wiener filter techniques introduced in Chapters 2 and 4. The numerical results are again computed using a modal model of the acoustic duct arrangement without including measurement noise. In the real-time experiments, the current estimate $\hat{e}_v(n|n)$ of the virtual error signal is minimised using the filtered-x LMS algorithm as described in Section 4.3. This algorithm is implemented using a convergence coefficient $\mu = 5 \cdot 10^{-6}$ and $l = 450$ control filter coefficients. The virtual filtered-reference signal that is needed to implement the filtered-x LMS algorithm is generated by filtering the reference signal $x(n)$ with the estimated state-space model $\hat{G}_{vu}$ of the virtual secondary transfer path defined in Eq. (8.11). The broadband feedforward control performance is analysed for the eight virtual distances $v$ defined in Eq. (8.13). In each real-time experiment, the primary and controlled sound pressure distributions inside the acoustic duct arrangement are measured with the traversing microphone. The broadband feedforward control performance that is obtained when minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal at the virtual locations for which $v > 0$ can then be compared with the broadband feedforward control performance that is obtained at these locations when the physical error signal $e_p(n)$ at the physical sensor located at $v = 0.000$ m is minimised.

**Broadband feedforward control performance for $v = 0.100$ m**

In this section, the broadband feedforward control performance that is obtained while minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal is compared with the broadband feedforward control performance that is obtained while minimising the true virtual error signal $e_v(n)$ for a virtual distance $v = 0.100$ m. The true virtual
8.5 Broadband adaptive feedforward control

The experimental power spectral density plots illustrated in Fig. 8.6(b) are generated by averaging 50 power spectral densities, which are computed from measurements taken by the traversing microphone positioned at the virtual location. In the numerical results shown in Fig. 8.6(a), an overall attenuation of 33.6 dB is obtained while minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal, compared to an overall attenuation of 35.0 dB obtained while minimising the true virtual error signal $e_v(n)$, which is a difference of 1.4 dB. In the experimental results shown in Fig. 8.6(b), an overall attenuation of 21.3 dB is obtained while minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal, compared to an overall attenuation of 24.0 dB obtained while minimising the true virtual error signal $e_v(n)$ directly measured by the traversing microphone positioned at the desired virtual location, which is a difference of 2.7 dB. This difference is very similar to the difference of 1.4 dB observed in the numerical results shown in Fig. 8.6(a), and is therefore predominantly determined by the broadband estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm and not by factors such as modelling errors and measurement noise. In other words, the main cause of the 2.7 dB difference in the real-time broadband
feedforward control performance is that the current estimate \( \hat{e}_v(n|n) \) of the virtual error signal is not perfect for \( v = 0.100 \text{ m} \), which has been observed in Section 8.3 where the broadband estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm was analysed.

The real-time broadband feedforward control performance obtained at the virtual location while minimising the current estimate \( \hat{e}_v(n|n) \) of virtual error signal is now compared with the real-time control performance obtained at this location while minimising the physical error signal \( e_p(n) \) at the physical sensor located at \( x_{p5} = 1.4750 \text{ m} \). The physical error signal \( e_p(n) \) is minimised using the filtered-x LMS algorithm, which is again implemented using a convergence coefficient of \( \mu = 5 \cdot 10^{-6} \) and \( I = 450 \) filter coefficients. The real-time broadband feedforward control performance that is obtained at the physical sensor is illustrated in Fig. 8.7(a), where power spectral density plots of the physical primary disturbance \( d_p(n) \) and the residual physical error signal \( e_p(n) \) measured after convergence of the filtered-x LMS algorithm are shown.

An overall attenuation of 27.5 dB is obtained at the physical sensor located at \( x_{p5} = 1.4750 \text{ m} \) while minimising the physical error signal \( e_p(n) \). Fig. 8.7(b) shows a power spectral density plot of the virtual primary disturbance \( d_v(n) \) at the virtual distance \( v = 0.100 \text{ m} \), and power spectral density plots of the residual virtual error signal \( e_v(n) \) at this virtual distance obtained while minimising either the current estimate \( \hat{e}_v(n|n) \) of the virtual error signal or the physical error signal \( e_p(n) \) at the physical sensor located at \( x_{p5} = 1.4750 \text{ m} \). Although an overall attenuation of 27.5 dB is
obtained at the physical sensor while minimising the physical error signal $e_p(n)$, the overall attenuation obtained at the desired virtual location is only 2.8 dB for this case. Thus, minimising the current estimate $\hat{e}_v(n|n)$ of the virtual error signal instead of the physical error signal $e_p(n)$, which results in an overall attenuation of 21.3 dB, increases the overall attenuation that is obtained at the virtual distance $v = 0.100$ m by 18.5 dB.

**Broadband feedforward control performance versus virtual distance**

In this section, the broadband feedforward control performance is analysed both numerically and experimentally for the eight virtual distances defined in Eq. (8.13). In the numerical analysis, the controlled sound pressure distributions inside the acoustic duct arrangement are computed using the causal Wiener filter techniques introduced in Chapters 2–4. In the real-time experiments, the primary and controlled sound pressure distributions inside the acoustic duct arrangement are measured with the traversing microphone. The measurements of the sound pressure distributions are performed similarly to the narrowband control case described in Section 8.5. The numerical and experimental results are shown in Fig. 8.8, where the primary and controlled sound pressure distributions inside the acoustic duct are plotted over a 0.500 m section located between a virtual distance of $-0.200$ m and 0.300 m. The black squares in Fig. 8.8 indicate the desired locations of maximum attenuation, i.e. the desired virtual locations.

![Numerical and experimental primary and controlled sound pressure distributions plotted against virtual distance $v$, with $- - -$ primary sound pressure distribution; controlled sound pressure distribution while minimising the $- - -$ physical error signal $e_p(n)$ at the physical sensor located at $v = 0.000$ m; $- - -$ current estimate $\hat{e}_v(n|n)$ of the virtual error signal at the virtual distance $v$. ■ desired spatially fixed locations of the zone of quiet.](image-url)
The numerical results in Fig. 8.8(a) show that the zone of quiet is moved away from the physical sensor located at \( v = 0.000 \) m to the desired virtual location for all of the virtual distances considered. It can also be seen that the overall numerical attenuation obtained at the desired virtual location decreases as the virtual distance increases. This is because the numerical estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm decreases for increasing virtual distance, which was observed in the numerical results shown in Fig. 8.3(a). The same trend is observed in the experimental results illustrated in Fig. 8.8(b), where the zone of quiet is also moved away from the physical sensor located at \( v = 0.000 \) m to the desired virtual location for all of the virtual distances considered, and where the overall attenuation also decreases for increasing virtual distance. Again, this is because the real-time estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm decreases for increasing virtual distance, which was observed in the experimental results illustrated in Fig. 8.3(b).

In Fig. 8.9, the numerical and experimental overall attenuations obtained at each of the virtual distances while minimising a current estimate \( \hat{e}_v(n|n) \) of the virtual error signal are compared with the overall attenuations obtained while minimising the physical error signal \( e_p(n) \) at the physical sensor located at \( v = 0.000 \) m.

![Figure 8.9: Numerical and experimental overall attenuation obtained at the virtual distance \( v \) while — minimising the physical error signal \( e_p(n) \) at the physical sensor located at \( v = 0.000 \) m; — minimising a current estimate \( \hat{e}_v(n|n) \) of the virtual error signal at the virtual distance \( v \).](image)

The numerical and experimental results in Fig. 8.9 show that for each of the virtual distances considered, the overall attenuation is always larger when minimising a current estimate \( \hat{e}_v(n|n) \) of the virtual error signal instead of the physical error signal \( e_p(n) \). This illustrates the potential benefits of employing the developed Kalman filter
based spatially fixed virtual sensing method over a conventional sensing method. The experimental results in Fig. 8.9(b) can also be compared to the experimental results shown in Fig. 7.21(b), where the overall attenuation that is obtained while minimising an estimate \( \hat{e}_v(n) \) of the virtual error signal computed by the adaptive LMS virtual microphone technique was plotted against virtual distance. This comparison indicates that using the Kalman filter based spatially fixed virtual sensing algorithm to compute an estimate of the virtual error signal instead of the adaptive LMS virtual microphone technique increases the overall attenuation that is obtained at the considered virtual distances, with an increased overall attenuation of almost 8 dB for a virtual distance \( v = 0.140 \text{ m} \).

### 8.6 Conclusion

The Kalman filter based spatially fixed virtual sensing algorithm developed in Chapter 3 has successfully been implemented on the acoustic duct arrangement introduced in Chapter 6. This algorithm has been used to compute an estimate of the virtual error signal at a virtual location that is spatially fixed inside the acoustic duct arrangement. It has been shown that the estimation performance decreases as the distance between the physical and virtual sensors located inside the acoustic duct arrangement increases. The reason for this is that the relationship between the stochastic primary disturbances at the physical and virtual sensors becomes increasingly non-causal as the distance between these two sensors increases. Because only that part of the virtual primary disturbance that is causally related to the physical primary disturbance can be causally estimated from the current and previous observations of the physical primary disturbance, the estimation performance is expected to decrease as the distance between the physical and virtual sensors located inside the acoustic duct arrangement increases.

Real-time narrowband and broadband feedforward control experiments, in which the estimate of the virtual error signal was minimised using the filtered-x LMS algorithm as described in Chapter 4, have successfully been conducted on the acoustic duct arrangement. The zone of quiet has effectively been moved away from the physical sensor to the spatially fixed virtual sensor located at the desired location of maximum attenuation. It has been shown that the narrowband and broadband feedforward control performance obtained at this location generally decreases as the distance between the physical and virtual sensors increases. This is because the estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm decreases as the distance between the physical and virtual sensors increases.
Chapter 9

Narrowband active noise control at a moving physical sensor

9.1 Introduction

The real-time experimental results presented in Chapters 7 and 8 illustrated that the implemented spatially fixed virtual sensing algorithms can be used to successfully move the zone of quiet away from a physical sensor to a virtual sensor that is spatially fixed within the sound field. These algorithms can therefore be used to improve the performance of a local active noise control system provided that the movement of the observer’s ear occurs around the intended spatially fixed virtual location and is relatively small in comparison to the size of the created zone of quiet. When this is not the case, the observer will generally experience a significant decrease in the performance when moving away from the spatially fixed virtual location for which the virtual sensing algorithm was originally designed. The performance of a local active noise control system can thus further be improved by creating a moving zone of quiet that tracks the observer’s ear.

In this chapter, the aim is to create a moving zone of quiet inside the acoustic duct arrangement that tracks the desired location of maximum attenuation for the case of narrowband disturbances. It is assumed that the error signal at this moving location, which is still referred to as the moving virtual location consistent with the definition introduced in Chapter 2, is directly measured during real-time control by a moving physical sensor. This error signal is still referred to as the virtual error signal even though it is directly measured during real-time control. The case of a moving virtual sensor is considered in the real-time experiments presented in Chapter 10. For this case, an estimate of the virtual error signal at the desired moving location of maximum attenuation is computed because the true virtual error signal is not directly measured during real-time control as is the case in this chapter.
As discussed in Chapter 5, the virtual primary disturbance that needs to be attenuated is a non-stationary signal when the moving physical sensor tracks a virtual location that is moving through the sound field inside the acoustic duct arrangement. An adaptive feedforward control approach is therefore adopted to be able to track the changes in the statistical properties of the virtual primary disturbance and adjust the controller accordingly. In this chapter, the tracking performance of the filtered-x LMS algorithm, the normalised filtered-x LMS algorithm and the filtered-x RLS algorithm is compared for narrowband disturbances inside the acoustic duct arrangement. These adaptive feedforward control algorithms are implemented on the acoustic duct arrangement using the implementation developed in Chapter 5 and illustrated in Fig. 5.3. The ability of these algorithms to track the non-stationarities that occur is thus compared while assuming that the true virtual error signal at the moving virtual location is measured directly during real-time control by a moving physical sensor. The comparison will show which of the considered adaptive algorithms provides the best tracking performance for the case of narrowband disturbances inside the acoustic duct arrangement. The adaptive algorithm that provides the best tracking performance will then be used in the real-time experiments presented in Chapter 10, where an estimate of the virtual error signal at the desired location of maximum attenuation is adaptively minimised instead of the true virtual error signal as is the case in this chapter.

In Section 9.2, a schematic diagram of the adaptive feedforward control implementation that is used in the presented real-time narrowband control experiments is introduced. In Sections 9.3, 9.4 and 9.5, the influence of the convergence coefficient $\mu$, the convergence coefficient $\alpha$, and the forgetting factor $\lambda$ on the tracking performance of the filtered-x LMS algorithm, normalised filtered-x LMS algorithm, and filtered-x RLS algorithm, respectively, is analysed. In Section 9.6, the tracking performance of these adaptive algorithms is compared for various speeds of the moving physical sensor and various spatial characteristics of the narrowband sound field inside the acoustic duct arrangement.

9.2 Acoustic duct arrangement

Fig. 9.1 shows a schematic diagram of the implementation that is used in the real-time experiments conducted on the acoustic duct arrangement introduced in Chapter 6. The acoustic duct has length $L_x = 4.830$ m, a primary loudspeaker located at $x_p = 4.730$ m that is excited by a narrowband disturbance source signal $s(n)$, and a control loudspeaker located at $x_c = 0.100$ m that is excited by the control signal $u(n)$. The aim of the implementation illustrated in Fig. 9.1 is to minimise the true virtual error signal $e_v(n)$ that is measured directly during real-time control by a moving physical sensor. This moving physical sensor is position controlled to track the moving virtual location...
9.2 Acoustic duct arrangement

Figure 9.1: Schematic diagram of the acoustic duct arrangement of length $L_x = 4.830 \text{ m}$, with a primary source located at $x_p = 4.730 \text{ m}$, a control source at $x_s = 0.100 \text{ m}$, and a moving physical sensor that tracks the moving virtual location $x_v(n) = x_{p5} + v(n)$, with $x_{p5} = 1.4750 \text{ m}$ and $v(n)$ the time-varying virtual distance.

$x_v(n) = x_{p5} + v(n)$, with $v(n)$ the time-varying virtual distance and $x_{p5} = 1.4750 \text{ m}$ as illustrated in Fig. 9.1. The aim of the real-time narrowband control experiments is therefore to create a moving zone of quiet at a moving physical sensor that tracks the desired location of maximum attenuation. In the real-time experiments, the traversing microphone that is located inside the acoustic duct arrangement is used as a moving physical sensor. As discussed in Chapter 6, this traversing microphone can be position controlled inside the acoustic duct arrangement to track the moving virtual location $x_v(n)$. A sample frequency $f_s = 1.6 \text{ kHz}$ is employed in the real-time experiments.

9.2.1 Adaptive feedforward control implementation

The virtual primary disturbance $d_v(n)$ that needs to be attenuated at the moving physical sensor illustrated in Fig. 9.1 is a non-stationary signal due to the movement of the virtual location $x_v(n)$ through the primary sound field inside the acoustic duct arrangement. As discussed in Chapter 5, when the primary disturbance is non-stationary, the common approach in active noise control is to use an adaptive control algorithm [25, 68]. In Chapter 2, the adaptive feedforward control algorithms known as the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms were introduced. These algorithms can be implemented as described in Chapter 2 provided that the virtual error signal $e_v(n)$ is directly measured during real-time control, which is the case in the implementation illustrated in Fig. 9.1, and the virtual location is spatially fixed.
within the sound field, which is not the case in the implementation illustrated in Fig. 9.1.
In Chapter 5, these adaptive feedforward control algorithms were therefore modified to
allow for a virtual location that is \textit{moving} through the sound field rather than being
\textit{spatially fixed}. The adaptive feedforward control implementation developed in Chapter 5
and illustrated in Fig. 5.3 is therefore implemented on the acoustic duct arrangement as
illustrated in Fig. 9.1.

\section*{Generating an estimate of the virtual filtered-reference signal}

In the adaptive feedforward control implementation illustrated in Fig. 9.1, an estimate
\( \hat{r}_v(n) \) of the virtual filtered-reference signal for the moving virtual location \( x_v(n) \) is
computed using spatial interpolation between the virtual filtered-reference signals \( \hat{x}_v(n) \)
for a number of spatially fixed virtual locations \( x_v \in \mathbb{R}^{M_v} \), with

\[
\hat{x}_v(n) = \begin{bmatrix} \hat{x}_{v1}(n) & \hat{x}_{v2}(n) & \cdots & \hat{x}_{vM_v}(n) \end{bmatrix}^T,
\]  

(9.1)

and

\[
x_v = \begin{bmatrix} x_{v1} & x_{v2} & \cdots & x_{vM_v} \end{bmatrix}^T.
\]  

(9.2)

The spatially fixed virtual locations \( x_v \) defined in Eq. (9.2) are suitably positioned
throughout the \textit{target zone} illustrated in Fig. 9.1, which defines the region of the acoustic
duct through which the physical sensor is moving. The virtual filtered-reference signals
\( \hat{r}_v(n) \) defined in Eq. (9.1) are generated by filtering the feedforward reference signal
\( x(n) \) with an estimate \( \hat{G}_{vu} \in \mathcal{RH}_{\infty}^{M_v \times 1} \) of the virtual secondary transfer paths between
the control source and the spatially fixed virtual locations \( x_v \). A state-space model of
these virtual secondary transfer paths is estimated in a preliminary identification stage
that will be described in Section 9.2.3, with the state-space model given by

\[
\hat{G}_{vu} \sim \begin{bmatrix} \begin{bmatrix} \hat{A}_v & \hat{B}_{vu} \\ \hat{C}_{vu} & \hat{D}_{vu} \end{bmatrix} \end{bmatrix}.
\]  

(9.3)

Using the state-space model of the virtual secondary transfer paths \( \hat{G}_{vu} \) defined in
Eq. (9.3), the virtual filtered-reference signals \( \hat{x}_v(n) \) are generated as

\[
\begin{align*}
\hat{x}(n+1) &= \hat{A}_v \hat{x}(n) + \hat{B}_u x(n) \\
\hat{r}_v(n) &= \hat{C}_{vu} \hat{x}(n) + \hat{D}_{vu} x(n).
\end{align*}
\]  

(9.4)

An estimate \( \hat{r}_v(n) \) of the virtual filtered-reference for the moving virtual location \( x_v(n) \)
is computed using \textit{linear spatial interpolation} between the virtual filtered-reference signals
\( \hat{x}_v(n) \) generated as defined in Eq. (9.4). The estimate \( \hat{r}_v(n) \) is therefore computed as

\[
\hat{r}_v(n) = \frac{x_v(n+1) - \hat{x}_v(n)}{\hat{x}_v(n+1) - \hat{x}_v(n)} \hat{r}_v(n) + \frac{\hat{x}_v(n) - \hat{x}_v(n)}{\hat{x}_v(n+1) - \hat{x}_v(n)} \hat{r}_v(n+1),
\]  

(9.5)

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where $\bar{x}_{vi}$ and $\bar{x}_{v(i+1)}$ are the two spatially fixed virtual locations defined in Eq. (9.2) that are closest to the moving virtual location $x_v(n)$, such that
\[
\bar{x}_{vi} \leq x_v(n) \leq \bar{x}_{v(i+1)},
\]
and with $\hat{r}_v(n)$ and $\hat{r}_v(n+1)$ the virtual filtered-reference signals for the spatially fixed virtual locations $\bar{x}_{vi}$ and $\bar{x}_{v(i+1)}$, which have been defined in Eq. (9.1). Note that to compute the spatial interpolation in Eq. (9.5), the moving virtual location $x_v(n)$ which defines the desired location of the zone of quiet is assumed known.

**Update equations for adaptive feedforward control algorithms**

The three adaptive algorithms that are investigated for the implementation illustrated in Fig. 9.1 are the filtered-$x$ LMS algorithm, the normalised filtered-$x$ LMS algorithm, and the filtered-$x$ RLS algorithm. These adaptive feedforward control algorithms were introduced in Chapter 2 and modified in Chapter 5 to account for a virtual location that is moving through the sound field. These algorithms are implemented here for the case of narrowband disturbances using $I = 2$ control filter coefficients $w(n) \in \mathbb{R}^2$, with
\[
w(n) = \begin{bmatrix} w_0(n) & w_1(n) \end{bmatrix}^T.
\]

The control signal $u(n)$ is then computed as
\[
u(n) = w_0(n)x(n) + w_1(n)x(n-1),
\]
with $x(n)$ the feedforward reference signal. When the filtered-$x$ LMS algorithm introduced in Chapter 2 is used, the control filter coefficients that are used in Eq. (9.8) to compute the control signal are updated as
\[
w(n+1) = w(n) - \mu \hat{r}_v(n)e_v(n),
\]
with $\mu$ the convergence coefficient, $e_v(n)$ the true virtual error signal that is directly measured by the moving physical sensor that tracks the moving virtual location $x_v(n)$, and $\hat{r}_v(n)$ defined as
\[
\hat{r}_v(n) = \begin{bmatrix} \hat{r}_v(n) & \hat{r}_v(n-1) \end{bmatrix}^T,
\]
with $\hat{r}_v(n)$ the estimate of the virtual filtered-reference signal for the moving virtual location $x_v(n)$ computed as defined in Eq. (9.5). When the normalised filtered-$x$ LMS algorithm introduced in Chapter 2 is used, the control filter coefficients $w(n)$ are updated as
\[
w(n+1) = w(n) - \mu(n)\hat{r}_v(n)e_v(n),
\]
with $\mu(n)$ the normalised convergence coefficient defined as

$$
\mu(n) = \frac{\alpha}{\hat{r}_v(n)^T \hat{r}_v(n)} + \epsilon,
$$

(9.12)

with $\alpha$ the convergence coefficient, and $\epsilon$ a small positive regularisation term. When the filtered-x RLS algorithm introduced in Chapter 2 is used, the control filter coefficients $w(n)$ are updated as

$$
w(n + 1) = w(n) - k(n) e_v(n),
$$

(9.13)

with $k(n) \in \mathbb{R}^2$ the gain matrix that is recursively computed as

$$
k(n) = \frac{\lambda^{-1} P(n) \hat{r}_v(n)}{1 + \lambda^{-1} \hat{r}_v(n)^T P(n) \hat{r}_v(n)},
$$

(9.14)

$$
P(n + 1) = \lambda^{-1} P(n) - \lambda^{-1} k(n) \hat{r}_v(n)^T P(n),
$$

(9.15)

where $\lambda$ is the forgetting factor, and $P(n) \in \mathbb{R}^{2 \times 2}$ the inverse correlation matrix initialised as $P(0) = \delta I$, with $\delta$ a regularisation parameter. The factor $1 - \lambda$ plays a similar role as the convergence coefficient $\mu$ in the filtered-x LMS, with the forgetting factor generally chosen as $0.9 \leq \lambda \leq 1 [51]$. Note that in the update equations for the filtered-x LMS and normalised filtered-x LMS algorithms defined in Eqs (9.9) and (9.11), the gain matrix $k(n)$ in Eq. (9.13) is effectively given by

$$
k_{LMS}(n) = \mu \hat{r}_v(n),
$$

(9.16)

$$
k_{nLMS}(n) = \frac{\alpha \hat{r}_v(n)}{\hat{r}_v(n)^T \hat{r}_v(n)} + \epsilon.
$$

(9.17)

The difference between the considered adaptive algorithms is thus the way in which the gain matrix is computed.

Tracking of non-stationarities

As discussed previously, the adaptive algorithms defined in Eqs (9.9), (9.11) and (9.13) are used in the implementation illustrated in Fig. 9.1 to provide the required tracking of the non-stationarities in the virtual primary disturbance $d_v(n)$ that is directly measured by the moving physical sensor during real-time control. Note that in Fig. 9.1, the estimate $\hat{r}_v(n)$ of the virtual filtered-reference signal for the moving virtual location $x_v(n)$ is also non-stationary due to the movement of the physical sensor through the sound field inside the acoustic duct arrangement. To successfully create a moving zone of quiet at the physical sensor, the adaptive algorithms considered thus have to adapt the control filter coefficients $w(n)$ such that tracking of the non-stationarities in both the virtual primary disturbance $d_v(n)$ and the estimate $\hat{r}_v(n)$ of the virtual filtered-reference signal for the moving virtual location $x_v(n)$ is obtained. The amount
and speed of tracking needed is dependent on both the temporal rate of change $\dot{x}_v(n)$ of the moving virtual location, i.e. the speed of the moving physical sensor, and the spatial rate of change of the relative magnitude and phase between the primary and secondary sound fields inside the acoustic duct arrangement over the target zone. This spatial rate of change determines how much the control filter coefficients $w(n)$ need to be adjusted over the region through which the physical sensor is moving. This information, together with the speed of the moving physical sensor, determines the amount and speed of tracking that is needed to successfully create a moving zone of quiet that tracks the desired location of maximum attenuation inside the acoustic duct arrangement.

As stated by Haykin [51], the tracking details of a time-variant dynamic system such as the acoustic duct implementation considered here are very problem specific. General statements on the tracking behaviour of the adaptive feedforward control implementation illustrated in Fig. 9.1 can therefore not easily be made and system-specific behaviour must be investigated explicitly. The tracking performance of the adaptive algorithms considered is therefore analysed and compared for various speeds of movement of the physical sensor and various spatial characteristics of the narrowband sound field inside the acoustic duct arrangement.

The convergence rate and the tracking capability of an adaptive algorithm are generally two different properties [51]. Whereas convergence is a transient phenomenon, tracking is a steady-state phenomenon. For an adaptive algorithm to exercise its tracking capability, it must therefore first transition from the transient mode to the steady-state mode [51]. In the real-time experiments presented here, the narrowband control performance that is obtained at the moving physical sensor is therefore measured after the adaptive algorithms have passed from the transient mode to the steady-state mode. Thus, it is the tracking capability of the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms that is investigated and compared here for narrowband disturbances at a moving physical sensor located inside the acoustic duct arrangement.

9.2.2 Speed of physical sensor and spatial characteristics of sound field

As discussed previously, the amount and speed of tracking needed is dependent on both the temporal rate of change $\dot{x}_v(n)$ of the moving virtual location, i.e. the speed of the moving physical sensor, and the spatial rate of change of the relative magnitude and phase between the primary and secondary sound fields inside the acoustic duct arrangement over the target zone.
Speed of physical sensor

In the presented experiments, the expression governing the desired location $x_v(n) = 1.4750 + v(n)$ of the moving physical sensor is given by

$$v(n) = 0.070 + 0.050 \sin\left(\frac{2\pi n}{T_v f_s}\right),$$

(9.18)

where $T_v$ is the period of the sinusoidally time-varying virtual distance $v(n)$. The moving physical sensor is thus tracking a moving virtual location $x_v(n)$ that changes sinusoidally between a virtual distance bounded by 0.020 m and 0.120 m. The narrowband control performance that is obtained at the moving physical sensor is analysed for three excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. These frequencies correspond to the sixth, seventh, and eighth resonance frequencies of the acoustic duct. For these excitation frequencies, the narrowband control performance at the moving physical sensor is measured for three different values of the period $T_v$ given by $T_v = 10$ s, $T_v = 5$ s, and $T_v = 2.5$ s. From Eq. (9.18), the maximum amplitude of the sinusoidally time-varying velocity of the moving physical sensor is thus given by $5 \text{ cm} \times \frac{2\pi}{2.5} \approx 12.6 \text{ cm/s}$. This is considered to be representative of the likely motion of an observer’s head in the intended applications. In total, nine real-time narrowband control experiments are thus conducted for each of the adaptive control algorithms considered in order to measure and compare the tracking performance for various speeds of the moving physical sensor, and various spatial characteristics of the sound field through which the physical sensor is moving.

Spatial characteristics of sound field

The measured spatial rate of change of the primary and secondary sound fields, and the relative spatial rate of change between these two sound fields over the target zone inside the acoustic duct have been plotted in Fig. 9.2 for the excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. In this figure, the relative magnitude and phase between the primary and secondary sound fields have been normalised against the relative magnitude and phase at a virtual distance $v = 0.000$ m. For an excitation frequency of 213 Hz, the relative magnitude and phase change by about 0.6 dB and 7.3° over the target zone, respectively. For an excitation frequency of 249 Hz, the relative magnitude and phase change by about 2.0 dB and 20.8° over the target zone, respectively. For an excitation frequency of 284 Hz, the relative magnitude and phase change relatively quickly at the smaller virtual distances $v < 0.060$ m, and by about 9.8 dB and 180°, respectively. As illustrated in Fig. 9.2(c), this relatively fast change is caused by a node in the sound field inside the acoustic duct. This means that for this frequency, the speed and the amount of tracking needed will be relatively large when the physical sensor moves through the node in the sound field.
Figure 9.2: Measured spatial rate of change of primary and secondary sound fields over the target zone inside the acoustic duct for the excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. — primary sound field — secondary sound field — relative magnitude and phase between primary and secondary sound fields.
9.2.3 Preliminary identification stage

To implement the adaptive control algorithms defined in Eqs (9.9), (9.11) and (9.13) as illustrated in Fig. 9.1, an estimate $\hat{G}_{vu} \in RH_{\infty}^{M_v \times 1}$ of the virtual secondary transfer paths between the control source and the spatially fixed virtual locations $\bar{x}_v$ defined in Eq. (9.2) needs to be computed. This is done in a preliminary identification stage in which the control loudspeaker is excited by a band-limited white noise signal $u(n)$ in the frequency range of 50–500 Hz. The traversing microphone is then sequentially positioned at the spatially fixed virtual locations $\bar{x}_v$ to measure an input/output data-set given by

$$\left\{u(n), \bar{y}_v(n)\right\}_{n=1}^{32000},$$

with $\bar{y}_v(n) \in \mathbb{R}^{M_v}$ the virtual secondary disturbances measured at the spatially fixed virtual locations $\bar{x}_v$. The recorded data-set is divided into a training data-set and a validation data-set each 16,000 samples long. The subspace model identification techniques [50, 124] described in Chapter 6 are then used to estimate a deterministic state-space model $\hat{G}_{vu}$ on the training data-set, with the resulting state-space model defined in Eq. (9.3). In the final implementation, a single input $M_v = 16$ output deterministic state-space model of order $N_1 = 32$ was estimated, with the spatially fixed virtual locations $\bar{x}_v = [x_v \ldots x_v]^T$. This resulted in a VAF = 99.9 % on the validation data-set, with the VAF value defined in Section 6.4, which indicates that a very accurate state-space model $\hat{G}_{vu}$ of the virtual secondary transfer paths between the control source and the spatially fixed virtual locations $\bar{x}_v$ has been obtained. Note that the spatially fixed virtual locations defined in Eq. (9.20) are positioned throughout the target zone illustrated in Fig. 9.1 at increments of 0.01 m.

9.3 Filtered-x LMS algorithm

In this section, the influence of the convergence coefficient $\mu$ on the real-time tracking performance of the filtered-x LMS algorithm defined in Eq. (9.9) is analysed on the acoustic duct arrangement. The physical sensor is moving sinusoidally as defined in Eq. (9.18) with the period of the sinusoidally time-varying virtual distance $v(n)$ given by $T_v = 5$ s. The real-time narrowband control performance that is obtained at the moving physical sensor is measured for the three considered excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. The results of the real-time experiments are illustrated in Fig. 9.3, where the narrowband attenuation and the control filter coefficients $w(n)$ computed as defined in Eq. (9.9) are plotted against time for each of the three excitation frequencies considered and for three convergence coefficients given by $\mu = 5 \cdot 10^{-3}$.
\[ \mu = 1 \cdot 10^{-4}, \text{ and } \mu = 1 \cdot 10^{-6}. \] The largest value for the convergence coefficient \( \mu \) is determined in an iterative stage in which the tracking performance of the filtered-x LMS algorithm is optimised. The real-time narrowband control results illustrated in Fig. 9.3 are generated by averaging the results of 30 data-sets each 10 s long, which are measured with the traversing microphone that is position controlled to track the moving virtual distance \( v(n) \). Furthermore, the average narrowband attenuations in dB are low-pass filtered in order to prevent noisy plots.

The results in Figs 9.3(a) and 9.3(c) show that for the largest convergence coefficient \( \mu = 5 \cdot 10^{-3} \) and excitation frequencies of 213 Hz and 249 Hz, a narrowband attenuation of over 40 dB is obtained at all times. For the lowest convergence coefficient \( \mu = 1 \cdot 10^{-6} \), the results in Figs 9.3(a) and 9.3(c) indicate that the obtained narrowband attenuations are much lower. The reason for this is illustrated in Figs 9.3(b) and 9.3(d), where it can be seen that for the lowest convergence coefficient \( \mu = 1 \cdot 10^{-6} \), the control filter coefficients \( w(n) \) are relatively constant over time. This is in contrast to the results for the largest convergence coefficient for which the control filter coefficients \( w(n) \) are time-varying as shown in Figs 9.3(b) and 9.3(d). This indicates that for the largest convergence coefficient \( \mu = 5 \cdot 10^{-3} \) and excitation frequencies of 213 Hz and 249 Hz, the filtered-x LMS algorithm is able to track the non-stationarities in the virtual primary disturbance \( d_v(n) \) at the moving physical sensor and the estimate \( \hat{r}_v(n) \) of the virtual filtered-reference signal for the moving virtual location \( x_v(n) \), which is not the case for the lowest convergence coefficient \( \mu = 1 \cdot 10^{-6} \). In other words, for a convergence coefficient \( \mu = 5 \cdot 10^{-3} \) and excitation frequencies of 213 Hz and 249 Hz, the filtered-x LMS algorithm is able to adjust the control filter coefficients \( w(n) \) fast enough to account for the variations in the relative magnitude and phase between the narrowband primary and secondary sound fields, which are illustrated in Figs 9.2(a) and 9.2(b), that occur as the physical sensor is moving through the sound field inside the acoustic duct arrangement. Note that for excitation frequencies of 213 Hz and 249 Hz, the narrowband attenuations for the lower convergence coefficients \( \mu = 1 \cdot 10^{-4} \) and \( \mu = 1 \cdot 10^{-6} \) are highest in Figs 9.3(a) and 9.3(c) when the control filter coefficients for these convergence coefficients intersect the control filter coefficients for \( \mu = 5 \cdot 10^{-3} \) in Figs 9.3(b) and 9.3(d).

For the largest convergence coefficient \( \mu = 5 \cdot 10^{-3} \) and excitation frequencies of 213 Hz and 249 Hz, the narrowband attenuations plotted in Figs 9.3(a) and 9.3(c) are highest in the vicinity of \( t = 0.0 \text{s}, 2.5 \text{s}, 5.0 \text{s}, 7.5 \text{s} \) and 10.0 s. This is because at these times the speed of the moving physical sensor \( \dot{x}_v(n) = 0 \text{m/s} \), which can be derived from the bottom part of the subfigures in the left column of Fig. 9.3, where the time-varying virtual distance \( v(n) \) defined in Eq. (9.18) is plotted against time. At and around these times, the temporal rate of change of the relative magnitude and phase between the narrowband primary and secondary sound fields, which can be derived
Figure 9.3: Influence of the convergence coefficient $\mu$ on the tracking performance of the filtered-x LMS algorithm for narrowband control at a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with $\mu = 5 \cdot 10^{-3}$, $\mu = 1 \cdot 10^{-4}$, $\mu = 1 \cdot 10^{-6}$. 
from Figs 9.2(a) and 9.2(b), is at its lowest. The tracking that is required from the filtered-x LMS algorithm to obtain good control performance is thus less demanding at these times. This explains why for the largest convergence coefficient $\mu = 5 \cdot 10^{-3}$, the narrowband attenuations in Figs 9.3(a) and 9.3(c) are highest when the moving physical sensor is slowing down to zero velocity at $t = 0.0 \text{s}, 2.5 \text{s}, 5.0 \text{s}, 7.5 \text{s}$ and $10.0 \text{s}$. Note that the presented discussion also explains why for a convergence coefficient $\mu = 5 \cdot 10^{-3}$, the narrowband attenuations in Figs 9.3(a) and 9.3(c) are smallest when the moving physical sensor reaches its maximum velocity.

The results in Fig. 9.3(e) show that for the lowest convergence coefficient $\mu = 1 \cdot 10^{-6}$ and an excitation frequency of 284 Hz, a narrowband attenuation of under 5 dB is obtained at all times. The reason for this low narrowband control performance is illustrated in Fig. 9.3(f), where it can be seen that for the low convergence coefficient $\mu = 1 \cdot 10^{-6}$, the control filter coefficients $w(n)$ are relatively constant over time and tracking is thus not achieved. For the largest convergence coefficient $\mu = 5 \cdot 10^{-3}$ and an excitation frequency of 284 Hz, the results in Fig. 9.3(e) indicate that the narrowband attenuation increases from about 5 dB around a virtual distance $v = 0.020 \text{m}$ to a maximum of 43 dB around a virtual distance $v = 0.120 \text{m}$. At these virtual distances, the moving physical sensor slows down to $\dot{x}_v(n) = 0 \text{ m/s}$. However, as the moving physical sensor slows down to reach zero velocity at $v = 0.020 \text{m}$ for an excitation frequency of 284 Hz, it moves through a node in the sound field as can be seen in Fig. 9.2(c). For this frequency, the relative magnitude and phase between the narrowband primary and secondary sound fields thus changes rapidly as the moving physical sensor passes through the node in the sound field. The tracking that is required from the filtered-x LMS algorithm to obtain good control performance is thus more demanding when the moving physical sensor approaches a virtual distance $v = 0.020 \text{m}$ than when it approaches $v = 0.120 \text{m}$. In other words, for a convergence coefficient $\mu = 5 \cdot 10^{-3}$ and an excitation frequency of 284 Hz, the filtered-x LMS algorithm is not able to adjust the control filter coefficients $w(n)$ fast enough to account for the rapid changes in the relative magnitude and phase between the narrowband primary and secondary sound fields, which can be derived from Fig. 9.2(c), that occur when the physical sensor is moving through the node in the sound field as it approaches $v = 0.020 \text{m}$. This explains why for a convergence coefficient $\mu = 5 \cdot 10^{-3}$, the narrowband attenuation is largest in Fig. 9.3(e) when the moving physical sensor is slowing down to zero velocity at a virtual distance $v = 0.120 \text{m}$ and smallest when it approaches a virtual distance $v = 0.020 \text{m}$.

9.4 Normalised filtered-x LMS algorithm

In this section, the influence of the convergence coefficient $\alpha$ on the real-time tracking performance of the normalised filtered-x LMS algorithm defined in Eq. (9.11) is analysed.
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on the acoustic duct arrangement. The real-time narrowband control results presented in this section are generated similarly to the results presented in Section 9.3, with the physical sensor again moving sinusoidally as defined in Eq. (9.18) with period \( T_o = 5 \) s. The results of the real-time experiments are illustrated in Fig. 9.4, where the narrowband attenuation and the control filter coefficients \( w(n) \) are plotted against time for each of the three excitation frequencies considered and for three convergence coefficients given by \( \alpha = 1 \cdot 10^{-1}, \alpha = 5 \cdot 10^{-5}, \) and \( \alpha = 1 \cdot 10^{-4} \). The largest value for the convergence coefficient \( \alpha \) is determined in an iterative stage in which the tracking performance of the normalised filtered-x LMS algorithm is optimised.

The results in Figs 9.4(a) and 9.4(c) show that for the largest convergence coefficient \( \alpha = 1 \cdot 10^{-1} \) and excitation frequencies of 213 Hz and 249 Hz, a narrowband attenuation of over 40 dB is obtained at all times. For the lowest convergence coefficient \( \alpha = 1 \cdot 10^{-4} \), the results in Figs 9.4(a) and 9.4(c) indicate that the obtained narrowband attenuation is much lower. As observed in the previous section, this is because the control filter coefficients \( w(n) \) are relatively constant over time for the low convergence coefficient \( \alpha = 1 \cdot 10^{-4} \), as illustrated in Figs 9.4(b) and 9.4(d), indicating that tracking is not achieved. For a convergence coefficient \( \alpha = 1 \cdot 10^{-1} \) and excitation frequencies of 213 Hz and 249 Hz, the normalised filtered-x LMS algorithm is thus able to adjust the control filter coefficients \( w(n) \) fast enough to account for the variations in the relative magnitude and phase between the narrowband primary and secondary sound fields that occur when the physical sensor is moving through the sound field inside the acoustic duct arrangement. Note that for excitation frequencies of 213 Hz and 249 Hz, the narrowband attenuations for the two lower convergence coefficients are highest in Figs 9.4(a) and 9.4(c) when the control filter coefficients \( w(n) \) for these convergence coefficients intersect the control filter coefficients for \( \alpha = 1 \cdot 10^{-1} \) in Figs 9.4(b) and 9.4(d), which was also observed in the previous section.

For a convergence coefficient \( \alpha = 1 \cdot 10^{-1} \) and an excitation frequency of 213 Hz, the narrowband attenuation plotted in Fig. 9.4(a) is relatively constant over time. For a convergence coefficient \( \alpha = 1 \cdot 10^{-1} \) and an excitation frequency of 249 Hz, the narrowband attenuation plotted in Fig. 9.4(c) is slightly larger in the vicinity of \( t = 0.0 \) s, 5.0 s and 10.0 s. As discussed in the previous section, this is because at these times the speed of the moving physical sensor \( \dot{x}_o(n) = 0 \) m/s. However, the speed of the moving physical sensor is zero at \( t = 2.5 \) s and 7.5 s as well. The reason why the narrowband attenuation at these times is slightly smaller in Fig. 9.4(c) than at \( t = 0.0 \) s, 5.0 s and 10.0 s can be found by examining the spatial rate of change of the relative magnitude and phase between the narrowband primary and secondary sound fields for a frequency of 249 Hz, which is illustrated in Fig. 9.2(b). It can be seen from this figure that when the moving physical sensor slows down to zero velocity as it approaches a virtual distance \( v = 0.120 \) m at \( t = 2.5 \) s and 7.5 s, the temporal rate of
Figure 9.4: Influence of the convergence coefficient $\alpha$ on the tracking performance of the normalised filtered-x LMS algorithm for narrowband control at a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with $\alpha = 1 \cdot 10^{-1}$, $\alpha = 5 \cdot 10^{-3}$, $\alpha = 1 \cdot 10^{-4}$. 
change of the relative magnitude and phase between the narrowband primary and secondary sound fields is slightly higher than when the moving physical sensor slows down to zero velocity as it approaches a virtual distance \( v = 0.020 \) m at \( t = 0.0 \) s, 5.0 s and 10.0 s. For an excitation frequency of 249 Hz, the tracking that is required from the normalised filtered-x LMS algorithm to obtain good control performance is thus slightly more demanding around \( t = 2.5 \) s and 7.5 s than around \( t = 0.0 \) s, 5.0 s and 10.0 s.

The results in Fig. 9.4(e) show that for the largest convergence coefficient \( \alpha = 1 \cdot 10^{-1} \) and an excitation frequency of 284 Hz, the narrowband attenuation increases from about 12 dB around a virtual distance \( v = 0.030 \) m to a maximum of 48 dB around a virtual distance \( v = 0.120 \) m. The difference in the narrowband control performance around these virtual distances can again be explained by noting that for an excitation frequency of 284 Hz, the moving physical sensor passes through a node in the sound field as it is slowing down to reach zero velocity at \( v = 0.020 \) m. When the physical sensor passes through this node, the normalised filtered-x LMS algorithm is not able to adjust the control filter coefficients \( w(n) \) fast enough to account for the rapid changes in the relative magnitude and phase between the narrowband primary and secondary sound fields.

### 9.5 Filtered-x RLS algorithm

In this section, the influence of the forgetting factor \( \lambda \) on the real-time tracking performance of the filtered-x RLS algorithm defined in Eq. (9.13) is analysed on the acoustic duct arrangement. The real-time narrowband control results presented in this section are generated similarly to the results presented in the previous sections, with the physical sensor again moving sinusoidally as defined in Eq. (9.18) with period \( T_v = 5 \) s. The results of the real-time experiments are illustrated in Fig. 9.5, where the narrowband attenuation and the control filter coefficients \( w(n) \) computed as defined in Eq. (9.11) are plotted against time for each of the three excitation frequencies considered and for three forgetting factors given by \( \lambda = 0.98 \), \( \lambda = 0.995 \), and \( \lambda = 0.999 \). The smallest value for the forgetting factor \( \lambda \) is determined in an iterative stage in which the tracking performance of the filtered-x RLS algorithm is optimised, with the inverse correlation matrix defined in Eq. (9.14) initialised as \( P(0) = \delta I \), with \( \delta = 1 \cdot 10^{-8} \).

The results in Figs 9.5(a) and 9.5(c) show that for the smallest forgetting factor \( \lambda = 0.98 \) and excitation frequencies of 213 Hz and 249 Hz, a narrowband attenuation of over 40 dB is obtained at all times. For a larger forgetting factor \( \lambda = 0.995 \), the results in Figs 9.5(a) and 9.5(c) indicate that the obtained narrowband attenuation is much lower. The reason for this is illustrated in Figs 9.5(b) and 9.5(d), where it can be seen that the control filter coefficients \( w(n) \) for the larger forgetting factor \( \lambda = 0.995 \)
Figure 9.5: Influence of the forgetting factor $\lambda$ on the tracking performance of the filtered-x RLS algorithm for narrowband control at a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with $\lambda = 0.98$, $\lambda = 0.995$, $\lambda = 0.999$. 
are lagging the control filter coefficients for $\lambda = 0.98$, which indicates that sufficient tracking is not achieved. Similarly to the results presented in the previous sections, note that for excitation frequencies of 213 Hz and 249 Hz, the narrowband attenuations for the two larger forgetting factors are highest in Figs. 9.5(a) and 9.5(c) when the control filter coefficients $w(n)$ for these convergence coefficients intersect the control filter coefficients for the smallest forgetting factor $\lambda = 0.98$ in Figs. 9.5(b) and 9.5(d).

For a forgetting factor $\lambda = 0.98$ and excitation frequencies of 213 Hz and 249 Hz, the narrowband attenuations plotted in Figs. 9.5(a) and 9.5(c) are slightly higher in the vicinity of $t = 0.0 \text{s}, 2.5 \text{s}, 5.0 \text{s}, 7.5 \text{s}$ and $10.0 \text{s}$, i.e. at times when the speed of the moving physical sensor $\dot{x}_v(n) = 0 \text{ m/s}$. As discussed in the previous section, the temporal rate of change of the relative magnitude and phase between the narrowband primary and secondary sound fields is lowest in the vicinity of these times, which means that the tracking required from the filtered-x RLS algorithm to obtain good control performance is less demanding.

For a forgetting factor $\lambda = 0.98$ and an excitation frequency of 284 Hz, the results in Fig. 9.5(e) indicate that the narrowband attenuation increases from about 20 dB around a virtual distance $v = 0.030 \text{ m}$ to a maximum of 50 dB around a virtual distance $v = 0.120 \text{ m}$. As discussed previously, the difference in the narrowband control performance obtained in the vicinity of these virtual distances can be explained by noting that for an excitation frequency of 284 Hz, the moving physical sensor passes through a node in the sound field as it slows down to reach zero velocity at $v = 0.020 \text{ m}$. The results in Fig. 9.5(e) also show that the narrowband attenuations obtained for the two larger forgetting factors are lower than the narrowband attenuation obtained for the smallest forgetting factor $\lambda = 0.98$. The reason for this is illustrated in Fig. 9.5(f), where it can be seen that the control filter coefficients for the two larger forgetting factors are lagging the control filter coefficients for the smallest forgetting factor $\lambda = 0.98$.

### 9.6 Tracking performance comparison

In this section, the real-time tracking performance of the filtered-x LMS algorithm, the normalised filtered-x LMS algorithm, and the filtered-x RLS algorithm is compared for various speeds of the moving physical sensor and various spatial characteristics of the sound field inside the acoustic duct arrangement illustrated in Fig. 9.1. The physical sensor included in this figure is again moving sinusoidally with an amplitude of 0.05 m as defined in Eq. (9.18), with the period of the sinusoidally time-varying virtual distance $v(n)$ given by $T_v = 10 \text{ s}$, $T_v = 5 \text{ s}$, and $T_v = 2.5 \text{ s}$. The real-time narrowband control performance that is obtained at the moving physical sensor is measured for the three considered excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. For a fair comparison, optimal values for the convergence coefficient $\mu$, the convergence coefficient $\alpha$, and
9.6 Tracking performance comparison

the forgetting factor \( \lambda \) are determined in an iterative stage in which the tracking performance of each adaptive algorithm is optimised. In the filtered-x LMS algorithm, the control filter coefficients \( w(n) \) are updated as defined in Eq. (9.9) using a convergence coefficient \( \mu = 5 \cdot 10^{-3} \). In the normalised filtered-x LMS algorithm defined in Eq. (9.11), a convergence coefficient \( \alpha = 1 \cdot 10^{-1} \) is used to update the control filter coefficients. In the filtered-x RLS algorithm defined in Eq. (9.13), the control filter coefficients are updated using a forgetting factor \( \lambda = 0.98 \), and with the inverse correlation matrix defined in Eq. (9.14) initialised as \( P(0) = \delta I \), with \( \delta = 1 \cdot 10^{-8} \). The real-time narrowband control results presented in the following sections have been generated similarly to the results presented in the previous sections.

9.6.1 Slow moving physical sensor with period \( T_v = 10 \text{s} \)

In this section, the moving physical sensor illustrated in Fig. 9.1 is tracking the moving virtual location \( x_v(n) = x_{p5} + v(n) \), with the period of the sinusoidally time-varying virtual distance \( v(n) \) defined in Eq. (9.18) given by \( T_v = 10 \text{s} \). The results of the real-time experiments are illustrated in Fig. 9.6, where the narrowband attenuation and the control filter coefficients \( w(n) \) are plotted against time for all of the three considered excitation frequencies.

For excitation frequencies of 213 Hz and 249 Hz, the results in Figs 9.6(a) and 9.6(c) show that very similar narrowband attenuations of over 48 dB and 45 dB, respectively, are obtained at all times when using either the filtered-x LMS, normalised filtered-x LMS, or filtered-x RLS algorithm to update the control filter coefficients. The narrowband attenuations obtained with these adaptive algorithms are very similar because the control filter coefficients \( w(n) \) computed as defined in Eqs (9.9), (9.11) and (9.13) are very similar for excitation frequencies of 213 Hz and 249 Hz. This can be seen in Figs 9.6(b) and 9.6(d), where the control filter coefficients are plotted against time for the three adaptive algorithms considered. For excitation frequencies of 213 Hz and 249 Hz and \( T_v = 10 \text{s} \), each of the three adaptive algorithms is thus able to adjust the control filter coefficients \( w(n) \) fast enough to account for the variations in the relative magnitude and phase between the narrowband primary and secondary sound fields, which are illustrated in Figs 9.2(a) and 9.2(b), that occur when the physical sensor is moving through the sound field inside the acoustic duct arrangement. Note that for an excitation frequency of 213 Hz, ringing occurs in the control filter coefficients illustrated in Fig. 9.6(b). This ringing phenomenon is observed because the adaptive algorithms are implemented using convergence parameters that achieve optimum tracking performance while just maintaining stability.

For an excitation frequency of 284 Hz, the results in Fig. 9.6(e) illustrate that for each of the three adaptive algorithms considered, a maximum narrowband attenuation of 48 dB is obtained when the physical sensor slows down to \( \dot{x}_v(n) = 0 \text{ m/s} \) at a virtual
Figure 9.6: Tracking performance of the filtered-x LMS algorithm, the normalised filtered-x LMS algorithm, and the filtered-x RLS algorithm for narrowband control at a physical sensor that moves sinusoidally with period $T_v = 10$ s for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with — filtered-x LMS algorithm -- normalised filtered-x LMS algorithm — filtered-x RLS algorithm.
distance $v = 0.140 \text{ m}$. However, when the physical sensor slows down to $\dot{x}_v(n) = 0 \text{ m/s}$ at a virtual distance $v = 0.020 \text{ m}$, the narrowband attenuation that is obtained decreases to 6 dB around a virtual distance $v = 0.030 \text{ m}$ for the filtered-x LMS algorithm, 16 dB for the normalised filtered-x LMS algorithm, and 24 dB for the filtered-x RLS algorithm. As discussed previously, this decrease in the narrowband attenuation for an excitation frequency of 284 Hz is caused by the node in the sound field inside the acoustic duct arrangement. When the physical sensor passes through this node as it slows down to $\dot{x}_v(n) = 0 \text{ m/s}$ at a virtual distance $v = 0.020 \text{ m}$, the narrowband attenuation decreases to 6 dB around a virtual distance $v = 0.030 \text{ m}$ for the filtered-x LMS algorithm, 16 dB for the normalised filtered-x LMS algorithm, and 24 dB for the filtered-x RLS algorithm.

As discussed previously, this decrease in the narrowband attenuation for an excitation frequency of 284 Hz is caused by the node in the sound field inside the acoustic duct arrangement. When the physical sensor passes through this node as it slows down to $\dot{x}_v(n) = 0 \text{ m/s}$ at a virtual distance $v = 0.020 \text{ m}$, the adaptive algorithms are not able to adjust the control filter coefficients $w(n)$ fast enough to completely account for the rapid changes in the relative magnitude and phase between the narrowband primary and secondary sound fields. Note that the decrease in the narrowband attenuation around a virtual distance $v = 0.030 \text{ m}$ is smaller for the normalised filtered-x LMS algorithm than for the filtered-x LMS algorithm, which indicates that normalising the convergence coefficient as defined in Eq. (9.12) improves the tracking performance. This is to be expected because the virtual filtered-reference signal is a non-stationary process due to the movement of the physical sensor. The decrease in the narrowband attenuation is smallest for the filtered-x RLS algorithm, which indicates that this algorithm provides the best tracking performance out of the three adaptive algorithms considered.

### 9.6.2 Medium pace moving physical sensor with period $T_v = 5 \text{ s}$

In this section, the moving physical sensor is tracking the moving virtual location $x_v(n) = x_{p5} + v(n)$, with the period of the sinusoidally time-varying virtual distance $v(n)$ defined in Eq. (9.18) given by $T_v = 5 \text{ s}$. The results of the real-time experiments are illustrated in Fig. 9.7, where the narrowband attenuation and the control filter coefficients $w(n)$ are plotted against time for all of the three considered excitation frequencies. Note that these real-time experimental results have also been presented in Sections 9.3–9.5, but that small differences will occur because these results have been generated from real-time experiments conducted on different days.

For excitation frequencies of 213 Hz and 249 Hz, the results in Figs 9.7(a) and 9.7(c) show that very similar narrowband attenuations of over 46 dB and 41 dB, respectively, are obtained at all times when using either the filtered-x LMS, normalised filtered-x LMS, or filtered-x RLS algorithm to update the control filter coefficients. This was also observed in the previous section and can be explained by noting that for excitation frequencies of 213 Hz and 249 Hz, the control filter coefficients computed with the considered adaptive algorithms are very similar, as shown in Figs 9.7(b) and 9.7(d). For excitation frequencies of 213 Hz and 249 Hz and a period $T_v = 5 \text{ s}$, each of the three adaptive algorithms is thus able to adjust the control filter coefficients $w(n)$ fast enough to account for the variations in the relative magnitude and phase between the
Figure 9.7: Tracking performance of the filtered-x LMS algorithm, the normalised filtered-x LMS algorithm, and the filtered-x RLS algorithm for narrowband control at a physical sensor that moves sinusoidally with period $T_v = 5$ s for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with — filtered-x LMS algorithm — normalised filtered-x LMS algorithm — filtered-x RLS algorithm.
narrowband primary and secondary sound fields that occur when the physical sensor is moving through the sound field inside the acoustic duct arrangement.

For an excitation frequency of 284 Hz, the results in Fig. 9.7(e) illustrate that for each of the three adaptive algorithms considered, a maximum narrowband attenuation of 48 dB is obtained when the physical sensor slows down to $\dot{x}_v(n) = 0$ m/s at a virtual distance $v = 0.140$ m. However, when the physical sensor slows down to $\dot{x}_v(n) = 0$ m/s at a virtual distance $v = 0.020$ m, the narrowband attenuation that is obtained decreases to 2 dB around a virtual distance $v = 0.030$ m for the filtered-x LMS algorithm, 11 dB for the normalised filtered-x LMS algorithms, and 18 dB for the filtered-x RLS algorithm. As discussed previously, this decrease in the narrowband attenuation for an excitation frequency of 284 Hz is caused by the node in the sound field inside the acoustic duct arrangement. Note that for all three adaptive algorithms considered, the decrease in the narrowband attenuation around a virtual distance $v = 0.030$ m is larger than in the previous section, where the physical sensor was moving slower with a period $T_v = 10$ s. This is to be expected as the temporal rate of change of the relative magnitude and phase between the narrowband primary and secondary sound fields is larger when the physical sensor moves through the sound field more quickly. Similar to the results presented in the previous section, the decrease in the narrowband attenuation is smallest for the filtered-x RLS algorithm, which again indicates that this algorithm provides the best tracking performance out of the three adaptive algorithms considered.

### 9.6.3 Fast moving physical sensor with period $T_v = 2.5$ s

In this section, the moving physical sensor is tracking the moving virtual location $x_v(n) = x_p5 + v(n)$, with the period of the sinusoidally time-varying virtual distance $v(n)$ defined in Eq. (9.18) given by $T_v = 2.5$ s. The results of the real-time experiments are illustrated in Fig. 9.8, where the narrowband attenuation and the control filter coefficients $w(n)$ are plotted against time for all of the three considered excitation frequencies.

For excitation frequencies of 213 Hz and 249 Hz, the results in Figs 9.8(a) and 9.8(c) show that very similar narrowband attenuations of over 44 dB and 36 dB, respectively, are obtained at all times when using either of the adaptive algorithms considered. Note that these narrowband attenuations are smaller than the results presented in the previous sections, i.e. 48 dB and 45 dB for a period $T_v = 10$ s, and 46 dB and 41 dB for a period $T_v = 5$ s. This is to be expected because the physical sensor was moving slower in the previous sections.

For an excitation frequency of 284 Hz, the results in Fig. 9.8(e) illustrate that for each of the three adaptive algorithms considered, a maximum narrowband attenuation of 45 dB is obtained when the physical sensor slows down to $\dot{x}_v(n) = 0$ m/s at a virtual distance $v = 0.140$ m. However, when the physical sensor slows down to $\dot{x}_v(n) = 0$ m/s
Figure 9.8: Tracking performance of the filtered-x LMS algorithm, the normalised filtered-x LMS algorithm, and the filtered-x RLS algorithm for narrowband control at a physical sensor that moves sinusoidally with period $T_v = 2.5$ s for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, with — filtered-x LMS algorithm — normalised filtered-x LMS algorithm — filtered-x RLS algorithm.
at a virtual distance $v = 0.020 \text{ m}$, the narrowband attenuation that is obtained decreases to $-1 \text{ dB}$ around a virtual distance $v = 0.030 \text{ m}$ for the filtered-x LMS algorithm, $7 \text{ dB}$ for the normalised filtered-x LMS algorithms, and $14 \text{ dB}$ for the filtered-x RLS algorithm. As observed in the previous sections, this decrease in the narrowband attenuation for an excitation frequency of $284 \text{ Hz}$ is caused by the node in the sound field inside the acoustic duct arrangement. The decrease in the narrowband attenuation is again smallest for the filtered-x RLS algorithm, which indicates that this algorithm provides the best tracking performance out of the three considered adaptive algorithms. Again, also note that for all three adaptive algorithms considered, the decrease in the narrowband attenuation around a virtual distance $v = 0.030 \text{ m}$ is larger than in the previous sections, where the physical sensor was moving slower, which is to be expected.

### 9.7 Conclusion

The filtered-x LMS algorithm, the normalised filtered-x LMS algorithm and the filtered-x RLS algorithm introduced in Chapter 2 have been implemented on the acoustic duct arrangement using the implementation developed in Chapter 5 and illustrated in Fig. 5.3 for local active noise control at a moving physical sensor. The tracking performance of these adaptive algorithms has been analysed and compared for various speeds of the moving physical sensor and various spatial characteristics of the narrowband sound field inside the acoustic duct arrangement. It has been observed that the tracking performance decreased when the temporal rate of change of the relative magnitude and phase between the primary and secondary sound fields increased due to an increase in either the speed of the moving physical sensor or the spatial variations in the relative magnitude and phase between the primary and secondary sound fields. The filtered-x RLS algorithm provided the best tracking performance in the conducted real-time narrowband control experiments, especially for a higher temporal rate of change of the relative magnitude and phase between the primary and secondary sound fields inside the acoustic duct arrangement. The real-time experimental narrowband control results indicated that a moving zone of quiet has successfully been created at a physical sensor that tracked the desired location of maximum attenuation.
Chapter 10

Narrowband active noise control at a moving virtual sensor

10.1 Introduction

In this chapter, the adaptive feedforward control approach developed in Chapter 5 and illustrated in Fig. 5.5 for local active noise control at a moving virtual sensor is implemented on the acoustic duct arrangement introduced in Chapter 6. The aim is to create a moving zone of quiet at a virtual sensor that tracks the desired location of maximum attenuation, i.e. the moving virtual location. An estimate of the virtual error signal at the moving virtual location is computed using the practical implementation of either the Kalman filter based moving virtual sensing algorithm or the adaptive LMS moving virtual microphone technique, which were developed in Chapter 5. As discussed in that chapter, the estimate of the virtual primary disturbance that needs to be attenuated at the moving virtual sensor is a non-stationary signal when the moving virtual sensor is tracking a virtual location that is moving through the sound field inside the acoustic duct arrangement. An adaptive feedforward control approach is therefore adopted to be able to track the changes in the statistical properties of the estimate of the virtual primary disturbance and adjust the controller accordingly. In Chapter 9, it has been shown that implementing the filtered-x RLS algorithm on the acoustic duct arrangement as illustrated in Fig. 9.1 provided better tracking performance for local active noise control at a moving physical sensor than the filtered-x LMS and normalised filtered-x LMS algorithms. The only difference between the implementation considered in Chapter 9 and the one considered in this chapter is that an estimate \( \hat{e}_v(n) \) of the virtual error signal at the moving virtual location \( x_v(n) \) is now minimised instead of the true virtual error signal \( e_v(n) \) directly measured by a moving physical sensor. In this chapter, the estimate \( \hat{e}_v(n) \) of the virtual error signal at the moving virtual location is therefore minimised using the filtered-x RLS algorithm.
In Section 10.2, a schematic diagram of the implementation that is used in the real-time experiments conducted on the acoustic duct arrangement is presented. In Sections 10.3 and 10.4, the narrowband control performance that is obtained when an estimate $\hat{e}_v(n)$ of the virtual error signal is computed using the practical implementation of either the Kalman filter based moving virtual sensing algorithm or the adaptive LMS moving virtual microphone technique, respectively, is analysed for various speeds of the moving virtual sensor and various spatial characteristics of the narrowband sound field inside the acoustic duct arrangement. In Section 10.5, the real-time experimental results obtained when using the two implemented moving virtual sensing algorithms are compared.

10.2 Acoustic duct arrangement

Fig. 10.1 shows a schematic diagram of the implementation that is used in the real-time experiments conducted on the acoustic duct arrangement. Note that this arrangement is equivalent to the arrangement used in Chapter 9 and illustrated in Fig. 9.1 with the exception that a moving virtual sensor is now located inside the acoustic duct arrangement instead of a moving physical sensor. The acoustic duct has length $L_x = 4.830\,\text{m}$, a primary loudspeaker located at $x_p = 4.730\,\text{m}$, and a control loudspeaker at $x_s = 0.100\,\text{m}$. The primary loudspeaker is excited by a narrowband disturbance source signal $s(n)$ and the control loudspeaker by the control signal $u(n)$. The aim of the implementation illustrated in Fig. 10.1 is to minimise the estimate $\hat{e}_v(n)$ of the virtual error signal at the moving virtual sensor. This moving virtual sensor is tracking the moving virtual location $x_v(n) = x_{p5} + v(n)$, with $v(n)$ the time-varying virtual distance and $x_{p5} = 1.4750\,\text{m}$ the position of the physical sensor located inside the acoustic duct arrangement. The estimate $\hat{e}_v(n)$ of the virtual error signal is computed in Section 10.3 using the practical implementation of the Kalman filter based moving virtual sensing algorithm developed in Chapter 5. In Section 10.4, the estimate $\hat{e}_v(n)$ of the virtual error signal is computed using the practical implementation of the adaptive LMS virtual microphone technique developed in Chapter 5. The aim of the real-time narrowband control experiments is thus to create a moving zone of quiet at a moving virtual sensor that tracks the desired location of maximum attenuation inside the acoustic duct arrangement. In the presented real-time experiments, the traversing microphone located inside the acoustic duct arrangement is used to measure the narrowband control performance that is obtained at the moving virtual location $x_v(n)$. As discussed in Chapter 6, this traversing microphone can be position controlled inside the acoustic duct arrangement to track the moving virtual location.
10.2 Acoustic duct arrangement

Figure 10.1: Schematic diagram of the acoustic duct arrangement of length \( L_x = 4.830 \text{ m} \), with a primary source located at \( x_p = 4.730 \text{ m} \), a control source located at \( x_s = 0.100 \text{ m} \), a physical sensor located at \( x_p5 = 1.4750 \text{ m} \), \( \bar{M}_v \) spatially fixed virtual sensors located at \( \bar{x}_v \), and a moving virtual sensor that tracks the moving virtual location \( x_v(n) = x_p5 + v(n) \), with \( v(n) \) the time-varying virtual distance.

10.2.1 Adaptive feedforward control implementation

The estimate \( \hat{d}_v(n) \) of the virtual primary disturbance that needs to be attenuated at the moving virtual sensor is a non-stationary signal due to the movement of the virtual location \( x_v(n) \) through the sound field inside the acoustic duct arrangement. An adaptive feedforward control approach is therefore adopted in order to track the changes in the statistical properties of the estimate of the virtual primary disturbance and adjust the controller accordingly. In Chapter 9, it was shown that implementing the filtered-x RLS algorithm on the acoustic duct arrangement as illustrated in Fig. 9.1 provides better tracking performance for local active noise control at a moving physical sensor than the filtered-x LMS and normalised filtered-x LMS algorithms. In this chapter, the estimate \( \hat{e}_v(n) \) of the virtual error signal at the moving virtual location is therefore minimised using the filtered-x RLS algorithm as illustrated in Fig. 10.1, with \( I = 2 \) control filter coefficients. The only difference with the implementation discussed in Chapter 9 and illustrated in Fig. 9.1 is that in Fig. 10.1, an estimate \( \hat{e}_v(n) \) of the virtual
error signal at the moving virtual location $x_v(n)$ is now minimised instead of the true virtual error signal $e_v(n)$ directly measured by a moving physical sensor. From Eqs (9.13)–(9.15), the update equations for the filtered-x RLS algorithm are thus given by

$$w(n + 1) = w(n) - k(n)\hat{e}_v(n),$$

$$k(n) = \frac{\lambda^{-1}P(n)\hat{r}_v(n)}{1 + \lambda^{-1}\hat{r}_v(n)^TP(n)\hat{r}_v(n)},$$

$$P(n + 1) = \lambda^{-1}P(n) - \lambda^{-1}k(n)\hat{r}_v(n)^TP(n),$$

with $\hat{e}_v(n)$ the estimate of the virtual error signal, $\lambda$ the forgetting factor, $w(n) \in \mathbb{R}^2$ the control filter coefficients, $k(n) \in \mathbb{R}^2$ the gain matrix, and $P(n) \in \mathbb{R}^{2 \times 2}$ the inverse correlation matrix, which is initialised as $P(0) = \delta I$, with $\delta$ a regularisation parameter. In the real-time experiments presented in this chapter, a forgetting factor $\lambda = 0.98$ is used and the inverse correlation matrix is initialised by setting $\delta = 1 \cdot 10^{-8}$. These values gave the best real-time narrowband control performance in an iterative process.

In Eqs (10.2) and (10.3), the vector $\hat{r}_v(n) \in \mathbb{R}^2$ of estimated virtual filtered-reference signals is defined as

$$\hat{r}_v(n) = \begin{bmatrix} \hat{r}_v(n) \\ \hat{r}_v(n-1) \end{bmatrix},$$

with $\hat{r}_v(n)$ the estimate of the virtual filtered-reference signal for the moving virtual location $x_v(n)$ computed as defined in Eq. (9.5) and illustrated in Fig. 10.1. This estimate $\hat{r}_v(n)$ is thus computed using linear spatial interpolation between the virtual filtered-reference signals $\hat{r}_v(n) \in \mathbb{R}^{M_v}$ for the spatially fixed virtual locations $\tilde{x}_v \in \mathbb{R}^{M_v}$, with

$$\hat{x}_v = \begin{bmatrix} \hat{x}_{v1} \\ \hat{x}_{v2} \\ \vdots \\ \hat{x}_{vM_v} \end{bmatrix}.$$

These spatially fixed virtual locations are again suitably positioned throughout the target zone illustrated in Fig. 10.1, which defines the region inside the acoustic duct arrangement through which the virtual sensor is moving. The virtual filtered-reference signals $\hat{x}_v(n)$ for the spatially fixed virtual locations $\hat{x}_v$ are generated as illustrated in Fig. 10.1 and defined in Eq. (9.4). In this equation, the feedforward reference signal $x(n)$ is filtered with a state-space model of the virtual secondary transfer paths $\hat{G}_{vu} \in \mathcal{RH}_\infty^{M_v \times 1}$ between the control source and the spatially fixed virtual locations $\hat{x}_v$ defined in Eq. (10.5). As described in Section 9.2.3, a state-space model $\hat{G}_{vu}$ of the virtual secondary transfer paths is estimated in a preliminary identification stage in which the control loudspeaker is excited by a band-limited white noise signal $u(n)$ in the frequency range of 50–500 Hz. The traversing microphone is then sequentially positioned at the considered spatially fixed virtual locations $\hat{x}_v$ to measure an input/output data-set as defined in Eq. (9.19). The subspace model identification techniques [50, 124] described
in Chapter 6 are then used to estimate a deterministic state-space model $\hat{G}_{vu}$ on the training data-set, with the estimated state-space model defined in Eq. (9.3). In the final implementation, a single input $\bar{M}_{v} = 16$ output deterministic state-space model of order $N_{l} = 32$ has been estimated, with the spatially fixed virtual locations given by

$$\bar{x}_{v} = \begin{bmatrix} 1.4750 & 1.4850 & \cdots & 1.6250 \end{bmatrix}^T. \quad (10.6)$$

Similarly to the preliminary identification results presented in Chapter 9, this resulted in a VAF = 99.9\% on the validation data-set, with the VAF value computed as defined in Eq. (6.30). This indicates that a very accurate state-space model $\hat{G}_{vu}$ of the virtual secondary transfer paths between the control source and the spatially fixed virtual locations $\bar{x}_{v}$ defined in Eq. (10.6) has been estimated.

### 10.2.2 Moving virtual sensing implementation

In the adaptive feedforward control implementation illustrated in Fig. 10.1, an estimate $\hat{e}_{v}(n)$ of the virtual error signal at the moving virtual location $x_{v}(n)$ is computed using spatial interpolation between the current estimate $\hat{e}_{v}(n|n)$ is computed using spatial interpolation between the current estimate $\hat{e}_{v}(n|n)$ of the virtual error signals at the spatially fixed virtual locations $\bar{x}_{v}$ defined in Eq. (10.5), with

$$\hat{e}_{v}(n|n) = \begin{bmatrix} \hat{e}_{v1}(n|n) & \hat{e}_{v2}(n|n) & \cdots & \hat{e}_{vM_{v}}(n|n) \end{bmatrix}^T. \quad (10.7)$$

As stated previously, the spatially fixed virtual locations $\bar{x}_{v}$ are suitably positioned throughout the target zone, which defines the region of the acoustic duct through which the physical sensor is moving. Using linear spatial interpolation between the current estimate $\hat{e}_{v}(n|n)$ of the virtual error signals at the spatially fixed virtual locations $\bar{x}_{v}$ defined in Eq. (10.7), the estimate $\hat{e}_{v}(n)$ of the virtual error signal at the moving virtual location $x_{v}(n)$ is computed as

$$\hat{e}_{v}(n) = \frac{\bar{x}_{v}(n) - x_{v}(n)}{\bar{x}_{v}(n+1) - \bar{x}_{v1}} \hat{e}_{v1}(n|n) + \frac{x_{v}(n) - \bar{x}_{v1}}{\bar{x}_{v}(n+1) - \bar{x}_{v1}} \hat{e}_{v1}(n|n), \quad (10.8)$$

where $\bar{x}_{v1}$ and $\bar{x}_{v(i+1)}$ are the two spatially fixed virtual locations defined in Eq. (10.5) that are closest to the moving virtual location $x_{v}(n)$, such that

$$\bar{x}_{v1} \leq x_{v}(n) \leq \bar{x}_{v(i+1)}, \quad (10.9)$$

and with $\hat{e}_{v1}(n|n)$ and $\hat{e}_{v(i+1)}(n|n)$ the current estimates of the virtual error signals at the spatially fixed virtual locations $\bar{x}_{v1}$ and $\bar{x}_{v(i+1)}$. Note that to compute the spatial interpolation in Eq. (10.8), the moving virtual location $x_{v}(n)$ that defines the desired location of the zone of quiet is assumed known. In Fig. 10.1 and Section 10.3, a current estimate $\hat{e}_{v}(n|n)$ of the virtual error signals at the spatially fixed virtual locations $\bar{x}_{v}$ is
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computed using the Kalman filter based spatially fixed virtual sensing algorithm developed in Chapter 3. In Section 10.4, an estimate of the virtual error signals \( \bar{e}_v(n) \) at the spatially fixed virtual locations \( \bar{x}_v \) is computed using the adaptive LMS virtual microphone technique, which was extended in Chapter 3 to the case of multiple spatially fixed virtual sensors.

10.2.3 Speed of virtual sensor and spatial characteristics of sound field

As discussed in Chapter 9, the tracking that is demanded from the filtered-x RLS algorithm to obtain effective local control performance is dependent on both the temporal rate of change \( \dot{x}_v(n) \) of the moving virtual location, i.e. the speed of the moving virtual sensor, and the spatial rate of change of the relative magnitude and phase between the primary and secondary sound fields inside the acoustic duct arrangement over the target zone. As in Chapter 9, the expression for the time-varying virtual distance that governs the desired moving virtual location \( x_v(n) = 1.4750 + v(n) \) is given, equivalent to Eq. (9.18), by

\[
v(n) = 0.070 + 0.050 \sin \left( \frac{2\pi n}{T_v f_s} \right),
\]

(10.10)

where \( T_v \) is the period of the sinusoidally time-varying virtual distance \( v(n) \). Similarly to Chapter 9, the narrowband control performance that is obtained at the moving virtual sensor is analysed for three excitation frequencies of 213 Hz, 249 Hz, and 284 Hz, and three values of the period \( T_v \) given by \( T_v = 10 \text{s} \), \( T_v = 5 \text{s} \), and \( T_v = 2.5 \text{s} \). The measured spatial rate of change of the primary and secondary sound fields and the relative spatial rate of change between these two sound fields over the target zone inside the acoustic duct arrangement have been plotted in Fig. 9.2 for the excitation frequencies considered.

10.3 Kalman filter based moving virtual sensing algorithm

In this section, a current estimate of the virtual error signals \( \bar{e}_v(n|n) \) at the spatially fixed virtual locations \( \bar{x}_v \) defined in Eq. (10.6) is computed using the Kalman filter based spatially fixed virtual sensing algorithm developed in Chapter 3. To implement this algorithm as described in Table 3.1 on page 123, an innovations model of the acoustic duct arrangement needs to be estimated in a preliminary identification stage that is described in Section 10.3.1. In Section 10.3.2, the Kalman filter based spatially fixed virtual sensing algorithm is implemented on the acoustic duct arrangement as illustrated in Fig. 10.1. The narrowband control performance that is obtained at the moving virtual sensor is analysed for various speeds \( \dot{x}_v(n) \) of the moving virtual location and various spatial characteristics of the narrowband sound field inside the acoustic duct arrangement. As in Chapter 8, a sample frequency \( f_s = 1.6 \text{kHz} \) is employed in the real-time experiments.
10.3 Kalman filter based moving virtual sensing algorithm

### 10.3.1 Preliminary identification stage

The Kalman filter based spatially fixed virtual sensing algorithm is computed in a preliminary identification stage in which the traversing microphone is sequentially positioned at the spatially fixed virtual locations \( \bar{x}_v \) defined in Eq. (10.6). An *innovations model* of the acoustic duct arrangement is then estimated from which the Kalman filter based spatially fixed virtual sensing algorithm can be computed as defined in Table 3.1 on page 123. The innovations model of the acoustic duct arrangement is estimated using the *subspace model identification techniques* [50, 124] described in Chapter 6. An example of the preliminary identification procedure was described in Chapter 6, where a *two-step* approach has been adopted to estimate the innovations model.

#### Deterministic part of the innovations model

In the *first step*, a deterministic state-space model is estimated that models the physical secondary transfer path \( G_{pu} \in \mathcal{RH}^{1 \times 1} \) between the control source and the physical sensor located at \( x_{p5} = 1.4750 \) m, and the virtual secondary transfer paths \( \bar{G}_{vu} \in \mathcal{RH}^{16 \times 1} \) between the control source and the spatially fixed virtual locations \( \bar{x}_v \) defined in Eq. (10.6). Similarly to Eq. (6.28), the estimated state-space model is given by

\[
\begin{align*}
\hat{z}_1(n+1) &= \hat{A}_1 \hat{z}_1(n) + \hat{B}_u u(n) \\
\hat{y}_p(n) &= \hat{C}_{p1} \hat{z}_1(n) + \hat{D}_{pu} u(n) \\
\bar{y}_v(n) &= \hat{C}_{v1} \hat{z}_1(n) + \hat{D}_{vu} u(n),
\end{align*}
\]

with \( \hat{y}_p(n) \) the estimated physical secondary disturbance, \( \bar{y}_v(n) \in \mathbb{R}^{16} \) the estimated virtual secondary disturbances at the spatially fixed virtual locations \( \bar{x}_v \), and \( \hat{A}_1, \hat{B}_u, \hat{C}_{p1}, \hat{C}_{v1}, \hat{D}_{pu}, \) and \( \hat{D}_{vu} \) the estimated state-space matrices. The state-space model defined in Eq. (10.11) is estimated while exciting the control loudspeaker with a band-limited white noise signal \( u(n) \) in the frequency range of 50–500 Hz. The traversing microphone is then sequentially positioned at the spatially fixed virtual locations \( \bar{x}_v \) to measure an input/output data-set given by

\[
\{u(n), \begin{bmatrix} y_p(n) \\ \bar{y}_v(n) \end{bmatrix}\}_{n=1}^{32000},
\]

with \( \bar{y}_v(n) \in \mathbb{R}^{16} \) the virtual secondary disturbances at the spatially fixed virtual locations \( \bar{x}_v \). The recorded data-set is divided into a *training data-set* and a *validation data-set* each 16 000 samples long. The PO-MOESP [50] subspace model identification method is then used as described in Chapter 6 to estimate a deterministic state-space model on the training data-set. In the final implementation, a deterministic state-space model of order \( N_1 = 32 \) was estimated. This resulted in a VAF = 99.9% on the
validation data-set, with the VAF value calculated as defined in Eq. (6.30). This indicates that an accurate state-space model of the physical and virtual secondary transfer paths has been obtained.

**Stochastic part of the innovations model**

In the second step of the two-step identification procedure, an estimate of the stochastic part of the innovations model of the acoustic duct arrangement is computed. Similarly to Eq. (6.31), this stochastic part is given by

\[
\begin{align*}
\hat{z}_2(n+1|n) &= \hat{A}_2\hat{z}_2(n|n-1) + \hat{K}_s[\varepsilon_p(n) \bar{\varepsilon}_v(n)]^T \\
\hat{d}_p(n) &= \hat{C}_{p2}\hat{z}_2(n|n-1) + \varepsilon_p(n) \\
\hat{d}_v(n) &= \hat{C}_{v2}\hat{z}_2(n|n-1) + \bar{\varepsilon}_v(n),
\end{align*}
\]

with \(\hat{d}_p(n)\) the estimated physical primary disturbance at the physical sensor located at \(x_{p5} = 1.4750\) m, \(\hat{d}_v(n) \in \mathbb{R}^{16}\) the estimated virtual primary disturbances at the spatially fixed virtual locations \(\bar{x}_v\) defined in Eq. (10.6), \(\hat{A}_2, \hat{C}_{p2},\) and \(\hat{C}_{v2}\) the estimated state-space matrices, \(\hat{K}_s\) the estimated Kalman gain matrix, and \(\varepsilon_p(n)\) and \(\bar{\varepsilon}_v(n) \in \mathbb{R}^{16}\) the white innovation signals. The covariance matrix of these innovation signals is also estimated, with the estimate defined, similarly to Eq. (3.224), as

\[
\hat{R}_s = \begin{bmatrix}
\hat{R}_{pe} & \hat{R}_{poe} \\
\hat{R}_{pe}^T & \hat{R}_{oe}
\end{bmatrix}.
\]

The state-space matrices and the Kalman gain matrix defined in Eq. (10.13), and the covariance matrix defined in Eq. (10.14), are estimated while exciting the primary loudspeaker with a band-limited white noise signal \(s(n)\) in the frequency range of 50–500 Hz. The traversing microphone is then sequentially positioned at the spatially fixed virtual locations \(\bar{x}_v\) to measure an output data-set given by

\[
\begin{bmatrix}
d_p(n) \\
\bar{d}_v(n)
\end{bmatrix}_{n=1}^{32000},
\]

with \(\bar{d}_v(n) \in \mathbb{R}^{16}\) the virtual primary disturbances at the spatially fixed virtual locations \(\bar{x}_v\). The recorded output data-set is divided into a training data-set and a validation data-set each 16000 samples long. The SSARX subspace identification algorithm [55] is then used as described in Chapter 6 to estimate a stochastic innovations model on the training data-set. In the final experiments, a stochastic innovations model of order \(N_2 = 40\) was estimated on the training data-set. For the physical primary disturbance \(d_p(n)\), a VAF = 99.6% was obtained on the validation data-set, while for the virtual primary disturbances \(\bar{d}_v(n)\), a VAF = 99.1% was obtained on the validation data-set, with the
VAF values calculated as described in Section 6.4. These VAF values indicate that an accurate innovations model of the stochastic part of the acoustic duct arrangement has been obtained.

Kalman filter based spatially fixed virtual sensing algorithm

Using the innovations model of the acoustic duct arrangement estimated in the described two-step identification procedure, the Kalman filter based spatially fixed virtual sensing algorithm can be computed as defined in step 3 of Table 3.1. The resulting state-space model computes a current estimate \( \hat{e}_s(n|n) \) of the virtual error signals at the spatially fixed virtual location \( \hat{x}_v \) given the current and previous observations of the physical error signal \( e_p(n) \) and the deterministic control signal \( u(n) \), and is given by

\[
\begin{bmatrix}
\hat{z}_1(n+1) \\
\hat{z}_2(n+1|n) \\
\hat{e}_s(n|n)
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_1 \\
0 \\
\hat{C}_p1 - \hat{M}_s \hat{C}_{p2}
\end{bmatrix}
\begin{bmatrix}
\hat{A}_2 \\
\hat{B}_u \\
\hat{D}_{yu} - \hat{M}_s \hat{D}_{pu}
\end{bmatrix}
\begin{bmatrix}
\hat{K}_{ps} \\
\hat{K}_{ps}
\end{bmatrix}
\begin{bmatrix}
\hat{e}_s(n|n) \\
\hat{e}_p(n)
\end{bmatrix},
\]

(10.16)

where the estimated Kalman gain matrix \( \hat{K}_{ps} \in \mathbb{R}^{40} \) and the estimated virtual gain matrix \( \hat{M}_{vs} \in \mathbb{R}^{16} \) are computed as

\[
\hat{K}_{ps} = \left( \hat{A}_2 X_s \hat{C}_{p2}^T + \hat{K}_s \begin{bmatrix} \hat{R}_{pe} \\ \hat{R}_{pe} \end{bmatrix} \right) \left( \hat{C}_{p2} X_s \hat{C}_{p2}^T + \hat{R}_{pe} \right)^{-1},
\]

(10.17)

\[
\hat{M}_{vs} = \left( \hat{C}_{p2} X_s \hat{C}_{p2}^T + \hat{R}_{pvs} \right) \left( \hat{C}_{p2} X_s \hat{C}_{p2}^T + \hat{R}_{pe} \right)^{-1},
\]

(10.18)

with \( X_s = X_s^T > 0 \) the unique stabilising solution to the DARE given by

\[
X_s = \hat{A}_2 X_s \hat{A}_2^T - \hat{K}_{ps} \left( \hat{C}_{p2} X_s \hat{C}_{p2}^T + \hat{R}_{pe} \right)^{-1} \hat{K}_{ps}^T + \hat{K}_s \hat{R}_s \hat{K}_s^T.
\]

(10.19)

Note that the estimated Kalman gain and virtual gain matrices are computed in Eqs (10.17) and (10.18), respectively, using the estimated innovations model of the stochastic part of the acoustic duct arrangement defined in Eq. (10.13), which has been computed in the second step of the two-step identification procedure.

10.3.2 Real-time experimental results

In this section, the real-time narrowband control performance that is obtained at the moving virtual location \( x_v(n) \) when using a moving virtual sensor is compared with the real-time narrowband control performance that is obtained when using either a moving physical sensor that tracks the moving virtual location or a spatially fixed virtual sensor located at a virtual distance \( v = 0.020 \) m. For the moving virtual sensor case, the practical implementation of the Kalman filter based moving virtual sensing algorithm
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illustrated in Fig. 10.1 is used, with the current estimate $\hat{e}_v(n|n)$ of the virtual error signals at the spatially fixed virtual locations $\hat{x}_v$ computed as defined in Eq. (10.16). For the moving physical sensor case, the implementation illustrated in Fig. 9.1 and discussed in Chapter 9 is used. The moving physical sensor thus directly measures the true virtual error signal $e_v(n)$ at the moving virtual location, which is then minimised using the filtered-x RLS algorithm. Note that very small differences between the experimental results presented in Chapter 9 and the experimental results presented in this section will be observed because these results have been derived from real-time experiments conducted on different days. For the case of a spatially fixed virtual sensor, the Kalman filter based spatially fixed virtual sensing algorithm is implemented for a virtual distance $v = 0.020 \text{ m}$ as described in Chapter 8 and illustrated in Fig. 8.1. This algorithm thus computes a current estimate $\hat{e}_v(n|n)$ of the virtual error signal at the spatially fixed virtual sensor located at $v = 0.020 \text{ m}$, which is then minimised using the filtered-x RLS algorithm. Note that for the case of a spatially fixed virtual sensor, small differences between the experimental results presented in Chapter 8 and the experimental results presented here will be observed for the same reasons mentioned previously. The real-time narrowband control results presented in the following are generated by averaging the results of 30 data-sets, each 10 s long. These data-sets are measured with the traversing microphone which is position controlled to track the moving virtual location $x_v(n)$ inside the acoustic duct arrangement. Furthermore, the average narrowband attenuations in dB are low-pass filtered in order to prevent noisy plots.

**Slow moving virtual sensor with period $T_v = 10 \text{ s}$**

In this section, the period of the sinusoidally time-varying virtual distance $v(n)$ defined in Eq. (10.10) that governs the speed $\dot{x}_v(n) = \dot{v}(n)$ of the moving virtual location is given by $T_v = 10 \text{ s}$. The results of the real-time experiments are illustrated in Fig. 10.2, where the narrowband attenuation and the control filter coefficients $w(n)$ are plotted against time for the three excitation frequencies considered.

The results in Fig. 10.2 illustrate that for all of the three excitation frequencies considered, the narrowband attenuation achieved when using a moving virtual sensor is always larger than when using a spatially fixed virtual sensor when the virtual location $x_v(n)$ moves away from the virtual distance $v = 0.020 \text{ m}$. This observation can be explained by noting that in Fig. 10.2, the control filter coefficients $w(n)$ for the case of a moving virtual sensor are time-varying while for the case of a spatially fixed virtual sensor, the control filter coefficients are relatively time-invariant. This indicates that as the virtual location $x_v(n)$ moves away from a virtual distance $v = 0.020 \text{ m}$, the tracking provided by the filtered-x RLS algorithm when implemented as illustrated in Fig. 10.1 is fast enough to account for the variations in the relative magnitude and phase between the estimated primary and secondary sound fields. When a spatially fixed virtual sensor
Figure 10.2: Narrowband control performance at the moving virtual location $x_v(n)$ when using either a spatially fixed virtual sensor at $v = 0.020$ m, a moving virtual sensor, or a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. For the moving virtual sensor case, the practical implementation of the Kalman filter based moving virtual sensing algorithm is used. The virtual location is moving sinusoidally with period $T_v = 10$ s, and the filtered-x RLS algorithm is implemented, with — moving virtual sensor — moving physical sensor — spatially fixed virtual sensor.
is used, the filtered-x RLS algorithm is implemented as illustrated in Fig. 8.1, and the
required tracking cannot be achieved for this case. As a result, when using a moving
virtual sensor instead of a spatially fixed virtual sensor, an additional narrowband
attenuation of 14 dB is achieved at \( v = 0.120 \text{ m} \) for an excitation frequency of 213 Hz,
23 dB for an excitation frequency of 249 Hz, and 26 dB for an excitation frequency of
284 Hz. This illustrates the improvement in the local control performance that can be
obtained when using the practical implementation of the Kalman filter based moving
virtual sensing algorithm developed in Chapter 5 instead of the Kalman filter based
spatially fixed virtual sensing algorithm developed in Chapter 3.

For all of the three excitation frequencies considered, the results in Fig. 10.2 illustrate
that the narrowband attenuation achieved when using a moving physical sensor is
generally larger than when using a moving virtual sensor. An exception occurs for an
excitation frequency of 249 Hz, where the narrowband attenuations for the moving
physical and virtual sensor cases are nearly identical, as shown in in Fig. 10.2(c), at times
when the control filter coefficients for these cases intersect, as shown in Fig. 10.2(d).
The observation that the narrowband attenuation that is achieved when using a moving
physical sensor is generally larger than when using a moving virtual sensor can be
explained as follows. It has been observed in Chapter 8 that the real-time estimation
performance of the Kalman filter based spatially fixed virtual sensing algorithm is limited
due to measurement noise on the sensors and the causality constraint that is put on the
estimation of the virtual primary disturbance given the physical primary disturbance.
In the practical implementation of the Kalman filter based moving virtual sensing
algorithm illustrated in Fig. 10.1, this thus limits the accuracy of the current estimate
\( \hat{e}_v(n|n) \) of the virtual error signals at the spatially fixed virtual locations \( \bar{x}_v \). As a result,
the estimate \( \hat{e}_v(n) \) of the virtual error signal at the moving virtual location \( x_v(n) \) is
not perfectly equal to the true virtual error signal \( e_v(n) \), which is directly measured
when using a moving physical sensor. Note that even if a perfect current estimate
\( \hat{e}_v(n|n) \) is computed, a perfect estimate \( \hat{e}_v(n) \) is still not obtained because this signal is
computed using linear spatial interpolation between the current estimate \( \hat{e}_v(n|n) \) of the
virtual error signals. This explains why the narrowband attenuation achieved when
using a moving physical sensor is generally larger in Fig. 10.2 than when using a moving
virtual sensor. However, the disadvantage of using a moving physical sensor is that
in a practical situation, an observer would need to wear actual microphones close to
their ears. These microphones also need to track the locations of the observer’s ear,
which is probably most easily achieved in practice by mounting the microphones on
the observer. This will generally not be possible or at least be very inconvenient in
which case the developed moving virtual sensing method can be used.
Medium pace moving virtual sensor with period $T_v = 5\,\text{s}$

In this section, the speed of the moving virtual location is increased by reducing the period of the sinusoidally time-varying virtual distance $v(n)$ defined in Eq. (10.10) to $T_v = 5\,\text{s}$. The results of the real-time experiments are illustrated in Fig. 10.3, where the narrowband attenuation and the control filter coefficients $w(n)$ are plotted against time for each of the three excitation frequencies considered.

The results in Fig. 10.3 again illustrate that when the virtual location $x_v(n)$ moves away from the virtual distance $v = 0.020\,\text{m}$, the narrowband attenuation achieved when using a moving virtual sensor is always larger than when using a spatially fixed virtual sensor for all of the excitation frequencies considered. This indicates that when the speed of the moving virtual location is increased by reducing the period to $T_v = 5\,\text{s}$, the tracking provided by the filtered-x RLS algorithm when implemented as illustrated in Fig. 10.1 is still fast enough to account for the variations in the relative magnitude and phase between the estimated primary and secondary sound fields. As a result, when using the moving physical sensor instead of a spatially fixed virtual sensor, an additional narrowband attenuation of $14\,\text{dB}$ is achieved at $v = 0.120\,\text{m}$ for an excitation frequency of $213\,\text{Hz}$, $23\,\text{dB}$ for an excitation frequency of $249\,\text{Hz}$, and $24\,\text{dB}$ for an excitation frequency of $284\,\text{Hz}$.

For all of the three excitation frequencies considered, the results in Fig. 10.3 again illustrate that the narrowband attenuation achieved when using a moving physical sensor is generally larger than when using a moving virtual sensor. As discussed previously, this is because the estimate $\hat{e}_v(n)$ of the virtual error signal at the moving virtual location $x_v(n)$ is not perfectly equal to the true virtual error signal $e_v(n)$, which is directly measured during real-time control when using a moving physical sensor. An exception again occurs for an excitation frequency of $249\,\text{Hz}$, for which the narrowband attenuations obtained for the moving physical and virtual sensor cases are nearly identical, as shown in Fig. 10.3(c), at times when the control filter coefficients for these cases intersect, as shown in Fig. 10.3(d). Another exception occurs for an excitation frequency of $284\,\text{Hz}$, for which the narrowband attenuation in Fig. 10.3(e) is larger when using a moving virtual sensor instead of a moving physical sensor in the vicinity of $t = 4.3\,\text{s}$ and $9.3\,\text{s}$. Note that at these times, the moving virtual location passes through a node in the sound field located around a virtual distance $v = 0.030\,\text{m}$ for an excitation frequency of $284\,\text{Hz}$. As discussed in Chapter 9, the tracking that is required from the filtered-x RLS algorithm is highly demanding when this occurs. This can also be observed in Fig. 10.3(f), where the control filter coefficients $w(n)$ for the moving physical and virtual sensor cases change rapidly when the moving virtual location passes through the node around $t = 4.3\,\text{s}$ and $9.3\,\text{s}$. The results thus indicate that when using a moving virtual sensor instead of a moving physical sensor, the filtered-x RLS algorithm provides a better approximation of the optimal time-varying
Chapter 10  Narrowband active noise control at a moving virtual sensor

Figure 10.3: Narrowband control performance at the moving virtual location $x_v(n)$ when using either a spatially fixed virtual sensor at $v = 0.020$ m, a moving virtual sensor, or a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. For the moving virtual sensor case, the practical implementation of the Kalman filter based moving virtual sensing algorithm is used. The virtual location is moving sinusoidally with period $T_v = 5$ s, and the filtered-x RLS algorithm is implemented, with — moving virtual sensor — moving physical sensor — spatially fixed virtual sensor.
solution for the control filter coefficients around these times. Note that this is not the case in the vicinity of \( t = 0.7 \) s and \( 5.7 \) s, when the moving virtual location also passes through the node in the sound field. Therefore, a likely explanation as to why a larger narrowband attenuation is obtained around \( t = 4.3 \) s and \( 9.3 \) s when using a moving virtual sensor instead of a moving physical sensor is that this is a lucky coincidence that can occur when the tracking that is required from the filtered-x RLS algorithm is highly demanding.

**Fast moving virtual sensor with period** \( T_v = 2.5 \) s

In this section, the speed of the moving virtual location is further increased by reducing the period of the sinusoidally time-varying virtual distance \( v(n) \) defined in Eq. (10.10) to \( T_v = 2.5 \) s. The results of the real-time narrowband control experiments are illustrated in Fig. 10.4, where the narrowband attenuation and the control filter coefficients \( w(n) \) are plotted against time for each of the excitation frequencies considered.

For all excitation frequencies considered, the results in Fig. 10.4 again illustrate that when the virtual location \( x_v(n) \) moves away from the virtual distance \( v = 0.020 \) m, the narrowband attenuation achieved when using a *moving virtual sensor* is always larger than when using a *spatially fixed virtual sensor* located at this virtual distance. This indicates that when the speed of the moving virtual location is further increased by reducing the period to \( T_v = 2.5 \) s, the tracking provided by the filtered-x RLS algorithm when implemented as illustrated in Fig. 10.1 is still fast enough to account for the variations in the relative magnitude and phase between the estimated primary and secondary sound fields. As a result, when using a moving virtual sensor instead of a spatially fixed virtual sensor, an additional narrowband attenuation of 14 dB is achieved at \( v = 0.120 \) m for an excitation frequency of 213 Hz, 23 dB for an excitation frequency of 249 Hz, and 23 dB for an excitation frequency of 284 Hz.

The results in Fig. 10.4 again illustrate that the narrowband attenuation achieved when using a *moving physical sensor* is generally larger than when using a *moving virtual sensor* for all excitation frequencies considered. Again, it can be seen in Fig. 10.3(c) that an exception occurs for an excitation frequency of 249 Hz, where the narrowband attenuations obtained for the moving physical and virtual sensor cases are nearly identical at times when the control filter coefficients for these cases intersect, as shown in Fig. 10.3(d). The exception that has been observed for a period \( T_v = 5 \) s and an excitation frequency of 284 Hz also occurs in Fig. 10.3(e), where the narrowband attenuation when using a moving virtual sensor instead of a moving physical sensor is larger in the vicinity of \( t = 2.0 \) s, 4.5 s, 7.0 s and 9.5 s. Similarly to the discussion presented previously, this is not the case around \( t = 0.5 \) s, 3.0 s, 5.5 s and 8.0 s when the moving virtual location also passes through the node in the sound field.
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Figure 10.4: Narrowband control performance at the moving virtual location $x_v(n)$ when using either a spatially fixed virtual sensor at $v = 0.020$ m, a moving virtual sensor, or a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. For the moving virtual sensor case, the practical implementation of the Kalman filter based moving virtual sensing algorithm is used. The virtual location is moving sinusoidally with period $T_v = 2.5$ s, and the filtered-x RLS algorithm is implemented, with — moving virtual sensor — moving physical sensor — spatially fixed virtual sensor.
Summary of the real-time narrowband control results

In Table 10.1, the time-average (mean) and the standard deviation of the real-time narrowband attenuations in dB that are obtained when using either a moving virtual sensor based on the practical implementation of the Kalman filter based moving virtual sensing algorithm illustrated in Fig. 10.1 or a moving physical sensor are compared for the three excitation frequencies and the three periods $T_v$ considered.

<table>
<thead>
<tr>
<th>$T_v$</th>
<th>213 Hz virtual</th>
<th>213 Hz physical</th>
<th>249 Hz virtual</th>
<th>249 Hz physical</th>
<th>284 Hz virtual</th>
<th>284 Hz physical</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 s</td>
<td>40(4.2)</td>
<td>57(2.0)</td>
<td>44(2.6)</td>
<td>52(2.8)</td>
<td>22(1.6)</td>
<td>37(8.1)</td>
</tr>
<tr>
<td>5 s</td>
<td>40(4.3)</td>
<td>55(2.3)</td>
<td>43(3.6)</td>
<td>49(3.7)</td>
<td>21(4.6)</td>
<td>32(10.3)</td>
</tr>
<tr>
<td>2.5 s</td>
<td>39(4.5)</td>
<td>52(2.9)</td>
<td>41(4.4)</td>
<td>45(4.1)</td>
<td>20(5.3)</td>
<td>27(11.1)</td>
</tr>
</tbody>
</table>

Table 10.1: Mean and standard deviation of the real-time narrowband attenuation in dB that is obtained when using either a moving physical sensor or a moving virtual sensor based on the practical implementation of the Kalman filter based moving virtual sensing algorithm.

The real-time experimental results listed in Table 10.1 show that for all of the combinations of the excitation frequencies and periods $T_v$ considered, the mean narrowband attenuation obtained at the moving virtual location is larger when using a moving physical sensor instead of a moving virtual sensor. As discussed previously, this is because the estimated primary and secondary sound fields at the moving virtual sensor, which are computed in this section using the practical implementation of the Kalman filter based moving virtual sensing algorithm illustrated in Fig. 10.1, are not exactly equal to the true primary and secondary sound fields, which are directly measured during real-time control when using a moving physical sensor.

The results in Table 10.1 indicate that when using a moving physical sensor, the time-average narrowband attenuation decreases and the standard deviation increases as the speed of the moving virtual location increases for all three excitation frequencies considered. This is because the tracking that is required from the filtered-x RLS algorithm is more demanding when the speed $\dot{x}_v(n)$ of the moving physical sensor that tracks the moving virtual location increases. The decrease in the mean narrowband attenuation is smallest for an excitation frequency of 213 Hz, i.e. 5 dB, because the spatial rate of change of the relative magnitude and phase between the primary and secondary sound fields is smallest for this frequency, as shown in Fig. 9.2. The standard deviations and the decrease in the mean narrowband attenuation are largest in Table 10.1 for an excitation frequency of 284 Hz, where the mean narrowband attenuation decreases by 10 dB as the speed of the moving physical sensor increases. This is because at this

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frequency, the moving physical sensor passes through a node in the sound field. When
this occurs, the real-time narrowband attenuations in Figs 10.2(e), 10.3(e) and 10.4(e)
decline significantly because the filtered-x RLS algorithm is not able to provide fast
enough tracking to account for the rapid variations in the relative magnitude and phase
between the primary and secondary sound fields.

The results in Table 10.1 indicate that when using a moving virtual sensor, a decrease
in the time-average narrowband attenuation and an increase in the standard deviation
is observed as the speed \( \dot{x}_{v}(n) \) of the moving virtual location increases. This is to be
expected because the temporal rate of change of the relative magnitude and phase
between the estimated primary and secondary sound fields increases when the virtual
location is moving at a higher speed through the acoustic duct arrangement. This
makes the tracking that is required from the filtered-x RLS algorithm more demanding,
which results in the observed decrease in the time-average narrowband attenuation and
increase in the standard deviation.

10.4  Adaptive LMS moving virtual microphone technique

In this section, the adaptive LMS virtual microphone technique is used in Fig. 10.1 instead
of the Kalman filter based spatially fixed virtual sensing algorithm to compute an estimate of
the virtual error signals \( \bar{e}_{v}(n) \) at the spatially fixed virtual locations \( \bar{x}_{v} \) defined
in Eq. (10.6). To implement this technique on the acoustic duct arrangement, optimal
physical sensor weights need to be estimated in a preliminary identification stage, which
is described in Section 10.4.1. In Section 10.4.2, the narrowband control performance
that is obtained at the moving virtual sensor is analysed for various speeds \( \dot{x}_{v}(n) \)
of the moving virtual location and various spatial characteristics of the narrowband
sound field inside the acoustic duct arrangement. As in Chapter 7, a sample frequency
\( f_s = 4.0 \text{ kHz} \) is employed in the real-time experiments.

10.4.1  Preliminary identification stage

The adaptive LMS virtual microphone technique, which is used in Fig. 10.1 instead of the
Kalman filter based spatially fixed virtual sensing algorithm to compute an estimate of the
virtual error signals \( \bar{e}_{v}(n) \) at the spatially fixed virtual locations \( \bar{x}_{v} \), is derived in a
preliminary identification stage. This preliminary identification stage was described in
Section 7.3.2, where experimental optimal physical sensor weights for the primary and
secondary sound fields were calculated while exciting either the primary or control
loudspeaker, respectively, with a bandlimited white noise signal in the frequency range
of 50–500 Hz. As discussed in that section, the experimental optimal weights illustrated
in Fig. 7.8 are very similar for the primary and secondary sound fields when using the
two physical sensor configuration. The adaptive LMS virtual microphone technique is
Adaptive LMS moving virtual microphone technique

therefore implemented to compute an estimate \( \hat{e}_v(n) \in \mathbb{R}^{16} \) of the virtual error signals at the spatially fixed virtual locations \( \hat{x}_v \) defined in Eq. (10.6) as

\[
\hat{e}_v(n) = \hat{H}_{uo}^T e_p(n),
\]

with \( \hat{H}_{uo} \in \mathbb{R}^{2 \times 16} \) the matrix of optimal physical sensor weights for the secondary sound field. These weights are thus directly applied to the physical error signals \( e_p(n) \in \mathbb{R}^2 \) at the two physical sensors located at \( x_{p1} = 1.4250 \text{ m} \) and \( x_{p5} = 1.4750 \text{ m} \) when computing the estimate \( \hat{e}_v(n) \). This implementation was successfully used in Chapter 7 to move the zone of quiet away from the physical sensors towards a spatially fixed virtual sensor.

10.4.2 Real-time experimental results

As in Section 10.3, the real-time narrowband control performance that is obtained at the moving virtual location \( x_v(n) \) when using a moving virtual sensor is compared with the real-time narrowband control performance that is obtained when using either a moving physical sensor that tracks the moving virtual location or a spatially fixed virtual sensor located at a virtual distance \( v = 0.020 \text{ m} \). The difference with the real-time experiments presented in Section 10.3 is that estimates of the virtual error signals at the spatially fixed virtual locations \( \hat{x}_v \), which includes the virtual distance \( v = 0.020 \text{ m} \), are now computed using the adaptive LMS virtual microphone technique instead of the Kalman filter based spatially fixed virtual sensing algorithm. For the spatially fixed virtual sensor case, small differences between the results presented here and the results presented in Chapter 7 will occur because these results have been derived from real-time experiments conducted on different days. For the same reason, for the case of a moving physical sensor, small differences between the results presented in Chapter 9 and Section 10.3 and the results presented here will occur. As in Section 10.3, each of the presented real-time narrowband control results is generated by averaging the results of 30 data-sets, each 10 s long. These data-sets are measured with the traversing microphone that is position controlled to track the moving virtual distance \( v(n) \). Furthermore, the average narrowband attenuations in dB are low-pass filtered in order to prevent noisy plots.

**Slow moving virtual sensor with period \( T_v = 10 \text{ s} \)**

In this section, the period of the sinusoidally time-varying virtual distance \( v(n) \) defined in Eq. (10.10) that governs the speed of the moving virtual location is given by \( T_v = 10 \text{ s} \). The results of the real-time narrowband control experiments are illustrated in Fig. 10.5, where the narrowband attenuation and the control filter coefficients \( w(n) \) are plotted against time for each of the excitation frequencies considered.

Fig. 10.5 shows that the obtained narrowband attenuations change with virtual distance in an almost identical way to the real-time experimental results presented.
Figure 10.5: Narrowband control performance at the moving virtual location $x_0(n)$ when using either a spatially fixed virtual sensor at $v = 0.020 \text{ m}$, a moving virtual sensor, or a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. For the moving virtual sensor case, the practical implementation of the adaptive LMS moving microphone technique is used. The virtual location is moving sinusoidally with period $T_v = 10 \text{ s}$, and the filtered-x RLS algorithm is implemented, with --- moving virtual sensor — moving physical sensor ••• spatially fixed virtual sensor.

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in Fig. 7.16(b), where the narrowband attenuations that were obtained for the case of a spatially fixed virtual sensor are plotted against the virtual distance for the three excitation frequencies considered. This indicates that when the speed of the moving virtual location is governed by a period $T_v = 10\, \text{s}$, the tracking provided by the filtered-x RLS algorithm is fast enough to account for the variations in the relative magnitude and phase between the estimated primary and secondary sound fields. As a result, when using a moving virtual sensor instead of the spatially fixed virtual sensor, an additional narrowband attenuation of 25 dB is achieved at $v = 0.120\, \text{m}$ for an excitation frequency of 213 Hz, 17 dB for an excitation frequency of 249 Hz, and 28 dB for an excitation frequency of 284 Hz. This illustrates the improvement in the local control performance that can be obtained when using the practical implementation of the adaptive LMS moving virtual microphone technique developed in Chapter 5 instead of the adaptive LMS spatially fixed virtual microphone technique [14] implemented on the acoustic duct arrangement in Chapter 7.

For all excitation frequencies considered, the results in Fig. 10.5 illustrate that the narrowband attenuation is larger when using a moving physical sensor instead of a moving virtual sensor, which was also observed in the previous section. This is because the accuracy of the estimated primary and secondary sound fields at the moving virtual sensor is limited by the estimation accuracy of the adaptive LMS virtual microphone technique, which has been analysed in Chapter 7, and by the accuracy of the linear spatial interpolation method that is used. However, as discussed previously, a disadvantage of using a moving physical sensor is that in a practical situation, an observer would need to wear actual microphones close to their ears. These microphones also need to track the locations of the observer’s ear, which is probably most easily achieved in practice by mounting the microphones on the observer. This will generally not be possible or at least be very inconvenient in which case the developed moving virtual sensing method can be used.

Finally, note that the control filter coefficients $w(n)$ for the moving physical sensor case illustrated in Fig. 10.5 are different to the ones illustrated in Fig. 10.2. This is because a sample frequency $f_s = 4.0\, \text{kHz}$ is used in the real-time experiments presented here, whereas a sample frequency $f_s = 1.6\, \text{kHz}$ was used in the real-time experiments presented in Section 10.3.

### Medium pace moving virtual sensor with period $T_v = 5\, \text{s}$

In this section, the speed of the moving virtual location is increased by reducing the period of the sinusoidally time-varying virtual distance $v(n)$ defined in Eq. (10.10) to $T_v = 5\, \text{s}$. The results of the real-time narrowband control experiments are illustrated in Fig. 10.6, where the narrowband attenuation and the control filter coefficients $w(n)$ are plotted against time for each of the excitation frequencies considered.
Figure 10.6: Narrowband control performance at the moving virtual location $x_v(n)$ when using either a spatially fixed virtual sensor at $v = 0.020$ m, a moving virtual sensor, or a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. For the moving virtual sensor case, the practical implementation of the adaptive LMS moving microphone technique is used. The virtual location is moving sinusoidally with period $T_v = 5$ s, and the filtered-x RLS algorithm is implemented, with moving virtual sensor — moving physical sensor — spatially fixed virtual sensor.
10.4 Adaptive LMS moving virtual microphone technique

As observed previously for a period $T_v = 10\,\text{s}$, it can be seen in Fig. 10.6 that the obtained narrowband attenuations change with virtual distance almost identically to the real-time experimental results presented in Fig. 7.16(b), where the narrowband attenuations that were obtained for the case of a spatially fixed virtual sensor were plotted against virtual distance for the three excitation frequencies considered. This indicates that when the speed of the moving virtual location is increased by reducing the period to $T_v = 5\,\text{s}$, the tracking provided by the filtered-x RLS algorithm when using a moving virtual sensor is still fast enough to account for the variations in the relative magnitude and phase between the estimated primary and secondary sound fields. As a result, when using a moving virtual sensor instead of the spatially fixed virtual sensor, an additional narrowband attenuation of $25\,\text{dB}$ is achieved at $v = 0.120\,\text{m}$ for an excitation frequency of $213\,\text{Hz}$, $17\,\text{dB}$ for an excitation frequency of $249\,\text{Hz}$, and $28\,\text{dB}$ for an excitation frequency of $284\,\text{Hz}$.

For all excitation frequencies considered, the results in Fig. 10.6 again illustrate that the narrowband attenuation achieved when using a moving physical sensor is larger than when a moving virtual sensor is used. As discussed previously, this is because the estimated primary and secondary sound fields at the moving virtual sensor are not perfectly equal to the true primary and secondary sound fields computed using the practical implementation of the adaptive LMS moving virtual microphone technique.

**Fast moving virtual sensor with period $T_v = 2.5\,\text{s}$**

In this section, the speed of the moving virtual location is further increased by reducing the period of the sinusoidally time-varying virtual distance $v(n)$ defined in Eq. (10.10) to $T_v = 2.5\,\text{s}$. The results of the real-time narrowband control experiments are illustrated in Fig. 10.7, where the narrowband attenuation and the control filter coefficients $w(n)$ are plotted against time for each of the excitation frequencies considered.

As observed before for a period $T_v$ of $10\,\text{s}$ and $5\,\text{s}$, it can be seen that the obtained narrowband attenuations illustrated in Fig. 10.7 change with virtual distance in an almost identical way as illustrated in Fig. 7.16(b), where the narrowband attenuation obtained at a spatially fixed virtual sensor were plotted against virtual distance. This indicates that when the speed of the moving virtual location is further increased by reducing the period to $T_v = 2.5\,\text{s}$, the tracking provided by the filtered-x RLS algorithm is still fast enough to account for the variations in the relative magnitude and phase between the estimated primary and secondary sound fields. When using a moving virtual sensor instead of the spatially fixed virtual sensor, this results in an additional narrowband attenuation of $25\,\text{dB}$ at $v = 0.120\,\text{m}$ for an excitation frequency of $213\,\text{Hz}$, $17\,\text{dB}$ for an excitation frequency of $249\,\text{Hz}$, and $28\,\text{dB}$ for an excitation frequency of $284\,\text{Hz}$. 

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Figure 10.7: Narrowband control performance at the moving virtual location $x_v(n)$ when using either a spatially fixed virtual sensor at $v = 0.020$ m, a moving virtual sensor, or a moving physical sensor for excitation frequencies of 213 Hz, 249 Hz, and 284 Hz. For the moving virtual sensor case, the practical implementation of the adaptive LMS moving microphone technique is used. The virtual location is moving sinusoidally with period $T_v = 2.5$ s, and the filtered-$x$ RLS algorithm is implemented, with $\text{---}$ moving virtual sensor $\text{—}$ moving physical sensor $\text{—-}$ spatially fixed virtual sensor.
10.4 Adaptive LMS moving virtual microphone technique

For all excitation frequencies considered, the results in Fig. 10.7 illustrate that the narrowband attenuation achieved when using a moving physical sensor is generally larger than when using a moving virtual sensor, which was also observed for the two periods \( T_v \) considered previously. Note that an exception occurs for an excitation frequency of 249 Hz, where the narrowband attenuations obtained for the moving physical and virtual sensor cases are nearly identical, as shown in Fig. 10.6(c), at times when the control filter coefficients for these cases intersect, as shown in Fig. 10.6(d). Another exception occurs for an excitation frequency of 284 Hz, for which the narrowband attenuation in Fig. 10.6(e) is larger in the vicinity of \( t = 2.0, 4.5, 7.0, 9.5 \) s when using a moving virtual sensor instead of a moving physical sensor. This was also observed in Section 10.3, where it was mentioned that at these times, the moving virtual location passes through a node in the sound field located in the vicinity of a virtual distance \( v = 0.030 \) m for an excitation frequency of 284 Hz. Note again that the narrowband attenuation when using a moving virtual sensor instead of a moving physical sensor is not larger in the vicinity of \( t = 0.5, 3.0, 5.5, 8.0 \) s, when the moving virtual location also passes through the node in the sound field.

**Summary of real-time narrowband control results**

In Table 10.2, the time-average (mean) and standard deviation of the narrowband attenuation in dB that is obtained at the moving virtual location when using either a moving virtual sensor based on the practical implementation of the adaptive LMS moving virtual microphone technique or a moving physical sensor are compared for all the combinations of the excitation frequencies and periods \( T_v \) considered.

<table>
<thead>
<tr>
<th>( T_v )</th>
<th>( 213 ) Hz</th>
<th>( 249 ) Hz</th>
<th>( 284 ) Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( v )</td>
<td>( p )</td>
<td>( v )</td>
</tr>
<tr>
<td>10 s</td>
<td>46(1.1)</td>
<td>59(1.7)</td>
<td>41(5.8)</td>
</tr>
<tr>
<td>5 s</td>
<td>46(1.3)</td>
<td>56(1.9)</td>
<td>40(5.9)</td>
</tr>
<tr>
<td>2.5 s</td>
<td>45(2.2)</td>
<td>55(2.1)</td>
<td>39(6.1)</td>
</tr>
</tbody>
</table>

Table 10.2: Mean and standard deviation of the real-time narrowband attenuation in dB that is obtained when using either a moving physical sensor or a moving virtual sensor based on the practical implementation of the adaptive LMS moving virtual microphone technique.

Note that the results for the moving physical sensor cases listed in Tables 10.1 and 10.2 are slightly different, but very similar, because these results have been computed from real-time experiments conducted on different days. The results in Table 10.2 show that for all of the nine combinations of the excitation frequencies and periods \( T_v \).
considered, the time-average narrowband attenuation obtained at the moving virtual location is larger when using a moving physical sensor instead of a moving virtual sensor. Note that this was also observed in Table 10.1 where the results for the practical implementation of the Kalman filter based moving virtual sensing algorithm were listed. Similarly to the discussion presented in that section, the time-average narrowband attenuation is larger when using a moving physical sensor because the estimated primary and secondary sound fields at the moving virtual location, which are computed in this section using the practical implementation of the adaptive LMS moving virtual microphone technique, are not perfectly equal to the true primary and secondary sound fields, which are directly measured during real-time control when using a moving physical sensor.

The results in Table 10.2 indicate that as the speed of the moving virtual location increases, the time-average narrowband attenuations slightly decrease and the standard deviations slightly increase. This was also observed in Table 10.1 and can be explained by noting that the tracking required from the filtered-x RLS algorithm is more demanding as the virtual location moves more quickly through the sound field inside the acoustic duct arrangement.

### 10.5 Comparison

In Table 10.3, the time-average (mean) and standard deviation of the narrowband attenuations in dB that are obtained at the moving virtual location when using the practical implementation of either the Kalman filter based moving virtual sensing algorithm or the adaptive LMS moving virtual microphone technique are compared for the nine combinations of the excitation frequencies and period $T_v$ considered.

<table>
<thead>
<tr>
<th>$T_v$</th>
<th>213 Hz</th>
<th>249 Hz</th>
<th>284 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kalman</td>
<td>LMS</td>
<td>Kalman</td>
<td>LMS</td>
</tr>
<tr>
<td>10 s</td>
<td>40(4.2)</td>
<td>46(1.1)</td>
<td>44(2.6)</td>
</tr>
<tr>
<td>5 s</td>
<td>40(4.3)</td>
<td>46(1.3)</td>
<td>43(3.6)</td>
</tr>
<tr>
<td>2.5 s</td>
<td>39(4.5)</td>
<td>45(2.2)</td>
<td>41(4.4)</td>
</tr>
</tbody>
</table>

Table 10.3: Mean and standard deviation of the real-time narrowband attenuation in dB that is obtained when using a moving virtual sensor based on the practical implementation of either the Kalman filter based moving virtual sensing algorithm or the adaptive LMS moving virtual microphone technique.

The results in Table 10.3 indicate that for an excitation frequency of 213 Hz, the practical implementation of the adaptive LMS moving virtual microphone technique
provides an additional mean narrowband attenuation of 6 dB over the practical implementation of the Kalman filter based moving virtual sensing algorithm, and a smaller standard deviation. For excitation frequencies of 249 Hz and 284 Hz, additional mean narrowband attenuations of about 2–4 dB and smaller standard deviations are obtained when using the practical implementation of the Kalman filter based moving virtual sensing algorithm instead of the adaptive LMS moving virtual microphone technique. This indicates that at these frequencies, a better estimate \( \hat{e}_v(n) \) of the virtual error signal at the moving virtual location is obtained when using the practical implementation of the Kalman filter based moving virtual sensing algorithm illustrated in Fig. 10.1. Note that the experimental broadband estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm, which was analysed in Chapter 8, is better than the experimental broadband estimation performance of the adaptive LMS spatially fixed virtual microphone technique, which was analysed in Chapter 7. It is therefore expected that for excitation frequencies other than the ones considered here, the practical implementation of the Kalman filter based moving virtual sensing algorithm illustrated in Fig. 10.1 will generally provide better narrowband control performance than the practical implementation of the adaptive LMS moving virtual microphone technique.

## 10.6 Conclusion

In this chapter, the adaptive feedforward control approach developed in Chapter 5 and illustrated in Fig. 5.5 for local active noise control at a moving virtual sensor has been implemented on the acoustic duct arrangement. The presented real-time experimental results illustrated that a moving zone of quiet at a virtual sensor that tracks the desired location of maximum attenuation has successfully been created for narrowband disturbances inside the acoustic duct arrangement. An estimate of the virtual error signal at the moving virtual location has been computed using the practical implementation of either the Kalman filter based moving virtual sensing algorithm or the adaptive LMS moving virtual microphone technique, which were developed in Chapter 5. The presented real-time narrowband control results illustrated the improvement in the local control performance that can be obtained when using these moving virtual sensing methods instead of either the Kalman filter based spatially fixed virtual sensing algorithm or the adaptive LMS spatially fixed virtual microphone technique [14], which were implemented on the acoustic duct arrangement in Chapters 7 and 8.

It has been shown that the narrowband control performance obtained at the moving virtual location inside the acoustic duct arrangement is generally larger when using a moving physical sensor instead of a moving virtual sensor. This is because the estimated primary and secondary sound fields at the moving virtual location, which have been computed using the practical implementation of either the Kalman filter based moving
virtual sensing algorithm or the adaptive LMS moving virtual microphone technique, are not perfectly equal to the true primary and secondary sound fields, which are measured directly during real-time control when using a moving physical sensor. However, the disadvantage of using a moving physical sensor is that in a practical situation, an observer would need to wear actual microphones close to their ears. These microphones also need to track the locations of the observer’s ears, which is probably most easily achieved in practice by mounting the microphones on the observer. This will generally not be possible or at least be a very inconvenient solution, in which case the moving virtual sensing method developed in Chapter 5 can be used. This method only requires a number of spatially fixed physical sensors that can be placed remotely from the target zone through which the observer’s head is moving. Furthermore, the presented experimental narrowband control results indicated that significant reductions have been obtained at the moving virtual sensor, and that a moving zone of quiet that tracks the desired location of maximum attenuation has successfully been created inside the acoustic duct arrangement without directly measuring the error signal at this location during real-time control.
Chapter 11

Conclusions & Future research

11.1 Conclusions

Local active noise control systems aim to create a zone of quiet at some desired location within an acoustic sound field. The created zone of quiet, in which the noise is reduced by 10 dB or more, is generally centred at the error sensor. For an observer to experience a significant reduction in the noise, the error sensor therefore usually has to be placed relatively close to an observer’s ear, which is not always a feasible solution or at least very inconvenient. This problem is further exacerbated by the fact that the size of the created zone of quiet tends to be small, especially for higher frequencies. Virtual sensing methods have been proposed to overcome these problems that have limited the scope of successful local active noise control applications. These methods require a non-intrusive sensor that is placed remotely from the desired location of maximum attenuation. This sensor is used to provide an estimate of the sound pressure at the remote location, which can then be minimised by a local active noise control system. This effectively moves the zone of quiet away from the physical location of the transducers to the desired location of maximum attenuation, such as a person’s ear.

A number of virtual sensing algorithms for local active noise control have been proposed in previous research, e.g. the virtual microphone arrangement [27], the remote microphone technique [104, 112], the adaptive LMS virtual microphone technique [14], the hybrid adaptive feedforward observer [119], and the secondary path equalisation method [66]. The common aim of these algorithms is to compute an accurate estimate of the sound pressure at the virtual location without directly measuring it during real-time control. The difference between these algorithms is the way in which the estimate of the sound pressure is computed, i.e. the assumed structure is what makes the virtual sensing algorithms proposed so far different from one another. Once the structure has been chosen, optimal solutions for the unknown parameters of the algorithms can be computed by optimising the estimation performance as described in this thesis. An
important question that arises is whether there is an optimal structure that can be used to solve the virtual sensing for active noise control problem, which amounts to a linear estimation problem. It is well-known that the Kalman filter provides an optimal structure for solving linear estimation problems [60]. The first main contribution of this thesis has been the derivation of an optimal solution to the virtual sensing for active noise control problem using Kalman filtering theory.

The previously proposed virtual sensing algorithms have all been developed with the aim to create a zone of quiet at a virtual location that is assumed spatially fixed within the sound field. Because an observer is very likely to move their head, the desired location of the zone of quiet is generally moving through the sound field rather than being spatially fixed. For effective control, a local active noise control system incorporating a virtual sensing method thus has to be able to create a moving zone of quiet that tracks the observer’s ears. The second main contribution of this thesis has been the development of a moving virtual sensing method that can be used to compute an estimate of the sound pressure at a virtual location that is moving through the sound field. It has been shown that an optimal solution to the moving virtual sensing problem can be derived using Kalman filtering theory.

In the first part of this thesis, optimal spatially fixed and moving virtual sensing algorithms have been derived using Kalman filtering theory, and have been combined with local active noise control algorithms. The developed algorithms have been implemented on an acoustic duct arrangement in the second part of this thesis in order to verify their numerical and real-time performance in a practical implementation. The main conclusions of this thesis are now summarised.

- An optimal solution to the spatially fixed virtual sensing for active noise control problem has been developed using Kalman filtering theory. The proposed algorithm has been implemented on an acoustic duct arrangement to demonstrate its effectiveness in a practical implementation. The presented experimental results indicated that the developed method can be used to effectively move the zone of quiet away from the physical sensor towards the location where maximum attenuation is required. It was observed that the narrowband and broadband feedforward control performance obtained at the virtual location decreased as the distance between the physical and virtual sensors increased. This is because the estimation performance of the Kalman filter based spatially fixed virtual sensing algorithm decreased as the distance between the physical and virtual sensors increased. The reason for this is that the relationship between the stochastic primary sound pressures at the physical and virtual sensors becomes increasingly non-causal as the distance between these two sensors increases. Because only that part of the virtual primary sound pressure that is causally related to the physical primary sound pressure can be causally estimated from the current and previous
observations of the physical primary sound pressure, the estimation performance decreased as the distance between the physical and virtual sensors located inside the acoustic duct arrangement increased.

- A comprehensive analysis of the previously proposed spatially fixed virtual sensing algorithms [14, 27, 66, 104, 112, 119] has been presented based on a general state-space model of the considered local active noise control system. For each of these algorithms, the assumed estimation structure has been introduced and an optimal solution for the unknown parameters of the algorithm has been derived by optimising the estimation performance. The presented analysis can for instance be used in an initial numerical comparison of the optimal estimation performance that can be obtained with the previously proposed virtual sensing algorithms. The presented analysis also provides a straightforward interpretation of the factors that determine the optimal estimation performance of the previously proposed virtual sensing algorithms. This is for example useful when determining the locations and the number of physical sensors that are required to obtain a certain estimation performance.

- The previously proposed spatially fixed virtual sensing methods for local active noise control [14, 27, 66, 104, 112, 119] have predominantly been implemented in combination with an adaptive feedforward control approach. The optimal narrowband and broadband control performance that can in theory be obtained at the virtual locations when using this approach, in which an estimate of the sound pressures at the virtual locations is adaptively minimised, has been derived in this thesis. It has been shown that this performance is always smaller than or equal to the optimal control performance that can in theory be obtained when adaptively minimising the true sound pressures at the virtual locations. The control performance that is lost has been theoretically quantified for each of the virtual sensing algorithms proposed previously [14, 27, 104, 112]. As expected, the presented analysis showed that the amount of control performance that is lost is determined by the estimation performance of the spatially fixed virtual sensing algorithm that is used.

- The optimal feedback control at virtual sensors problem has been solved by recognising that this problem is equivalent to the general feedforward/feedback control problem, which is a standard problem often encountered in active noise control [25].

- The adaptive LMS virtual microphone technique [14] was originally introduced for the case of only one spatially fixed virtual sensor. This technique has been extended to the case of multiple spatially fixed virtual sensors. Furthermore, an optimal solution for the unknown parameters of this technique has been derived given
a state-space model of the local active noise control system under consideration. This solution is especially useful in an initial numerical analysis of this technique because the unknown parameters no longer need to be determined in a computationally more intensive adaptive manner as was the case in previous research [14, 80]. The adaptive LMS virtual microphone technique has been implemented on an acoustic duct arrangement for both narrowband and broadband disturbances to demonstrate its effectiveness. Although the case of narrowband disturbances was also investigated in previous research [80], some additional insights have been presented in this thesis that explain the disagreement between the numerical and experimental results reported by these researchers for the case of using a three or five physical sensor configuration. It was found that the estimation problem was ill-conditioned for these configurations.

- **Forward difference prediction techniques**, which have been proposed and investigated previously [13, 80], have been analytically compared to the adaptive LMS virtual microphone technique for the case of single tone disturbances inside the acoustic duct arrangement. The analysis showed that the forward difference prediction weights approach the analytical optimal solutions for the weights for low frequencies and small distances between the physical and virtual sensors. The presented discussion explained the experimental results that were reported in previous research [80].

- An analysis of the *hybrid adaptive feedforward observer* [119] has been presented in which it has been shown that this method is only suitable for rejecting non-stationary disturbances at the physical sensors and not for spatially fixed virtual sensing purposes.

- An *optimal solution* to the moving virtual sensing for local active noise control problem has been derived using Kalman filtering theory. A practical implementation of the developed algorithm has been combined with an adaptive feedforward control algorithm and implemented on an acoustic duct arrangement. The experimental results showed that a moving zone of quiet at a moving virtual sensor that tracks the desired location of maximum attenuation has successfully been created for the case of narrowband disturbances.

- The narrowband control performance obtained at the moving virtual location inside the acoustic duct arrangement was generally found to be larger when using a *moving physical sensor* instead of a *moving virtual sensor*. This was because the *estimated* primary and secondary sound fields at the moving virtual locations, which were computed using the practical implementation of the Kalman filter based moving virtual sensing algorithm, were not perfectly equal to the true
primary and secondary sound fields directly measured by the moving physical sensor. However, the disadvantage of using a moving physical sensor is that in a practical situation, an observer would need to wear actual microphones close to their ears. These microphones also need to track the locations of the observer’s ear, which is probably most easily achieved in practice by mounting the microphones on the observer. This will generally not be possible or at least a very inconvenient solution in which case the developed moving virtual sensing methods can be used.

- The previously proposed spatially fixed virtual sensing algorithms [14, 27, 104, 112] have been modified to account for a virtual location that is moving through the sound field rather than being spatially fixed.

- It has been observed that when the virtual location is moving through the sound field, the primary sound pressure that needs to be attenuated is a non-stationary process. To account for this, an adaptive control algorithm that has the ability to track the statistical changes in the primary sound pressure at the moving virtual location is needed. In this thesis, the ability of a number of adaptive feedforward control algorithms, i.e. the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms, to track the non-stationarities in the true primary sound pressure at the moving virtual location, which was measured with a moving physical sensor, has been analysed and compared for various speeds of the moving physical sensor and various spatial characteristics of the narrowband sound field inside the acoustic duct arrangement. The experimental results showed that for the physical system examined here, the filtered-x RLS algorithm provided the best tracking performance. As a result, a narrowband moving zone of quiet at a moving physical sensor that tracks the desired location of maximum attenuation has effectively been created inside an acoustic duct arrangement.

- A more succinct expression for a previously proposed travelling wave model of an acoustic duct with arbitrary termination conditions [129] has been derived. It has been shown that for totally reflective termination conditions, the proposed travelling wave model reduces to the well-known expressions for the case of plane waves inside a rigidly terminated duct [88].

In summary, this thesis has proposed optimal spatially fixed and moving virtual sensing algorithms that can be used to improve the performance of local active noise control systems. The optimal virtual sensing algorithms have been derived using Kalman filtering theory and have been combined with local active noise control algorithms. The developed methods have been implemented on an acoustic duct arrangement. The presented real-time experimental results showed that the proposed spatially fixed
virtual sensing algorithm can be used to effectively move the zone of quiet away from the physical sensor to a desired location of maximum attenuation that is spatially fixed. The developed moving virtual sensing algorithm can be used to create a moving zone of quiet that tracks the desired location of maximum attenuation. The presented real-time experimental results illustrated that this has been achieved for narrowband disturbances inside an acoustic duct arrangement.

11.2 Future research

The research presented in this thesis has led to further research questions that are outlined in the following.

- In a practical application of the developed moving virtual sensing method, an important issue that needs to be addressed is how to determine the desired location of maximum attenuation, i.e. the moving virtual location. This could for instance be done using a 3D head tracking system based on camera vision, or a head tracking system based on ultrasonic position sensing such as the Logitech® head tracker. Such a measurement system then needs to be incorporated into the developed moving virtual sensing methods.

- In Chapter 9, the ability of the filtered-x LMS, normalised filtered-x LMS, and filtered-x RLS algorithms to track the non-stationarities in the primary sound pressure at a moving physical sensor that tracks the desired location of maximum attenuation has been analysed for the case of a narrowband sound field inside the acoustic duct arrangement. The ability of these algorithms to track these non-stationarities also needs to be analysed for the case of a broadband sound field. If sufficient tracking is obtained for this case, similar experiments as presented in Chapter 10, where a moving zone of quiet was successfully created at a virtual sensor that moved through a a narrowband sound field inside an acoustic duct arrangement, need to be conducted for the case of a broadband sound field.

- The Kalman filter based spatially fixed virtual sensing algorithm has been derived in Chapter 3 given a standard state-space model of the local active noise control system under consideration, which has been defined as

\[
\begin{align*}
\mathbf{z}(n+1) &= \mathbf{A} \mathbf{z}(n) + \mathbf{B}_u \mathbf{u}(n) + \mathbf{B}_s \mathbf{s}(n) \\
\mathbf{e}_p(n) &= \mathbf{C}_p \mathbf{z}(n) + \mathbf{D}_{pu} \mathbf{u}(n) + \mathbf{D}_{ps} \mathbf{s}(n) + \mathbf{v}_p(n) \\
\mathbf{e}_v(n) &= \mathbf{C}_v \mathbf{z}(n) + \mathbf{D}_{vu} \mathbf{u}(n) + \mathbf{D}_{vs} \mathbf{s}(n) + \mathbf{v}_v(n),
\end{align*}
\]

with \( \mathbf{e}_p(n) \) the physical error signals that are directly measured during real-time control, and \( \mathbf{e}_v(n) \) the virtual error signals that are not directly measured during
real-time control. In the acoustic duct experiments presented in Chapter 8, a virtual sound pressure sensor was effectively created using the developed Kalman filter based spatially fixed virtual sensing algorithm, which provided an estimate of the sound pressure $p_v(n)$ at the desired location of maximum attenuation. For this case, the virtual error signal $e_v(n)$ in Eq. (11.1) thus represents a sound pressure signal. However, the Kalman filter based spatially fixed virtual sensing algorithm has been derived in Chapter 3 given the general state-space model defined in Eq. (11.1), where the virtual error signals $e_v(n)$ can represent any unmeasured outputs of the considered acoustic system, and the physical error signals can represent any directly measured outputs of this system. As an example, suppose the state-space model in Eq. (11.1) describes the input-output behaviour of an acoustic system for which the virtual error signal is given by

$$e_v(n) = \begin{bmatrix} p_v(n) & \frac{\partial p_v(n)}{\partial x} & \frac{\partial p_v(n)}{\partial y} & \frac{\partial p_v(n)}{\partial z} \end{bmatrix}^T,$$

(11.2)

with $p_v(n)$ the sound pressure at the desired location of maximum attenuation, and the remaining signals the sound pressure gradients at this location in the three orthogonal directions. The derived Kalman filter based spatially fixed virtual sensing algorithm can then be used to compute an estimate of these signals, which effectively creates a virtual energy density sensor. Such a sensor can for instance be used to not only move the zone of quiet to the desired location of maximum attenuation, but to also enlarge the zone of quiet created at this location. To implement such a virtual energy density sensor, the Phone-Or optical energy density probe [15, 59] can for instance be used to measure the sound pressure and sound pressure gradients at the desired location of maximum attenuation in a preliminary identification stage.

- One of the arguments for using energy density sensing methods for global active noise control inside acoustic enclosures is that the acoustic energy density generally has a smaller spatial variance than the acoustic potential energy density in these types of sound fields [23]. If an estimate of the acoustic energy density cost function is obtained for these types of sound fields by measuring the acoustic energy density at a number of points throughout the sound field, observability problems, which are often encountered when employing a traditional sound pressure squared cost function as an estimate of the acoustic potential energy, are less likely to occur. However, it is important to note that this argument assumes that the particle velocity is directly measured and not estimated using an energy density sensor based on multiple sound pressure measurements [16, 115]. The question now arises as to whether the observability is still improved when using an energy density sensor that provides an estimate of the acoustic energy density based on multiple sound pressure measurements.
• In Chapter 6, a modal model [88] and a travelling wave model [129] of a rectangular acoustic duct have been presented, where the travelling wave model assumed arbitrary termination conditions while the modal model assumed rigid termination conditions. To the author’s knowledge, it has not been shown whether the modal model and travelling wave model are equivalent when rigid terminations are assumed in the travelling wave model.
Appendix A

Proof of causal Wiener solutions

A.1 Proof of causal Wiener Theorem 2.1

Theorem A.1 (Causal Wiener filter [34]).
Given the transfer function matrices $G_{vs} \in \mathcal{RH}_{\infty}^{M_v \times S}$, $G_{vu} \in \mathcal{RH}_{\infty}^{M_v \times L}$, and $G_{xs} \in \mathcal{RH}_{\infty}^{K \times S}$, and assuming that $G_{vu}$ and $G_{xs}$ do not have any zeros on the unit circle, the following inner-outer and outer-inner factorisations can be defined

\[ G_{vu} = G_{vu},i G_{vu},o \]  \hspace{1cm} (A.1)
\[ G_{xs} = G_{xs},co G_{xs},ci \]  \hspace{1cm} (A.2)

where $G_{vu},o$ has a stable right-inverse $G_{vu}^\dagger$, and $G_{xs},co$ has a stable left-inverse $G_{xs}^\dagger$. Furthermore, let $G_{vu},i$ and $G_{xs},ci$ be such that $[G_{vu},i \ G_{vu},i]^*$ and $[G_{xs},ci \ G_{xs},ci]^*$ are unitary. Then

\[ W_0 = -G_{vu,\rho}^\dagger \left( G_{vu,i}^* G_{vu} G_{xs,ci}^* \right) + G_{xs,co}^\dagger \]  \hspace{1cm} (A.3)

minimises

\[ J = \| G_{vs} + G_{vu} W G_{xs} \|_2^2, \quad \text{subject to } W \in \mathcal{RH}_{\infty}^{L \times K}, \]  \hspace{1cm} (A.4)

and its minimum value is given by

\[ J_{\text{min}} = \| G_{vu} G_{xs,ci}^\dagger \|_2^2 + \| G_{vu} G_{vs} G_{xs,ci}^* \|_2^2 + \| [G_{vu,i} G_{vs} G_{xs,ci}^*]^* - \|_2^2. \]  \hspace{1cm} (A.5)

Proof. A proof can be found in Vidyasagar [126], and an alternative proof presented by Fraanje [34] is included here. The cost function defined in Eq. (A.4) can be written in the frequency domain as

\[ J = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{vs} + G_{vu} W G_{xs}) (G_{vs} + G_{vu} W G_{xs})^* d\omega. \]  \hspace{1cm} (A.6)

Because $[G_{xs,ci}^* G_{xs,ci}^\dagger]^*$ is unitary, such that

\[ G_{xs,ci}^* G_{xs,ci} + G_{xs,ci}^\dagger G_{xs,ci} = I, \]  \hspace{1cm} (A.7)

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the following expression can be derived

\[ G_{\text{ds}} = G_{\text{ds}} G_{\text{xs,ci}}^* G_{\text{xs,ci}} + G_{\text{ds}} G_{\text{xs,ci}}^* G_{\text{xs,ci}} = G_{\text{ds},1} G_{\text{xs,ci}} + G_{\text{ds},2} G_{\text{xs,ci}} \]

where

\[ G_{\text{ds},1} = G_{\text{ds}} G_{\text{xs,ci}}^* G_{\text{xs,ci}} \quad G_{\text{ds},2} = G_{\text{ds}} G_{\text{xs,ci}}^*. \tag{A.8} \]

Using this expression, Eq. (A.6) can now also be written as

\[
J = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{\text{ds},1} G_{\text{xs,ci}} + G_{\text{ds},2} G_{\text{xs,ci}}^* + G_{\text{vu}} W G_{\text{xs,ci}} G_{\text{xs,ci}} \left( \cdot \right)^* d\omega
\]

\[
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{\text{ds},2} G_{\text{ds},2}^* d\omega + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{\text{ds},1} + G_{\text{vu}} W G_{\text{xs,ci}} \left( \cdot \right)^* d\omega
\]

\[
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{\text{ds},2} G_{\text{ds},2}^* d\omega + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{\text{ds},1} G_{\text{xs,ci}}^* G_{\text{xs,ci}}^* \left( \cdot \right)^* d\omega
\]

\[
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{\text{ds},1} G_{\text{xs,ci}}^* G_{\text{xs,ci}}^* \left( \cdot \right)^* d\omega
\]

\[
\tag{A.9}
\]

where \((G)(G)^*\) is denoted by \((G)(\cdot)^*\) for notational convenience, and where use has been made of the fact that \(\text{tr}(AB) = \text{tr}(BA)\). Because \(G_{\text{vu},i} G_{\text{vu},j}^*\) is also unitary, the following expression can be formulated

\[
G_{\text{ds}} = G_{\text{vu},i} G_{\text{vu},j} G_{\text{ds}} + G_{\text{vu},j}^* G_{\text{vu},i} G_{\text{ds}}
\]

\[
= G_{\text{vu},i} \tilde{G}_{\text{ds},1} + G_{\text{vu},j} \tilde{G}_{\text{ds},2}, \tag{A.10}
\]

where

\[
\tilde{G}_{\text{ds},1} = G_{\text{vu},i} G_{\text{ds}} \quad \tilde{G}_{\text{ds},2} = G_{\text{vu},j} G_{\text{ds}}. \tag{A.11}
\]

Using the expression defined in Eq. (A.10), Eq. (A.9) can now be written as

\[
J = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{\text{ds},1} G_{\text{vx,ci}}^* G_{\text{vx,ci}} d\omega +
\]

\[
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \left( \cdot \right)^* \left( G_{\text{vu},i} \tilde{G}_{\text{vx},1} G_{\text{vx,ci}}^* + G_{\text{vu},j} \tilde{G}_{\text{vx},2} G_{\text{vx,ci}}^* + G_{\text{vu},j} G_{\text{vu},j} W G_{\text{vx,ci}} \right) d\omega
\]

\[
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \left( G_{\text{vx,ci}}^* G_{\text{vx,ci}} G_{\text{vx,ci}} G_{\text{vx,ci}}^* \right) d\omega +
\]

\[
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \left( \cdot \right)^* \left( G_{\text{vx,ci}}^* G_{\text{vx,ci}} + G_{\text{vx,ci}} G_{\text{vx,ci}}^* + G_{\text{vu},i} W G_{\text{vx,ci}} \right) d\omega
\]

\[
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \left( G_{\text{vx,ci}}^* G_{\text{vx,ci}} G_{\text{vx,ci}} G_{\text{vx,ci}}^* \right) d\omega +
\]

\[
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \left( \cdot \right)^* \left( G_{\text{vu},i} G_{\text{vx,ci}} G_{\text{vx,ci}}^* + G_{\text{vu},j} G_{\text{vu},j} W G_{\text{vx,ci}} \right) d\omega. \tag{A.12}
\]
Because $W$ is constrained to be stable, $G_{vu,o}WG_{xs,co}$ is stable, such that
\[ G_{vu,o}WG_{ps,co} = [G_{vu,o}WG_{ps,co}]_+, \quad \forall W \in \mathcal{RH}_\infty^{L \times K}. \quad (A.13) \]
Eq. (A.12) can therefore also be written as
\[
J = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{xs,cl}^*G_{vs}G_{xs,cl}^* + G_{xs,cl}G_{vs}^*G_{vu,l}G_{vs}G_{xs,cl}^*) d\omega + \\
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* ([G_{vu,l}G_{vs}G_{xs,cl}^*]_+ + [G_{vu,l}G_{vs}G_{xs,cl}^*]_+ + G_{vu,o}WG_{xs,co}) d\omega \\
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{xs,cl}^*G_{vs}G_{xs,cl}^* + G_{xs,cl}G_{vs}^*G_{vu,l}G_{vs}G_{xs,cl}^* + \\
\quad [G_{vu,l}G_{vs}G_{xs,cl}^*]_+[G_{vu,l}G_{vs}G_{xs,cl}^*]_-) d\omega + \\
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* ([G_{vu,l}G_{vs}G_{xs,cl}^*]_+ + G_{vu,o}WG_{xs,co}) d\omega, \quad (A.14)
\]
where use has been made of the fact that for two transfer function matrices $G_1$ and $G_2$, which have the same number of inputs and outputs, and where $\{G_1\}_+\{G_2\}_-$ has no poles on the unit circle, the following relationship exists [34]
\[
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} ([G_1]_+ + [G_2]_-)^* ([G_1]_+ + [G_2]_-) d\omega = \\
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} ([G_1]^*G_1]_+ + [G_2]^*G_2]_-) d\omega, \quad (A.15)
\]
because for this case [34]
\[
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} [G_1]^*_+[G_2]_- d\omega = 0. \quad (A.16)
\]
Eq. (A.14) can therefore be derived by using Eq. (A.15) with
\[
G_1 = [G_{vs}G_{xs,cl}^*]_+ + G_{vu,o}WG_{xs,co} \\
G_2 = [G_{vs}G_{xs,cl}^*]_+ + G_{vu,o}WG_{xs,co} \\
G_2 = [G_{vs}G_{xs,cl}^*]_-, \quad (A.17)
\]
where use has been made of Eq. (A.13). Hence, Eq. (A.14) is minimised subject to $W \in \mathcal{RH}_\infty^{L \times K}$ if and only if $W$ satisfies
\[
G_{vu,o}WG_{xs,co} = [G_{vu,o}G_{vs}G_{xs,cl}^*]_+. \quad (A.18)
\]
Because $G_{xs,co}$ has a stable left-inverse $G_{xs,co}^+$ such that $G_{xs,co}G_{xs,co}^+ = I$, and $G_{vu,o}$ has a stable right-inverse $G_{vu,o}^+$ such that $G_{vu,o}G_{vu,o}^+ = I$, a stable and causal transfer function matrix $W$ that minimises Eq. (A.14), and thus Eq. (A.4), is given by
\[
W_0 = -G_{vu,o}^+ [G_{vu,l}G_{vs}G_{xs,cl}^*]_+ G_{xs,co}^+ \quad (A.19)
\]
Substituting Eq. (A.19) into Eq. (A.14) yields the minimum value of the cost function given in Eq. (A.5). \qed

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Adaptive LMS virtual microphone technique

It has been shown in Chapter 4 that when using the adaptive LMS virtual microphone technique [14] as a spatially fixed virtual sensing algorithm in the adaptive feedforward control implementation illustrated in Fig. 4.3, the response of the feedforward controller $W$ is adapted such that it approximates the impulse response of the causal Wiener solution defined in Eq. (4.36) as

$$\hat{W}_o = -\hat{G}_{vu,o}^\dagger \hat{G}_{vu,i}^* + G_{xs,co}^\dagger.$$  \hspace{1cm} (A.20)

Substituting Eq. (A.20) into the cost function defined in Eq. (A.14) results in the minimum value $J(\hat{W}_o)$ defined in Eq. (4.37) as

$$J(\hat{W}_o) = J_{\text{min}} + \|G_{vu,i}^* G_{xs,ci}^* - G_{vu,o}^\dagger \hat{G}_{vu,i}^* G_{xs,ci}^* + G_{xs,co}^\dagger\|^2,$$ \hspace{1cm} (A.21)

where $J_{\text{min}}$ is the minimum value defined in Eq. (A.5).

Remote microphone technique

It has been shown in Chapter 4 that when using the remote microphone technique [104, 112] as a spatially fixed virtual sensing algorithm in the adaptive feedforward control implementation illustrated in Fig. 4.3, the response of the feedforward controller $W$ is adapted such that it approximates the impulse response of the causal Wiener solution defined in Eq. (4.42) as

$$\hat{W}_o = -G_{vu,o}^\dagger \hat{G}_{vu,i}^* HG_{ps} G_{xs,ci}^* + G_{xs,co}^\dagger.$$ \hspace{1cm} (A.22)

Substituting Eq. (A.22) into the cost function defined in Eq. (A.14) results in the minimum value $J(\hat{W}_o)$ defined in Eq. (4.43) as

$$J(\hat{W}_o) = J_{\text{min}} + \|G_{vu,i}^* (G_{vs} - HG_{ps}) G_{xs,ci}^* + G_{xs,co}^\dagger\|^2,$$ \hspace{1cm} (A.23)

where $J_{\text{min}}$ is the minimum value defined in Eq. (A.5). This can be derived by first noting that the term in the second integral on the right-hand side of Eq. (A.14) can be written, substituting the controller defined in Eq. (A.22), as

$$G_{vu,i}^* G_{vs} G_{xs,ci}^* + G_{vu,o}^\dagger \hat{W}_o G_{xs,co} = [G_{vu,i}^* G_{vs} G_{xs,ci}^* - G_{vu,i}^* HG_{ps} G_{xs,ci}^*] + [G_{vu,i}^* (G_{vs} - HG_{ps}) G_{xs,ci}^*].$$ \hspace{1cm} (A.24)

This can be derived by first noting that the following holds [34]

$$G_{vu,i}^* G_{vs} G_{xs,ci}^* = [G_{vu,i}^* G_{vs}] G_{xs,ci}^* + [G_{vu,i}^* G_{vs}] - G_{xs,ci}^*$$ \hspace{1cm} (A.25)

$$G_{vu,i}^* HG_{ps} G_{xs,ci}^* = [G_{vu,i}^* HG_{ps}] G_{xs,ci}^* + [G_{vu,i}^* HG_{ps}] - G_{xs,ci}^*.$$ \hspace{1cm} (A.26)
and because $G_{xs,ci}^*$ is non-causal up to a direct feedthrough term, the following expressions can be derived

$$[G_{vu,i}^* G_{vs} G_{xs,ci}^*] + = [[G_{vu,i}^* G_{vs} + G_{xs,ci}^*] +$$  \hspace{1cm} (A.27)

$$[G_{vu,i} H G_{ps} G_{xs,ci}^*] + = [[G_{vu,i} H G_{ps} + G_{xs,ci}^*] +.$$  \hspace{1cm} (A.28)

Also note that it can be derived from Eq. (B.24) that

$$[G_1 G_3^*]_+ - [G_2 G_3^*]_+ = [(G_1 - G_2) G_3^*]_+.$$  \hspace{1cm} (A.29)

Using Eqs (A.27), (A.28) and (A.29), Eq. (A.24) can thus also be written as

$$[G_{vu,i}^* G_{vs} G_{xs,ci}^*] + - [G_{vu,i}^* H G_{ps} G_{xs,ci}^*] + = [[(G_{vu,i}^* G_{vs}) + - [G_{vu,i}^* H G_{ps}] +] G_{xs,ci}^*]_+.$$  \hspace{1cm} (A.30)

Also note that it can be derived from Eq. (A.27) that

$$[G_3 G_1]_+ - [G_3 G_2]_+ = [G_3 (G_1 - G_2)]_+.$$  \hspace{1cm} (A.31)

Using Eq. (A.31), Eq. (A.30) can thus also be written as

$$[[G_{vu,i}^* G_{vs}] + - [G_{vu,i}^* H G_{ps}] +] G_{xs,ci}^*] + = [[G_{vu,i}^* (G_{vs} - H G_{ps})] + G_{xs,ci}^*]_+.$$  \hspace{1cm} (A.32)

Because the following holds

$$G_{vu,i}^* (G_{vs} - H G_{ps}) G_{xs,ci}^* = [G_{vu,i}^* (G_{vs} - H G_{ps})] + G_{xs,ci}^* + [G_{vu,i}^* (G_{vs} - H G_{ps})] - G_{xs,ci}^*$$  \hspace{1cm} (A.33)

and because $G_{xs,ci}^*$ is non-causal up to a direct feedthrough term, it can be derived that the right-hand side of Eq. (A.32) can also be written as

$$[[G_{vu,i}^* (G_{vs} - H G_{ps})] + G_{xs,ci}^*]_+ = [G_{vu,i}^* (G_{vs} - H G_{ps}) G_{xs,ci}^*]_+,$$  \hspace{1cm} (A.34)

thereby arriving at Eq. (A.23).

**A.2 Proof of Theorem 3.2**

**Theorem A.2** (Causal Wiener solution for filter $H$).

Given the transfer function matrices $G_{ps} \in \mathcal{RH}_{\infty}^{M_p \times S}$ and $G_{vs} \in \mathcal{RH}_{\infty}^{M_v \times S}$, and assuming that $G_{ps}$ does not have any zeros on the unit circle, the following outer-inner factorisation can be defined

$$G_{ps} = G_{ps,co} G_{ps,ci},$$  \hspace{1cm} (A.35)

where $G_{ps,co}$ has a stable left-inverse $G_{ps,co}^+$. Furthermore, let $G_{ps,ci}^+$ be such that $[G_{ps,ci}^* G_{ps,ci}]^*$ is unitary. Then

$$H_0 = [G_{vs} G_{ps,ci}^*]_+ + G_{ps,co}^+$$  \hspace{1cm} (A.36)
minimises
\[ J_\ell = \|G_{vs} - HG_{ps}\|_2^2, \quad \text{subject to } H \in \mathcal{RH}_\infty^{M_x \times M_p}, \] (A.37)
and its minimum value is given by
\[ J_{\ell, \text{min}} = \|G_{vs} - H G_{ps}\|_2 = \left\| G_{ps,ci} \right\|^2_2 + \left\| \left[ G_{ps,ci}^* - G_{ps,ci} \right] \right\|^2_2. \] (A.38)

**Proof.** The proof follows the proof of the Causal Wiener Theorem presented by Fraanje [34], and the strategy is to complete the squares. The cost function in Eq. (A.37) can be written in the frequency domain as
\[ J_\ell = \|G_{vs} - HG_{ps}\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (G_{vs} - HG_{ps}) (G_{vs} - HG_{ps})^* d\omega. \] (A.39)
Because \([G_{ps,ci}^* G_{ps,ci}]^*\) is unitary, the following expression can be derived
\[ G_{vs} = G_{vs} G_{ps,ci} G_{ps,ci}^* G_{ps,ci} + G_{vs} G_{ps,ci} G_{ps,ci}^* G_{ps,ci} = G_{vs,1} G_{ps,ci} + G_{vs,2} G_{ps,ci}^*, \]
where
\[ G_{vs,1} = G_{vs} G_{ps,ci}^*, \quad G_{vs,2} = G_{vs} G_{ps,ci}^*. \] (A.40)
Using this expression, Eq. (A.39) can now also be written as
\[ J_\ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( G_{vs,1} G_{ps,ci} + G_{vs,2} G_{ps,ci}^* - HG_{ps,co} G_{ps,ci} \right) (\cdot)^* d\omega \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_{vs,2} G_{ps,ci}^* d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} (G_{vs,1} - HG_{ps,co}) (\cdot)^* d\omega \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_{ps,ci} G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^* d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} (G_{vs} G_{ps,ci}^* - HG_{ps,co}) (\cdot)^* d\omega \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_{ps,ci} G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^* d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cdot)^* (G_{vs} G_{ps,ci}^* - HG_{ps,co}) d\omega, \] (A.41)
where use has been made of the fact that \(\text{tr}(AB) = \text{tr}(BA)\). Because \(H\) is constrained to be stable, \(H G_{ps,co}\) is stable, such that
\[ H G_{ps,co} = [H G_{ps,co}]^+ \quad \forall H \in \mathcal{RH}_\infty^{M_x \times M_p}. \] (A.42)
Eq. (A.41) can therefore also be written as
\[ J_\ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^* d\omega + \]
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cdot)^* \left( [G_{ps} G_{ps,ci}^*] + [G_{ps} G_{ps,ci}^*] - H G_{ps,co} \right) d\omega \] (A.43)
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} (G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^* + G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^*) d\omega + \]
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cdot)^* \left( [G_{ps} G_{ps,ci}^*] + H G_{ps,co} \right) d\omega, \] (A.44)
where use has been made of Eq. (A.15), with

\[
G_1 = [G_{vs}G_{ps,cl}^+] - HG_{ps,co} \\
= [G_{vs}G_{ps,ci}^+ - HG_{ps,co}]_+ ,
\]

(A.45)

\[
G_2 = [G_{vs}G_{ps,ci}^+] ,
\]

(A.46)

where use has been made of Eq. (A.42). Hence, Eq. (A.44) is minimised subject to \( H \in RH_{\infty}^{M_z \times M_p} \) if and only if \( H \) satisfies

\[
HG_{ps,co} = [G_{vs}G_{ps,ci}^+]_+ .
\]

(A.47)

Because \( G_{ps,co} \) has a stable left-inverse \( G_{ps,co}^\dagger \), such that \( G_{ps,co}^\dagger G_{ps,co} = I \), a stable and causal transfer function matrix \( H \) that minimises Eq. (A.44), and thus Eq. (A.37), is given by

\[
H_0 = [G_{vs}G_{ps,ci}^+]_+ G_{ps,co}^\dagger .
\]

(A.48)

Substituting Eq. (A.48) into Eq. (A.44) yields the minimum value of the cost function given in Eq. (A.38). \( \square \)
Appendix B

State-space solutions of virtual sensing algorithms

B.1 Calculus with state-space realisations

A discrete-time state-space system is given by the equations

\[ \begin{align*}
x(n+1) &= Ax(n) + Bu(n) \\
y(n) &= Cx(n) + Du(n),
\end{align*} \tag{B.1} \]

where \( x(n) \in \mathbb{R}^N \) are the states of the system, \( u(n) \in \mathbb{R}^L \) the inputs, and \( y(n) \in \mathbb{R}^M \) the outputs. The state-space matrices \( A, B, C, \) and \( D \) are real-valued and of appropriate dimensions. The state-space realisation \( G \) of the system in Eq. (B.1) will be denoted by

\[ G \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{B.2} \]

The input-output behaviour of the system in Eq. (B.1) is described by the transfer function matrix \( G(z) \) given by

\[ G(z) = D + C(zI - A)^{-1}B = D + CBz^{-1} + CABz^{-2} + CA^2Bz^{-3} + \ldots, \tag{B.3} \]

such that

\[ y(n) = G(z)u(n), \tag{B.4} \]

with \( z^{-1} \) the unit delay operator in the discrete time-domain, such that

\[ u(n-1) = z^{-1}u(n). \tag{B.5} \]
The variable $z$ is also used as complex variable in the $z$-transform, which is generally used in a frequency domain analysis. The transfer function matrix $G(z)$ in Eq. (B.3) then defines the relationship between the $z$-transforms of the input and output signals, such that

$$Y(z) = G(z)U(z), \quad (B.6)$$

with $U(z)$ the $z$-transform of the input signal $u(n)$ defined as [51]

$$U(z) = \sum_{n=-\infty}^{\infty} u(n)z^{-n}, \quad (B.7)$$

and with $Y(z)$ the $z$-transform of the output signal $y(n)$ defined in a similar way. In this thesis, the variable $z$ is used as both the unit shift forward operator in the discrete time-domain and as a complex variable in the $z$-transform.

**Causal and non-causal representation**

The discrete-time state-space system in Eq. (B.1) is given in causal, or direct-time, form. If the matrix $A$ is invertible, the discrete-time system is said to be time-reversible [54]. As a result, the system can be described by equivalent causal and non-causal, or indirect-time, representations. An equivalent non-causal representation of the system $G$ in Eq. (B.1) is given by

$$x(n) = A^{-1}x(n+1) - A^{-1}Bu(n)$$

$$y(n) = CA^{-1}x(n+1) + (D - CA^{-1}B)u(n), \quad (B.8)$$

which will be denoted by

$$G \sim \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{bmatrix}_{ac}. \quad (B.9)$$

The input-output behavior of this system is described by the transfer function matrix $G(z)$, which is now given by

$$G(z) = -CA^{-1}z^{-1}I - A^{-1}B + (D - CA^{-1}B)$$

$$= \cdots - CA^{-3}Bz^2 - CA^{-2}Bz + (D - CA^{-1}B). \quad (B.10)$$

**Adjoint operator**

The adjoint of the transfer function matrix $G(z)$ in Eq. (B.3) is defined as

$$G^\ast(z) \triangleq G^T(z^{-1}). \quad (B.11)$$
A state-space realisation of the adjoint is defined as

\[
G^* \sim \begin{bmatrix}
A^T & C^T \\
B^T & D^T
\end{bmatrix}_{ac},
\]  

(B.12)
such that the transfer function matrix \(G^*(z)\) in Eq. (B.11) is given by

\[
G^*(z) = D^T + B^T(z^{-1}I - A^T)^{-1}C^T.
\]  

(B.13)

If the matrix \(A\) is invertible, an equivalent causal state-space realisation of \(G^*(z)\) is given by

\[
G^* \sim \begin{bmatrix}
A^{-T} & -A^{-T}C^T \\
B^T A^{-T} & D^T - B^T A^{-T} C^T
\end{bmatrix}.
\]  

(B.14)

**Pseudo-inverse model**

Let \(D^\dagger\) denote a right (left) inverse of \(D\) if \(D\) has full row (column) rank. Then the pseudo-inverse of the system \(G\) in Eq. (B.2) is defined as [130]

\[
G^\dagger \sim \begin{bmatrix}
A - BD^\dagger C & BD^\dagger \\
-D^\dagger C & D^\dagger
\end{bmatrix},
\]  

(B.15)

with \(G^\dagger\) a right (left) inverse of the system \(G\). If the matrix \(D\) is invertible, the pseudo-inverse is simply equivalent to the inverse, such that \(G^\dagger = G^{-1}\).

**Similarity transformation**

Let a linear transformation of the states of the system \(G\) in Eq. (B.1) be defined by

\[
\tilde{x}(n) = T x(n),
\]  

(B.16)

where \(T \in \mathbb{R}^{N \times N}\) is an invertible matrix. A state-space realisation of the resulting system \(\tilde{G}\) is then given by

\[
\tilde{G} \sim \begin{bmatrix}
T A T^{-1} & TB \\
CT^{-1} & D
\end{bmatrix}.
\]  

(B.17)

The resulting transfer function matrix \(\tilde{G}(z) = G(z)\), and \(\tilde{G}\) is said to be equal to \(G\) up to a *similarity transformation* \(T\).
Parallel connection

Let the transfer function matrices \( G_1(z) \) and \( G_2(z) \) be defined by the state-space realisations

\[
G_1 \sim \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \quad G_2 \sim \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}.
\]  \( \text{(B.18)} \)

If the two systems have the same number of inputs and outputs, the parallel connection of \( G_1(z) \) and \( G_2(z) \) is defined as

\[
G(z) = G_1(z) + G_2(z),
\]  \( \text{(B.19)} \)

and a state-space realisation of the resulting transfer function matrix \( G(z) \) is given by

\[
G \sim \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_1 + D_2 \end{bmatrix}.
\]  \( \text{(B.20)} \)

Serial connection

Let the transfer function matrices \( G_1(z) \) and \( G_2(z) \) be defined by the state-space realisations in Eq. (B.18). If the number of outputs of \( G_2(z) \) is equal to the number of inputs of \( G_1(z) \), the serial connection of \( G_1(z) \) and \( G_2(z) \), in this specific order, is defined as

\[
G(z) = G_1(z)G_2(z).
\]  \( \text{(B.21)} \)

A state-space realisation of the resulting transfer function matrix \( G(z) \) is then given by

\[
G \sim \begin{bmatrix} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ D_1C_2 & C_1 & D_1D_2 \end{bmatrix}.
\]  \( \text{(B.22)} \)

Causality and non-causality operators

Let the state-space realisations \( G_1 \) and \( G_2 \) defined in Eq. (B.18) be strictly stable, such that the eigenvalues of the matrices \( A_1 \) and \( A_2 \) are inside the unit circle. For this case, the following holds

\[
G_1G_2^* = [G_1G_2^*]_+ + [G_1G_2^*]_-, \tag{B.23}
\]

where \([·]_+\) and \([·]_-\) denote the causal and non-causal components of the term inside the brackets, respectively. State-space realisations of these components are given by

\[
[G_1G_2^*]_+ \sim \begin{bmatrix} A_1 & B_1D_2^T + A_1X_{12}C_2^T \\ C_1 & D_1D_2^T + C_1X_{12}C_2^T \end{bmatrix}, \tag{B.24}
\]

\[
[G_1G_2^*]_- \sim \begin{bmatrix} A_1^T \\ D_1B_2^T + C_1X_{12}A_2^T \end{bmatrix}.
\]
with $X_{12}$ the solution to the discrete-time Lyapunov equation \[ A_1 X_{12} A_1^T + B_1 B_2^T = X_{12}. \] (B.25)

Similarly, the following holds:

\[
G_i^* G_2 = [G_i^* G_2]_+ + [G_i^* G_2]_-, \tag{B.26}
\]

where state-space realisations of the causal and non-causal components are given by

\[
[G_i^* G_2]_+ \sim \begin{bmatrix}
A_1 & B_1 \\
D_2^T C_1 + B_2^T X_{21} A_1 & D_2^T D_1 + B_2^T X_{21} B_1
\end{bmatrix},
\]

\[
[G_i^* G_2]_- \sim \begin{bmatrix}
A_2^T & C_2^T D_1 + A_2^T X_{21} B_1 \\
B_2^T & 0
\end{bmatrix}_+, \tag{B.27}
\]

with $X_{21}$ the solution to the discrete-time Lyapunov equation \[ A_2^T X_{21} A_1 + C_2^T C_1 = X_{21}. \] (B.28)

**Inner-outer factorisation**

An inner-outer factorisation of the system $G$ in Eq. (B.2) is given by

\[
G = G_i G_o, \tag{B.29}
\]

with $G_i^* G_i = I$, and where $G_o$ has a stable right-inverse. State-space realisations of the inner factor $G_i$, and outer factor $G_o$ are denoted by

\[
G_i \sim \begin{bmatrix}
A_i & B_i \\
C_i & D_i
\end{bmatrix}, \quad G_o \sim \begin{bmatrix}
A^o & B^o \\
C^o & D^o
\end{bmatrix}. \tag{B.30}
\]

Assuming Eq. (B.2) defines a minimal realisation of the system $G$, the matrices $A_i, B_i, C_i, D_i, C^o, \text{ and } D^o$ defined in Eq. (B.30) can be calculated as follows \[54\]. Let the matrix $P = P^T \geq 0$ be a stabilising solution to the discrete time algebraic Ricatti equation given by

\[
P = A^T P A - (A^T P B + S)(B^T P B + R)^{-1}(B^T P A + S^T) + Q, \tag{B.31}
\]

with

\[
Q = C^T C, \quad R = D^T D, \quad S = C^T D. \tag{B.32}
\]

Furthermore, let the matrix $F$ be defined as

\[
F = (B^T P B + R)^{-1}(B^T P A + S^T), \tag{B.33}
\]
and let the matrix $\Gamma$ be an upper triangular matrix calculated from a Cholesky factorisation of $B^T PB + R$, such that

$$\Gamma^T \Gamma = B^T PB + R. \quad (B.34)$$

Then state-space realisations of the inner factor $G_i$ and outer factor $G_o$ are given by

$$G_i \sim \begin{bmatrix} A - BF & B \Gamma^T \\ C - DF & D \Gamma^T \end{bmatrix}, \quad G_o \sim \begin{bmatrix} A & B \\ \Gamma F & \Gamma \end{bmatrix}, \quad (B.35)$$

where $\Gamma^T$ is a right-inverse of $\Gamma$.

Matrix relationships associated with inner-outer factorisation

Let the matrix $Y$ be an upper triangular matrix calculated from a Cholesky factorisation of $P$ in Eq. (B.31), such that

$$Y^T Y = P. \quad (B.36)$$

The following QR-factorisation can now be defined

$$\begin{bmatrix} D & C \\ YB & YA \end{bmatrix} = \begin{bmatrix} D^i & C^i \\ B^i & A^i \end{bmatrix} \begin{bmatrix} D^o & C^o \\ 0 & Y \end{bmatrix}, \quad (B.37)$$

where the first matrix on the right-hand side is orthogonal, such that

$$\begin{bmatrix} D^i & C^i \\ B^i & A^i \end{bmatrix}^T \begin{bmatrix} D^i & C^i \\ B^i & A^i \end{bmatrix} = I. \quad (B.38)$$

Therefore, the following matrix relationships can be defined

$$D^T D + B^T YB = D^o \quad (B.39)$$

$$D^T C + B^T YA = C^o \quad (B.40)$$

$$C^T D + A^T YB = 0 \quad (B.41)$$

$$C^T C + A^T YA = Y. \quad (B.42)$$

Furthermore, by evaluating the expression

$$\begin{bmatrix} D & CY \\ B & AY \end{bmatrix}^T \begin{bmatrix} D & CY \\ B & AY \end{bmatrix}, \quad (B.43)$$

the following matrix relationships can be defined

$$D^T D + B^T PB^T = D^o T D^o \quad (B.44)$$

$$D^T C + B^T PA^T = D^o T C^o \quad (B.45)$$

$$C^T D + A^T PB^T = C^o T D^o \quad (B.46)$$

$$C^T C + A^T PA^T = C^o T C^o + P. \quad (B.47)$$
An outer-inner factorisation of the system $G$ in Eq. (B.2) is given by

$$G = G_{co} G_{ci},$$

(B.48)

with $G_{ci} G_{ci}^{*} = I$, and where $G_{co}$ has a stable left-inverse. State-space realisations of the co-inner factor $G_{ci}$, and co-outer factor $G_{co}$ are denoted by

$$G_{co} \sim [A^{co} B^{co} C^{co} D^{co}]$$

and

$$G_{ci} \sim [A^{ci} B^{ci} C^{ci} D^{ci}],$$

(B.49)

Assuming Eq. (B.2) defines a minimal realisation of the system $G$, the matrices $A^{ci}$, $B^{ci}$, $C^{ci}$, $D^{ci}$, $C^{co}$, and $D^{co}$ defined in Eq. (B.49) can be calculated as follows [54]. Let the matrix $P = P^{T} \geq 0$ be a stabilising solution to the discrete time algebraic Ricatti equation given by

$$P = APA^{T} - (APC^{T} + S)(CPC^{T} + R)^{-1}(CPA^{T} + S^{T}) + Q,$$

(B.50)

with

$$Q = BB^{T}, \quad R = DD^{T}, \quad S = BD^{T}.$$  

(B.51)

Furthermore, let the matrix $K$ be defined as

$$K = (APC^{T} + S)(CPC^{T} + R)^{-1},$$

(B.52)

and let the matrix $\Gamma$ be a lower triangular matrix calculated from a Cholesky factorisation of $CPC^{T} + R$, such that

$$\Gamma^{T} = CPC^{T} + R.$$  

(B.53)

Then state-space realisations of the co-inner factor $G_{ci}$ and co-outer factor $G_{co}$ are given by

$$G_{co} \sim \begin{bmatrix} A & B^{co} \\ C & D^{co} \end{bmatrix}, \quad G_{ci} \sim \begin{bmatrix} A^{ci} & B^{ci} \\ C^{ci} & D^{ci} \end{bmatrix},$$

(B.54)

where $\Gamma^{T}$ is a left-inverse of $\Gamma$.

**Matrix relationships associated with outer-inner factorisation**

Let the matrix $Y$ be a lower triangular matrix calculated from a Cholesky factorisation of $P$ in Eq. (B.50), such that

$$YY^{T} = P.$$  

(B.55)

The following LQ-factorisation can now be defined

$$\begin{bmatrix} D & CY \\ B & AY \end{bmatrix} = \begin{bmatrix} D^{co} & 0 \\ B^{co} & Y \end{bmatrix} \begin{bmatrix} D^{ci} & C^{ci} \\ B^{ci} & A^{ci} \end{bmatrix},$$

(B.56)
where the second matrix on the right-hand side is orthogonal, such that

\[
\begin{bmatrix}
D^{ci} & C^{ci} \\
B^{ci} & A^{ci}
\end{bmatrix} \begin{bmatrix}
D^{ci} & C^{ci} \\
B^{ci} & A^{ci}
\end{bmatrix}^T = I.
\] (B.57)

Therefore, the following matrix relationships can be defined

\[
\begin{align*}
DD^{cT} + CYC^{cT} &= D^o \\
DB^{cT} + CYA^{cT} &= 0 \\
BD^{cT} + AYC^{cT} &= B^o \\
BB^{cT} + AYA^{cT} &= Y.
\end{align*}
\] (B.58) (B.59) (B.60) (B.61)

Furthermore, by evaluating the expression

\[
\begin{bmatrix}
D & CY \\
B & AY
\end{bmatrix} \begin{bmatrix}
D & CY \\
B & AY
\end{bmatrix}^T,
\] (B.62)

the following matrix relationships can be defined

\[
\begin{align*}
DD^T + CPC^T &= D^o D^{oT} \\
DB^T + CPA^T &= D^o B^{oT} \\
BD^T + APC^T &= B^{co} D^{oT} \\
BB^T + APA^T &= B^{co} B^{oT} + P.
\end{align*}
\] (B.63) (B.64) (B.65) (B.66)

### B.2 State-space solution of the remote microphone technique

A transfer function matrix \(G_{RMT} \in \mathcal{RH}_{\infty}^{M_s \times (L+M_p)}\) that defines the input-output behaviour of the remote microphone technique [104, 112] has been defined in Eq. (3.82) on page 90 as

\[
\hat{e}_p(n) = [G_{v} - HG_{pu} H] \begin{bmatrix} u(n) \\ e_p(n) \end{bmatrix} = G_{RMT} \begin{bmatrix} u(n) \\ e_p(n) \end{bmatrix}. \quad \text{ (B.67)}
\]

A state-space realisation of the transfer function matrix \(G_{RMT}\) is now derived.

**Minimal realisation of filter H**

A causal Wiener solution for the transfer function matrix \(H \in \mathcal{RH}_{\infty}^{M_s \times M_p}\) has been defined in Theorem 3.2 as

\[
H_o = [G_{vs} G_{ps,ci}] + G_{ps,co}^+. \quad \text{ (B.68)}
\]
with
\[ G_{os} \sim \begin{bmatrix} A & B_s \\ C_v & D_{vs} \end{bmatrix}, \] (B.69)

and where \( G_{ps,ci} \) and \( G_{ps,co} \) can be calculated from an outer-inner factorisation of \( G_{ps} \) as described in Section B.1, such that
\[ G_{ps,co} \sim \begin{bmatrix} A \\ C_p \end{bmatrix} B_{ps}^{co} \] \( \begin{bmatrix} D_{ps}^{co} \end{bmatrix} \), \[ G_{ps,ci} \sim \begin{bmatrix} A_{ci}^{ps} \\ C_{ci}^{ps} \end{bmatrix} B_{ci}^{ps} \] \( \begin{bmatrix} D_{ci}^{ps} \end{bmatrix} \). (B.70)

A state-space realisation of \( G_{ps,co}^{+} \) is then given by
\[ G_{ps,co}^{+} \sim \begin{bmatrix} A - B_{ps}^{co} D_{ps}^{co} & C_v \\ -D_{ps}^{co} & B_{ps}^{co} \end{bmatrix} \begin{bmatrix} D_{ps}^{co} & C_v \end{bmatrix}. \] (B.71)

Furthermore, a state-space realisation of \( [G_{vs} G_{ps,ci}^{+}] \) can be shown to be given by
\[ [G_{vs} G_{ps,ci}^{+}] \sim \begin{bmatrix} A \\ C_v \end{bmatrix} B_{ps}^{ciT} D_{ps}^{ciT} + A X_{ps} C_{ciT} \] \( \begin{bmatrix} D_{ps}^{ciT} + C_v X_{ps} C_{ciT} \end{bmatrix} \), (B.72)

with \( X_{ps} \) the solution to the discrete-time Lyapunov equation [54]
\[ A X_{ps} A_{ciT} + B_{ps} B_{ciT}^{T} = X_{ps}. \] (B.73)

Using the matrix relationships in Eqs (B.60) and (B.61), it can be shown that
\[ B_{ps} D_{ps}^{ciT} + A X_{ps} C_{ciT} = B_{ps}^{co}. \] (B.74)

The state-space realisation in Eq. (B.72) can therefore also be written as
\[ [G_{vs} G_{ps,ci}^{+}] \sim \begin{bmatrix} A \\ C_v \end{bmatrix} B_{ps}^{co} \] \( \begin{bmatrix} D_{ps}^{co} \end{bmatrix} \), (B.75)

with
\[ D_{ps}^{co} = D_{vs} D_{ps}^{ciT} + C_v X_{ps} C_{ciT}. \] (B.76)

Using Eqs (B.71) and (B.75), a state-space realisation of the causal Wiener solution \( H_{o} \) defined in Eq. (B.68) is now given by
\[ H_{o} \sim \begin{bmatrix} A - B_{ps}^{co} D_{ps}^{co} C_p & 0 & B_{ps}^{co} D_{ps}^{co} \\ -B_{ps}^{co} D_{ps}^{co} C_p & A & B_{ps}^{co} D_{ps}^{co} \\ -D_{ps}^{co} D_{ps}^{co} C_p & C_v & D_{ps}^{co} D_{ps}^{co} \end{bmatrix}. \] (B.77)
Appendix B State-space solutions of virtual sensing algorithms

Next, by performing a similarity transformation given by

\[
T = \begin{bmatrix}
I & 0 \\
-1 & I
\end{bmatrix},
\]  
(B.78)

Eq. (B.77) can also be written as

\[
H_0 \sim \begin{bmatrix}
A - B_{ps}^c D_{ps}^{c\dagger} C_P & 0 & B_{ps}^c D_{ps}^{c\dagger} \\
0 & A & 0 \\
C_v - D_{vs}^{c\dagger} D_{ps}^{c\dagger} C_P & C_v & D_{vs}^{c\dagger} D_{ps}^{c\dagger}
\end{bmatrix}.
\]  
(B.79)

Because the second part of the states of the system in Eq. (B.79) is uncontrollable, a minimal realisation of \(H_0\) is given by

\[
H_0 \sim \begin{bmatrix}
A - B_{ps}^c D_{ps}^{c\dagger} C_P & B_{ps}^c D_{ps}^{c\dagger} \\
C_v - D_{vs}^{c\dagger} D_{ps}^{c\dagger} C_P & D_{vs}^{c\dagger} D_{ps}^{c\dagger}
\end{bmatrix}.
\]  
(B.80)

Minimal realisation of \(HG_{pu}\)

A state-space realisation of the transfer function matrix \(G_{pu} \in \mathcal{RH}_{\infty}^{M_p \times L}\) is defined by

\[
G_{pu} \sim \begin{bmatrix}
A & B_u \\
C_P & D_{pu}
\end{bmatrix},
\]  
(B.81)

such that a state-space realisation of \(H_0 G_{pu}\) in Eq. (B.67) is given by

\[
H_0 G_{pu} \sim \begin{bmatrix}
A & 0 & B_u \\
B_{ps}^c D_{ps}^{c\dagger} C_P & A - B_{ps}^c D_{ps}^{c\dagger} C_P & B_{ps}^c D_{ps}^{c\dagger} D_{pu} \\
D_{vs}^{c\dagger} D_{ps}^{c\dagger} C_P & C_v - D_{vs}^{c\dagger} D_{ps}^{c\dagger} C_P & D_{vs}^{c\dagger} D_{ps}^{c\dagger} D_{pu}
\end{bmatrix}.
\]  
(B.82)

Next, by performing a similarity transformation given by

\[
T = \begin{bmatrix}
I & 0 \\
-1 & I
\end{bmatrix},
\]  
(B.83)

Eq. (B.82) can also be written as

\[
HG_{pu} \sim \begin{bmatrix}
A & 0 & B_u \\
0 & A - B_{ps}^c D_{ps}^{c\dagger} C_P & B_{ps}^c D_{ps}^{c\dagger} D_{pu} - B_u \\
C_v & C_v - D_{vs}^{c\dagger} D_{ps}^{c\dagger} C_P & D_{vs}^{c\dagger} D_{ps}^{c\dagger} D_{pu}
\end{bmatrix}.
\]  
(B.84)
Minimal realisation of $G_{vu} - H G_{pu}$

A state-space realisation of the transfer function matrix $G_{vu} \in \mathcal{RH}_{\infty}^{M_v \times L}$ is defined by

$$G_{vu} \sim \begin{bmatrix} A & B_u \\ C_v & D_{vu} \end{bmatrix}, \quad (B.85)$$

such that a state-space realisation of the term $G_{vu} - H_o G_{pu}$ in Eq. (B.67) is given by

$$G_{vu} - H_o G_{pu} \sim \begin{bmatrix} A & 0 & 0 & B_u \\ 0 & A & 0 & B_u \\ 0 & 0 & A - B_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p & B_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} - B_u \\ C_v & -C_v & D_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p - C_v & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} \end{bmatrix}. \quad (B.86)$$

Next, by performing a similarity transformation given by

$$T = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}, \quad (B.87)$$

Eq. (B.86) can also be written as

$$G_{vu} - H_o G_{pu} \sim \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & B_u \\ 0 & 0 & A - B_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p & B_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} \\ C_v & 0 & C_v - D_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} \end{bmatrix}. \quad (B.88)$$

Because the first part of the states of the system in Eq. (B.88) is uncontrollable, and the second part is unobservable, a minimal realisation of $G_{vu} - H_o G_{pu}$ is given by

$$G_{vu} - H_o G_{pu} \sim \begin{bmatrix} A - B_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p & B_u - B_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} & B_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} \\ C_v - D_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} \end{bmatrix}. \quad (B.89)$$

Minimal realisation of $G_{RMT}$

From Eqs (B.80) and (B.89), a minimal state-space realisation of the remote microphone technique, with input-output behaviour defined by the transfer function matrix $G_{RMT} \in \mathcal{RH}_{\infty}^{M_v \times (M_p + L)}$ in Eq. (B.67), is now given by

$$G_{RMT} \sim \begin{bmatrix} A - B_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p & B_u - B_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} & B_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} \\ C_v - D_{co}^{ps} D_{co}^{\dagger} D_{ps} C_p & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{ps} D_{pu} \end{bmatrix}. \quad (B.90)$$

The order of the derived minimal state-space realisation of the remote microphone technique is equal to the order $N$ of the standard state-space model $G$ of the considered active noise control system that has been defined in Eq. (2.18) on page 46.
B.3 Comparison to Kalman filter based algorithm

A state-space realisation of the Kalman filter based spatially fixed virtual sensing algorithm, which gives an optimal current estimate \( \hat{e}_v(n|n) \) of the virtual error signals given observations \( e_p(i) \) of the physical error signals up to \( i = n \), has been defined in Theorem 3.4 on page 117 as

\[
\begin{bmatrix}
\hat{z}(n+1|n) \\
\hat{e}_v(n|n)
\end{bmatrix} =
\begin{bmatrix}
A - K_{ps}C_p & B_u - K_{ps}D_{pu} & K_{ps} \\
C_v - M_{vs}C_p & D_{vu} - M_{vs}D_{pu} & M_{vs}
\end{bmatrix}
\begin{bmatrix}
\hat{z}(n|n-1) \\
\hat{u}(n) \\
e_p(n)
\end{bmatrix},
\] (B.91)

where the Kalman gain \( K_{ps} \), and the virtual gain matrix \( M_{vs} \) are given by

\[
K_{ps} = (AP_{ps}C_p^T + \tilde{S}_{ps})(C_pP_{ps}C_p^T + \tilde{R}_p)^{-1}
\]

\[
M_{vs} = (C_vP_{ps}C_p^T + \tilde{R}_p)(C_pP_{ps}C_p^T + \tilde{R}_p)^{-1},
\] (B.92)

with \( P_{ps} = P_{ps}^T > 0 \) the unique stabilising solution to the DARE given by

\[
P_{ps} = AP_{ps}A^T - (AP_{ps}C_p^T + \tilde{S}_{ps})(C_pP_{ps}C_p^T + \tilde{R}_p)^{-1}(AP_{ps}C_p^T + \tilde{S}_{ps})^T + \tilde{Q}_s,
\] (B.93)

Assuming there is no measurement noise on the physical sensors, it can be derived from Eqs (3.132)–(3.135) that

\[
\tilde{Q}_s = B_sB_s^T \\
\tilde{S}_{ps} = D_{ps}B_s^T \\
\tilde{R}_p = D_{ps}D_{ps}^T \\
\tilde{R}_{ps} = D_{ps}D_{ps}^T
\]

such that the Kalman and virtual gain matrices defined in Eq. (B.92) then reduce to

\[
K_{ps} = (AP_{ps}C_p^T + B_sD_{ps}^T)(C_pP_{ps}C_p^T + D_{ps}D_{ps}^T)^{-1}
\]

\[
M_{vs} = (C_vP_{ps}C_p^T + D_{vs}D_{ps}^T)(C_pP_{ps}C_p^T + D_{ps}D_{ps}^T)^{-1}.
\] (B.97)

Comparing the state-space realisations in Eqs (B.91) and (B.90), it can be seen that the Kalman filter based spatially fixed virtual sensing algorithm and the remote microphone technique are equivalent, assuming the measurement noise on the physical sensors is zero, if the following equalities hold

\[
K_{ps} = B_{ps}^C D_{ps}^{C\dagger},
\] (B.98)

\[
M_{vs} = D_{vs}^C D_{ps}^{C\dagger}.
\] (B.99)

The matrices \( B_{ps}^C \) and \( D_{ps}^C \) are calculated from an outer-inner factorisation of \( G_{ps} \), which can be calculated as described in Section B.1. Comparing Eqs (B.50) and (B.93), it can be seen that the DAREs that are solved to compute the Kalman filter based spatially fixed
virtual sensing algorithm and the outer-inner factorisation of $G_{ps}$ are equivalent, with the unique stabilising solution denoted by $P_{ps} = P_{ps}^T > 0$. Using the matrix relationships defined in Eqs (B.63) and (B.65), which are now applied to an outer-inner factorisation of $G_{ps}$, Eqs (B.96) and (B.97) can therefore also be written as

$$K_{ps} = B_{ps}^c D_{ps}^{c'i} (D_{ps}^{co} D_{ps}^{coT})^{-1} = B_{ps}^c D_{ps}^{c'i}, \quad (B.100)$$

$$M_{vs} = (C_v P_{ps} C_p^T + D_{vs} D_{ps}^T) (D_{ps}^{co} D_{ps}^{coT})^{-1}, \quad (B.101)$$

and the equality in Eq. (B.98) thus holds. From Eq. (B.56), it can be seen that

$$C_p Y_{ps} = D_{ps}^{co} C_{ci}^T \quad (B.102)$$

$$D_{ps} = D_{ps}^{co} D_{ps}^{coT}, \quad (B.103)$$

with $Y_{ps} Y_{ps}^T = P_{ps}$, such that Eq. (B.101) can be written as

$$M_{vs} = (C_v Y_{ps} C_p^T D_{ps}^{coT} + D_{vs} D_{ps}^T) (D_{ps}^{co} D_{ps}^{coT})^{-1} = (C_v Y_{ps} C_{ci}^T + D_{vs} D_{ps}^{coT}) D_{ps}^{coT}. \quad (B.104)$$

The equality in Eq. (B.99) now holds if

$$C_v Y_{ps} C_{ci}^T + D_{vs} D_{ps}^{coT} = D_{vs}^{co}, \quad (B.105)$$

with $D_{vs}^{co}$ defined in Eq. (B.76) as

$$D_{vs}^{co} = D_{vs} D_{ps}^{coT} + C_v X_{ps} C_{ci}^T, \quad (B.106)$$

with $X_{ps}$ the solution to the discrete-time Lyapunov equation

$$A X_{ps} A_{ci}^T + B_c B_{ci}^T = X_{ps}. \quad (B.107)$$

From Eq. (B.61), it can be seen that $X_{ps} = Y_{ps}$ and the equality in Eq. (B.99) thus holds. It has thus been shown that, assuming there is no measurement noise on the physical sensors, the state-space realisation of the remote microphone technique derived in Eq. (B.90) is thus equivalent to the developed Kalman filter based spatially fixed virtual sensing algorithm defined in Eq. (B.91). Note that the presented derivations can be extended to the case of measurement noise on the physical sensors, including the ones that are positioned at the virtual locations during a preliminary identification stage.
B.4 State-space solution of the virtual microphone arrangement

A transfer function matrix \( G_{VMA} \in \mathcal{RH}_{\infty}^{M_v \times (L+M_p)} \) that defines the input-output behaviour of the virtual microphone arrangement \cite{27} has been defined in Eq. (3.114) on page 99 as

\[
\hat{e}_v(n) = \begin{bmatrix} G_{uu} - G_{pu} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}(n) \\ \mathbf{e}_p(n) \end{bmatrix} = G_{VMA} \begin{bmatrix} \mathbf{u}(n) \\ \mathbf{e}_p(n) \end{bmatrix}.
\] (B.108)

A state-space realisation of the virtual microphone arrangement is given by

\[
G_{VMA} \sim \begin{bmatrix} A & B_u & 0 \\ C_v - C_p & D_{vu} - D_{pu} & \mathbf{I} \end{bmatrix}.
\] (B.109)

The order of the derived minimal state-space realisation of the virtual microphone arrangement is equal to the order \( N \) of the standard state-space model \( G \) of the considered active noise control system that has been defined in Eq. (2.18) on page 46. It can also be noted that the virtual microphone arrangement is a simplified version of the Kalman filter based spatially fixed virtual sensing algorithm defined in Eq. (B.91), where it is assumed that the Kalman gain matrix \( K_{ps} = 0 \) and the virtual gain matrix \( M_{vs} = \mathbf{I} \).
References


References


References


References


References


References


(Cited on page 100)


(Cited on pages 9, 10, 16, and 150)


References


References


